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Cyclic Admissible Multivalued Contraction and an Application to Cyclic mappings

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ABSTRACT: We prove the existence of coincidence point for a hybrid pair as well as a pair of single valued mappings satisfying certain cyclic type contractive conditions in the framework of b-metric spaces. Our results are illustrated with examples and an application to cyclic mappings.

Key Words: Cyclic (α, β)-admissible, (α, β)-(ψ, ϕ)-contractive mapping, cyclic (α, β)-(ξ, ψ, ϕ) - contractive mapping, coincidence point, fixed point, b-metric spaces.

Contents

1. Introduction and Preliminaries

The Banach contraction principle, one of the most important tools in fixed point theory states that every contraction on a complete metric space has a unique fixed point. Several authors have extended or generalized this result in various directions. Our interest is on the space of b-metric space introduced by Czerwik [5] in 1993. Throughout this paper, let X be a nonempty set, $\mathbb{R}^+$ denote the set of non-negative real numbers and $s \geq 1$ a fixed real number.

Definition 1.1 *[2] A function $d : X \times X \to \mathbb{R}^+$ (nonnegative real numbers) is called a b-metric provided that, for all $x, y, z \in X$, the following conditions are satisfied:*

1. $d(x, y) = 0$ *if and only if* $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space with parameter s.

Definition 1.2 *[4] Let (X, d) be a b-metric space. A sequence $\{x_n\}$ in X is called a Cauchy sequence if for every $\epsilon > 0$, there exists $K(\epsilon) \in \mathbb{N}$, such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq K(\epsilon)$. A sequence $\{x_n\}$ in X is said to converge to $x \in X$ if for every $\epsilon > 0$, there exists $K(\epsilon) \in \mathbb{N}$, such that $d(x_n, x) < \epsilon$ for all $n \geq K(\epsilon)$. In this case, we write $\lim_{n\to\infty} x_n = x$. The b-metric space (X, d) is complete if every Cauchy sequence in X converges in X.*

Remark 1.1 *In a b-metric space (X, d) the following assertions hold:*

1. *A convergent sequence has a unique limit.*
2. *Every convergent sequence is Cauchy.*

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Let (X,d) be a b-metric space with parameter $s \geq 1$. Let $N(X)$ be the collection of nonempty subsets of X, $\mathcal{CL}(X)$ be the collection of all nonempty closed subsets of X and $\mathcal{CB}(X)$ be the collection of all nonempty closed and bounded subsets of X. For $A, B \in \mathcal{CB}(X)$, define

$$H(A,B) = \max\{\delta(A,B), \delta(B,A)\},$$

where $\delta(A,B) = \sup\{D(a,B); a \in A\}$, $\delta(B,A) = \sup\{D(b,A); b \in B\}$ with $D(a,C) = \inf\{d(a,x); x \in C\}, C \in \mathcal{CB}(X)$.
The mapping H is said to be a b-Hausdorff metric induced by the b-metric d.

Let X be any nonempty set. An element $x \in X$ is said to be a fixed point of a multi-valued mapping $T : X \to N(X)$ if $x \in Tx$. Let (X,d) be a b-metric space with parameter $s \geq 1$. A point $x \in X$ is said to be a coincidence point of hybrid pair $f : X \to X$ and $T : X \to N(X)$ if $fx \in Tx$. The set of coincidence points of f and T is denoted by $C(f,T)$. If f and T are both self mappings, then $x \in X$ is called a coincidence point of f and T if $fx = Tx$. A point $y \in X$ is called a point of coincidence of f and T if there exists a point $x \in X$ such that $y = fx = Tx$.

Let (X,d) be a b-metric space with parameter $s \geq 1$. The self mappings $f : X \to X$ and $T : X \to X$ are compatible [7] if and only if $d(Tfx_n, fTx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $Tx_n \to z$ and $fx_n \to z$ for some $z \in X$. The mappings $f : X \to X$ and $T : X \to \mathcal{CB}(X)$ are compatible [7] if and only if $fTx \subseteq \mathcal{CB}(X)$ for all $x \in X$ and $H(Tfx_n, fTx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $Tx_n \to A \in \mathcal{CB}(X)$ and $fx_n \to y \in A$. The mappings $f : X \to X$ and $T : X \to \mathcal{CL}(X)$ are weakly compatible [6] if they commute at their coincidence points, i.e. if $fTx = Tfx$ whenever $fx \in Tx$.

We denote by Ψ_b the family of all strictly increasing functions $\psi : [0,\infty) \to [0,\infty)$ such that the series $\sum_{n=1}^{\infty} s^n \psi^n(t)$ converges for any $t \in [0,\infty)$ where ψ^n is the n^{th} iterate of ψ. Also, let Φ be the family of functions $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) = 0 \Leftrightarrow t = 0$. It is easy to see that if $\psi \in \Psi_b$, then $\psi(t) < t$ for all $t > 0$. Infact, if there is a $t_0 > 0$ such that $\psi(t_0) \geq t_0$, then we have $\psi^2(t_0) \geq \psi(t_0) \geq t_0$ (since ψ is increasing). Continuing like this, we get $\psi^n(t_0) \geq t_0 > 0$, $n \in \mathbb{N}$. This contradicts the fact that $\psi \in \Psi_b$.

We denote by Ξ the family of functions $\xi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

1. ξ is continuous.
2. ξ is nondecreasing on $[0,\infty)$.
3. $\xi(0) = 0$ and $\xi(t) > 0$ for all $t \in (0,\infty)$.
4. ξ is subadditive and $\xi(ct) \leq c\xi(t)$ where $c \geq 1$.

It is easy to show that if (X,d) is a b-metric space and $\xi \in \Xi$, then $(X, \xi \circ d)$ is a b-metric space.

Several authors have dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces. In 2012, Samet et al. [9] presented the concepts of α-ψ-contractive and α-admissible mappings. Asl et al. [3] extended these concepts to multi-valued mappings by presenting the notions of α_*-ψ-contractive and α_*-admissible mappings and proved some fixed point results for these mappings. In 2014, Alizadeh et al. [1] offered the notion of cyclic (α,β)-admissible mapping and proved some new fixed point results in complete metric spaces which generalize some recent results in the literature.

Definition 1.3 *[9] Let X be a non empty set. Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be mappings. We say that T is α-admissible if $\alpha(Tx,Ty) \geq 1$ whenever $\alpha(x,y) \geq 1$ for all $x, y \in X$.*

Definition 1.4 *[3] Let X be a non empty set. Let $T : X \to N(X)$ and $\alpha : X \times X \to [0,\infty)$ be mappings. We say that T is α_*-admissible if $\alpha_*(Tx,Ty) \geq 1$ whenever $\alpha(x,y) \geq 1$ for all $x, y \in X$, where $\alpha_*(Tx,Ty) = \inf\{\alpha(a,b) : a \in Tx, b \in Ty\}$.*

Definition 1.5 *[1] Let X be a non empty set. Let $T : X \to X$ be a mapping and $\alpha, \beta : X \to [0,\infty)$ be two functions. We say that T is a cyclic (α,β)-admissible mapping if $\beta(Tx) \geq 1$ whenever $\alpha(x) \geq 1$ for some $x \in X$ and $\alpha(Tx) \geq 1$ whenever $\beta(x) \geq 1$ for some $x \in X$.*

Kaushik and Kumar [8] in 2016, introduced the notion of (α,ψ,ξ)-contractive multi-valued mappings for a pair of mappings and proved some fixed point theorems for such mappings under certain conditons. Also, Yamaoda and Sintunavarata [10] introduced and studied the notion of (α,β)-(ψ,ϕ)-contractive mappings in the setting of b-metric space. In this paper, we introduce the notion of (α_*,β_*)-(ξ,ψ,ϕ)- contractive mappings and establish the existence of coincidence point for continuous and compatible hybrid pair of mappings in b-metric spaces satisfying contractive type conditions. We will also prove the existence of unique point of coincidence for a pair of self mappings satisfying certain conditions and the existence of unique common fixed point for a pair of weakly compatible self mappings. The existence of coincidence point and common fixed point for continuous and compatible pair of self mappings satisfying contractive type conditions will also be established. Consequently, we apply our main results to establish coincidence point theorems for cyclic mappings. Examples are also furnished to illustrate main results.

2. Main Results

In this section, we first define cyclic (α_*,β_*)-admissible mapping, cyclic (α,β)-admissible mapping with respect to a self mapping and cyclic (α_*,β_*)-admissible with respect to a self mapping. We extend these definitions further to introduce the notion of (α_*,β_*)-(ξ,ψ,ϕ)- contractive mappings and establish the existence of coincidence point and common fixed point for a hybrid pair of mappings as well as pair of self mappings satisfying various sets of conditions.

Definition 2.1 *Let X be a non empty set. Let $T : X \to N(X)$ be a mapping and $\alpha,\beta : X \to [0,\infty)$ be two functions. We say that T is a cyclic (α_*,β_*)-admissible mapping if $\beta_*(Tx) \geq 1$ whenever $\alpha(x) \geq 1$ for some $x \in X$ and $\alpha_*(Tx) \geq 1$ whenever $\beta(x) \geq 1$ for some $x \in X$, where $\alpha_*(Tx) = \inf\{\alpha(a) : a \in Tx\}$ and $\beta_*(Tx) = \inf\{\beta(b) : b \in Tx\}$.*

Definition 2.2 *Let X be a non empty set. Let $T : X \to X$ and $f : X \to X$ be mappings and $\alpha,\beta : X \to [0,\infty)$ be two functions. We say that T is a cyclic (α,β)-admissible mapping with respect to f if $\beta(Tx) \geq 1$ whenever $\alpha(fx) \geq 1$ for some $x \in X$ and $\alpha(Tx) \geq 1$ whenever $\beta(fx) \geq 1$ for some $x \in X$.*

Definition 2.3 *Let X be a non empty set. Let $T : X \to N(X)$ and $f : X \to X$ be mappings and $\alpha,\beta : X \to [0,\infty)$ be two functions. We say that T is a cyclic (α_*,β_*)-admissible mapping with respect to f if $\beta_*(Tx) \geq 1$ whenever $\alpha(fx) \geq 1$ for some $x \in X$ and $\alpha_*(Tx) \geq 1$ whenever $\beta(fx) \geq 1$ for some $x \in X$, where $\alpha_*(Tx) = \inf\{\alpha(a) : a \in Tx\}$ and $\beta_*(Tx) = \inf\{\beta(b) : b \in Tx\}$.*

We give below an example of a cyclic (α_*,β_*)-admissible mapping with respect to a self mapping f.

Example 2.1 *Let $X = \mathbb{R}$, $T : X \to N(X)$ be defined by $Tx = \{-x^5\}$.*
Suppose that $\alpha,\beta : X \to [0,\infty)$ are given by $\alpha(x) = e^x$ $\forall x \in \mathbb{R}$ and $\beta(x) = e^{-x}$ $\forall x \in \mathbb{R}$. Let $f : X \to X$ be defined by $f(x) = ax$ $\forall x \in X, a > 0$.
Now if $\alpha(fx) = e^{ax} \geq 1$, then $x \geq 0$. This implies $-Tx \geq 0$ and hence $\beta_*(Tx) = \beta(Tx) = e^{-Tx} \geq 1$. Again if $\beta(fx) = e^{-ax} \geq 1$, then $x \leq 0$. This implies $Tx \geq 0$ and $\alpha_*(Tx) = \alpha(Tx) = e^{Tx} \geq 1$. Thus T is a cyclic (α_*,β_*)-admissible mapping with respect to f.

We now define the notion of (α_*,β_*)-(ξ,ψ,ϕ)-contractive mappings in b-metric spaces.

Definition 2.4 *Let (X,d) be a b-metric space with $s \geq 1$.Then the hybrid pair $T : X \to \mathcal{CB}(X)$ and $f : X \to X$ is called an (α_*,β_*)-(ξ,ψ,ϕ)- contractive mappings if there exist functions $\alpha,\beta : X \to \mathbb{R}^+$, $\xi \in \Xi$, $\psi \in \Psi_b$ and $\phi \in \Phi$ such that for all $x,y \in X$ with $\alpha(fx)\beta(fy) \geq 1$, we have $\xi(sH(Tx,Ty)) \leq \psi(\xi(M(x,y))) - \phi(M(x,y))$ where $M(x,y) = \max\{d(fx,fy), D(fx,Tx), D(fy,Ty), \frac{D(fx,Ty)+D(fy,Tx)}{2s}\}$.*

We state and prove the following lemma which will be used in the subsequent theorem.

Lemma 2.1 *Let (X,d) be a b-metric space, let $\xi \in \Xi$ and let $A \in N(X)$. Assume that there exists $x \in X$ such that $\xi(D(x,A)) > 0$. Then there exists $y \in A$ such that $\xi(d(x,y)) < r\xi(D(x,A))$, where $r > 1$.*

Proof: By hypothesis we have $\xi(D(x, A)) > 0$. Choose $\epsilon = (r-1)\xi(D(x, A)$. Now, since $\xi \circ d$ is a b-metric space, it follows by definition of infimum that there exists $y \in A$ such that $\xi(d(x, y)) \leq \xi(D(x, A)) + \epsilon = r\xi(D(x, A))$. □

In the following theorem, we establish the existence of coincidence point and common fixed point for continuous and compatible hybrid pair of mappings satisfying contractive type conditions in the framework of b-metric spaces.

Theorem 2.1 *Let (X, d) be a complete b-metric space with parameter $s \geq 1$ and $f : X \to X$ and $T : X \to \mathcal{CB}(X)$ be a continuous and compatible hybrid pair such that $Tx \subseteq f(X)$ for all x in X. Suppose that the following conditions hold:*
(i) T is a cyclic (α_*, β_*)-admissible mapping with respect to f.
(ii) T and f are (α_*, β_*)-(ξ, ψ, ϕ)-contractive mappings.
(iii) There exist $x_0 \in X$ and such that $\alpha(fx_0) \geq 1$ and $\beta(fx_0) \geq 1$.
Then f and T have a coincidence point. Further if $ffa = fa$ for some $a \in C(f, T)$, then f and T have a common fixed point.

Proof: Let $x_0 \in X$ such that $\alpha(fx_0) \geq 1$ and $\beta(fx_0) \geq 1$. Since $Tx_0 \subseteq f(X)$, there exists $x_1 \in X$ such that $fx_1 \in Tx_0$. Since T is cyclic (α_*, β_*)-admissible mapping with respect to f, we obtain $\beta(fx_1) \geq \beta_*(Tx_0) \geq 1$ and $\alpha(fx_1) \geq \alpha_*(Tx_0) \geq 1$. Therefore, $\alpha(fx_0)\beta(fx_1) \geq 1$. If $fx_0 = fx_1$ or $fx_1 \in Tx_1$, then x_0 or x_1 is a coincidence point of f and T and hence we are done. So we assume $fx_0 \neq fx_1$ or $fx_1 \notin Tx_1$. Now, by (ii) we have

$$\begin{aligned} 0 < \xi(D(fx_1, Tx_1)) \leq \xi(H(Tx_0, Tx_1)) &\leq \xi(sH(Tx_0, Tx_1)) \\ &\leq \psi(\xi(M(x_0, x_1))) - \phi(M(x_0, x_1)) \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} M(x_0, x_1) &= \max\{d(fx_0, fx_1), D(fx_0, Tx_0), D(fx_1, Tx_1), \frac{D(fx_0, Tx_1) + D(fx_1, Tx_0)}{2s}\} \\ &= \max\{d(fx_0, fx_1), D(fx_1, Tx_1), \frac{D(fx_0, Tx_1)}{2s}\} \\ &\leq \max\{d(fx_0, fx_1), D(fx_1, Tx_1), \frac{d(fx_0, fx_1) + D(fx_1, Tx_1)}{2}\} \\ &= \max\{d(fx_0, fx_1), D(fx_1, Tx_1)\}. \end{aligned}$$

In case, $\max\{d(fx_0, fx_1), D(fx_1, Tx_1)\} = D(fx_1, Tx_1)$, we obtain from equation 2.1 that

$$0 < \xi(D(fx_1, Tx_1)) \leq \psi(\xi(D(fx_1, Tx_1))) - \phi(D(fx_1, Tx_1)) < \psi(\xi(D(fx_1, Tx_1)))$$

which is a contradiction. Thus, $M(x_0, x_1) = d(fx_0, fx_1)$. Therefore,

$$0 < \xi(D(fx_1, Tx_1)) \leq \psi(\xi(d(fx_0, fx_1))) - \phi(d(fx_0, fx_1)) < \psi(\xi(d(fx_0, fx_1))).$$

Now, for $r > 1$, by lemma 2.1, there exists $fx_2 \in Tx_1$ such that

$$0 < \xi(d(fx_1, fx_2)) < r\xi(D(fx_1, Tx_1)) < r\psi(\xi(d(fx_0, fx_1))).$$

By applying ψ, we get
$0 < \psi(\xi(d(fx_1, fx_2))) < \psi(r\psi(\xi(d(fx_0, fx_1))))$.
Put $r_1 = \frac{\psi(r\psi(\xi(d(fx_0, fx_1))))}{\psi(\xi(d(fx_1, fx_2)))} > 1$.
Since T is cyclic (α_*, β_*)-admissible mapping with respect to f and $fx_2 \in Tx_1$, we have $\alpha(fx_1) \geq 1$ which implies $\beta(fx_2) \geq \beta_*(Tx_1) \geq 1$ and $\beta(fx_1) \geq 1$ which implies that $\alpha(fx_2) \geq \alpha_*(Tx_1) \geq 1$. So, $\alpha(fx_1)\beta(fx_2) \geq 1$. Now if $fx_2 \in Tx_2$, then x_2 is a coincidence point of f and T. So, assume that $fx_2 \notin Tx_2$. We have

$$\begin{aligned} 0 < \xi(D(fx_2, Tx_2)) \leq \xi(H(Tx_1, Tx_2)) &\leq \xi(sH(Tx_1, Tx_2)) \\ &\leq \psi(\xi(M(x_1, x_2))) - \phi(M(x_1, x_2)) \end{aligned}$$

where

$$\begin{aligned}M(x_1,x_2) &= \max\{d(fx_1,fx_2), D(fx_1,Tx_1), D(fx_2,Tx_2), \frac{D(fx_1,Tx_2)+D(fx_2,Tx_1)}{2s}\}\\ &= \max\{d(fx_1,fx_2), D(fx_2,Tx_2), \frac{D(fx_1,Tx_2)}{2s}\}\\ &\le \max\{d(fx_1,fx_2), D(fx_2,Tx_2), \frac{d(fx_1,fx_2)+D(fx_2,Tx_2)}{2}\}\\ &= \max\{d(fx_1,fx_2), D(fx_2,Tx_2)\}.\end{aligned}$$

If $\max\{d(fx_1,fx_2), D(fx_2,Tx_2)\} = D(fx_2,Tx_2)$, we get

$$0 < \xi(D(fx_2,Tx_2)) \le \psi(\xi(D(fx_2,Tx_2))) - \phi(D(fx_2,Tx_2)) < \psi(\xi(D(fx_2,Tx_2)))$$

which is a contradiction. Thus $M(x_1,x_2) = d(fx_1,fx_2)$. Therefore,

$$0 < \xi(D(fx_2,Tx_2)) \le \psi(\xi(d(fx_1,fx_2))) - \phi(d(fx_1,fx_2)) < \psi(\xi(d(fx_1,fx_2))).$$

For $r_1 > 1$, by lemma 2.1, there exists $fx_3 \in Tx_2$ such that

$$0 < \xi(d(fx_2,fx_3)) < r_1\xi(D(fx_2,Tx_2)) < r_1\psi(\xi(d(fx_1,fx_2))) = \psi(r\psi(\xi(d(fx_0,x_1)))).$$

By applying ψ, we obtain

$$0 < \psi(\xi(d(fx_2,fx_3))) < \psi^2(r\psi(\xi(d(fx_0,fx_1)))).$$

Continuing this process, we can construct a sequence $\{fx_n\}$ in X such that $fx_n \in Tx_n$, $\alpha(fx_n)\beta(fx_{n+1}) \ge 1$ and $0 < \xi(d(fx_{n+1},fx_{n+2}))) < \psi^n(r\psi(\xi(d(fx_0,fx_1))))$ for $n \in \mathbb{N}\cup\{0\}$. Now, we show that $\{fx_n\}$ is a Cauchy sequence in X.
Let $m,n \in \mathbb{N}$, $m > n \ge 2$. We have

$$\begin{aligned}\xi(d(fx_n,fx_m)) &\le \xi(s[d(fx_n,fx_{n+1}) + d(fx_{n+1},fx_m)])\\ &\le \xi(sd(fx_n,fx_{n+1}) + s^2[d(fx_{n+1},fx_{n+2}) + d(fx_{n+2},fx_m)])\\ &\le \frac{1}{s^{n-2}}\sum_{i=n}^{m-1} s^{i-1}\xi(d(fx_i,fx_{i+1}))\\ &< \frac{1}{s^{n-2}}\sum_{i=n}^{m-1} s^{i-1}\psi^{i-1}(r\psi(\xi(d(fx_0,fx_1))))\\ &= \frac{1}{s^{n-2}}\sum_{i=n-1}^{m-2} s^{i}\psi^{i}(r\psi(\xi(d(fx_0,fx_1)))).\end{aligned}$$

Let $\{S_n\}$ be the sequence of partial sums of the series $\sum_{k=0}^{\infty} s^k\psi^k(r\psi(\xi(d(fx_0,fx_1))))$.
Then

$$\xi(d(fx_n,fx_m)) \le \frac{1}{s^{n-2}}[S_{m-1} - S_{n-1}], n \ge 2. \tag{2.2}$$

By definition of Ψ_b, $\sum_{k=0}^{\infty} s^k\psi^k(r\psi(\xi(d(fx_0,fx_1))))$ is convergent.
Let $\lim S_n = S$. Since $s \ge 1$, using eq (2.2) we obtain $\xi(d(fx_n,fx_m)) \to 0$ as $n \to \infty$. Using properties of ξ, we have $d(fx_n,fx_m) \to 0$ as $n \to \infty$.
That is, $\{fx_n\}$ is a Cauchy sequence in the b-metric space (X,d). Since (X,d) is complete, there exists $x^* \in X$ such that $fx_n \to x^*$ as $n \to \infty$.
Further, above inequalities show that $\xi(H(Tx_n,Tx_{n-1})) < \psi(\xi(fx_n,fx_{n-1}))$.
This implies that $\{Tx_n\}$ is a Cauchy sequence in the complete b-metric space $(\mathcal{CB}(X),H)$.
Let $Tx_n \to A \in \mathcal{CB}(X)$. Consider
$D(x^*,A) \le s[d(x^*,fx_n) + D(fx_n,A)] \le s[d(x^*,fx_n) + H(Tx_{n-1},A)] \to 0$ as $n \to \infty$.

Since A is closed, we have $x^* \in A$ and by compatibility of f and T, we conclude that $H(Tfx_n, fTx_n) \to 0$ as $n \to \infty$. Further, we have

$$\begin{aligned} D(fx^*, Tx^*) &\leq s[d(fx^*, ffx_{n+1}) + D(ffx_{n+1}, Tx^*)] \\ &\leq s[d(fx^*, ffx_{n+1}) + H(fTx_n, Tx^*)] \\ &\leq s[d(fx^*, ffx_{n+1}) + sH(fTx_n, Tfx_n) + sH(Tfx_n, Tx^*)] \end{aligned}$$

$\to 0$ as $n \to \infty$ (Since f and T are continuous). Since Tx^* is closed, we have $fx^* \in Tx^*$.
That is, x^* is a coincidence point of f and T. Further compatibility implies that f and T commute at their coincidence point, that is, $fTx^* = Tfx^*$.
So, if $ffx^* = fx^*$, then $y = fx^* = ffx^* \in fTx^* = Tfx^* = Ty$, that is, $fy = y \in Ty$. Thus, f and T have a common fixed point. □

Corollary 2.1 *Let (X, d) be a complete b-metric space with parameter $s \geq 1$ and $f : X \to X$ and $T : X \to \mathcal{CB}(X)$ be a continuous and compatible hybrid pair such that $Tx \subseteq f(X)$ for all x in X. Suppose that the following conditions hold:*
(i) T is a cyclic (α_*, β_*)-admissible mapping with respect to f.
(ii) T and f satisfies $\alpha(x)\beta(y)\xi(sH(Tx, Ty)) \leq \psi(\xi(M(x, y))) - \phi(M(x, y))$ where $M(x, y) = \max\{d(fx, fy), D(fx, Tx), D(fy, Ty), \frac{D(fx,Ty)+D(fy,Tx)}{2s}\}$ and $\alpha, \beta : X \to \mathbb{R}^+$, $\xi \in \Xi$, $\psi \in \Psi_b$ and $\phi \in \Phi$.
(iii) There exist $x_0 \in X$ and $fx_1 \in Tx_0$ such that $\alpha(fx_0) \geq 1$ and $\beta(fx_0) \geq 1$.
Then f and T have a coincidence point. Further if $ffa = fa$ for some $a \in C(f, T)$ then f and T have a common fixed point.

Proof: Let $\alpha(x)\beta(y) \geq 1$ for all $x, y \in X$. We have

$$\xi(sH(Tx, Ty)) \leq \alpha(x)\beta(y)\xi(sH(Tx, Ty)) \leq \psi(\xi(M(x, y))) - \phi(M(x, y)).$$

This shows that T and f are (α_*, β_*)-(ξ, ψ, ϕ)-contractive mappings. Thus by Theorem 2.1, we obtain the desired result. □

Corollary 2.2 *Let (X, d) be a complete b-metric space with parameter $s \geq 1$ and $f : X \to X$ and $T : X \to \mathcal{CB}(X)$ be a continuous and compatible hybrid pair such that $Tx \subseteq f(X)$ for all x in X. Suppose that the following conditions hold:*
(i) T is a cyclic (α_*, β_*)-admissible mapping with respect to f.
(ii) T and f satisfies $H(Tx, Ty)) \leq k(M(x, y)$ whenever $\alpha(x)\beta(y) \geq 1$, where $k \in [0, 1)$, $M(x, y) = \max\{d(fx, fy), D(fx, Tx), D(fy, Ty), \frac{D(fx,Ty)+D(fy,Tx)}{2s}\}$ and $\alpha, \beta : X \to \mathbb{R}^+$.
(iii) There exist $x_0 \in X$ and $fx_1 \in Tx_0$ such that $\alpha(fx_0) \geq 1$ and $\beta(fx_0) \geq 1$.
Then f and T have a coincidence point. Further if $ffa = fa$ for some $a \in C(f, T)$ then f and T have a common fixed point.

Proof: In Theorem 2.1, take $\xi(t) = \psi(t) = t$ and $\phi(t) = (1 - k)t$ for all t in $[0, \infty)$. □

Corollary 2.3 *Let (X, d) be a complete b-metric space with parameter $s \geq 1$ and $T : X \to \mathcal{CB}(X)$ be a continuous multivalued mapping. Suppose that the following conditions hold:*
(i) T is a cyclic (α_*, β_*)-admissible mapping.
(ii) T satisfies $\xi(sH(Tx, Ty)) \leq \psi(\xi(M(x, y))) - \phi(M(x, y))$ whenever $\alpha(x)\beta(y) \geq 1$, $x, y \in X$, where $M(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x,Ty)+D(y,Tx)}{2s}\}$ and $\alpha, \beta : X \to \mathbb{R}^+$, $\xi \in \Xi$, $\psi \in \Psi_b$ and $\phi \in \Phi$.
(iii) There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.
Then T has a fixed point in X.

Proof: Let $f(x) = x$ for all x in X in Theorem 2.1. □

Following theorem deals with the existence of coincidence point for a hybrid pair of mappings with different set of conditions.We relax the condition of continuity and compatibility of the hybrid pair and assume that the range of the self mapping f is closed.

Theorem 2.2 *Let (X, d) be a complete b-metric space with parameter $s \geq 1$ and $f : X \to X$ and $T : X \to \mathcal{CB}(X)$ be a hybrid pair such that $Tx \subseteq f(X)$ for all x in X and $f(X)$ is a closed subset of X. Suppose that the following conditions hold:*
(i) T is a cyclic (α_*, β_*)-admissible mapping with respect to f.
(ii) T and f are (α_*, β_*)-(ξ, ψ, ϕ)-contractive mapping.
(iii) There exist $x_0 \in X$ and $fx_1 \in Tx_0$ such that $\alpha(fx_0) \geq 1$ and $\beta(fx_0) \geq 1$.
(iv) If $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$.
Then f and T have a coincidence point.

Proof: Proceeding as in Theorem 2.1, we can define sequences $\{fx_n\}$ and $\{Tx_n\}$ such that $\lim fx_n = x^*$ and $\lim Tx_n = A$ where $A \in \mathcal{CB}(X)$. Since $f(X)$ is closed, we obtain $x^* \in f(X)$, that is, there exists $u \in X$ such that $x^* = fu$. Suppose that $D(fu, Tu) \neq 0$. Now as $y_n = fx_n \to x^*$ and $\beta(y_n) = \beta(fx_n) \geq 1$ $\forall n$, by condition (iv), $\beta(x^*) = \beta(fu) \geq 1$. Thus, $\alpha(fx_n)\beta(fu) \geq 1$ for all n in $\mathbb{N}$. Consider

$$\begin{aligned} \xi(sD(fx_{n-1}, Tu)) &\leq \xi(sH(Tx_n, Tu)) \leq \psi(\xi(M(x_n, u))) - \phi(M(x_n, u)) \\ &\leq \psi(\xi(M(x_n, u))) \end{aligned}$$

where $M(x_n, u) = \max\{d(fx_n, fu), D(fx_n, Tx_n), D(fu, Tu), \frac{D(fx_n,Tu)+D(fu,Tx_n)}{2s}\}$
Taking lim sup as $n \to \infty$, we have

$$\xi(D(fu, Tu)) \leq \psi(\xi(D(fu, Tu))) < \xi(D(fu, Tu))$$

which is a contradiction. Therefore, $D(fu, Tu) = 0$. Since Tu is closed, we conclude that $fu \in Tu$. That is, f and T have a coincidence point. □

Corollary 2.4 *Let (X, d) be a complete b-metric space with parameter $s \geq 1$ and $f : X \to X$ and $T : X \to \mathcal{CB}(X)$ be a hybrid pair such that $Tx \subseteq f(X)$ for all x in X and $f(X)$ is a closed subset of X. Suppose that the following conditions hold:*
(i) T is a cyclic (α_*, β_*)-admissible mapping with respect to f.
(ii) T and f satisfies $\alpha(x)\beta(y)\xi(sH(Tx, Ty)) \leq \psi(\xi(M(x, y))) - \phi(M(x, y))$ where
$M(x, y) = \max\{d(fx, fy), D(fx, Tx), D(fy, Ty), \frac{D(fx,Ty)+D(fy,Tx)}{2s}\}$ and $\alpha, \beta : X \to \mathbb{R}^+$, $\xi \in \Xi$, $\psi \in \Psi_b$ and $\phi \in \Phi$.
(iii) There exist $x_0 \in X$ and $fx_1 \in Tx_0$ such that $\alpha(fx_0) \geq 1$ and $\beta(fx_0) \geq 1$.
(iv) If $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$.
Then f and T have a coincidence point.

Proof: Let $\alpha(x)\beta(y) \geq 1$ for all $x, y \in X$. Then

$$\xi(sH(Tx, Ty)) \leq \alpha(x)\beta(y)\xi(sH(Tx, Ty)) \leq \psi(\xi(M(x, y))) - \phi(M(x, y)).$$

This shows that T and f are (α_*, β_*)-(ξ, ψ, ϕ)- contractive mappings. Thus, by Theorem 2.2, we obtain the desired result. □

Corollary 2.5 *Let (X, d) be a complete b-metric space with parameter $s \geq 1$ and $f : X \to X$ and $T : X \to \mathcal{CB}(X)$ be a hybrid pair such that $Tx \subseteq f(X)$ for all x in X and $f(X)$ is a closed subset of X. Suppose that the following conditions hold:*
(i) T is a cyclic (α_*, β_*)-admissible mapping with respect to f.
(ii) T and f satisfies $H(Tx, Ty)) \leq k(M(x, y)$ whenever $\alpha(x)\beta(y) \geq 1$, where $k \in [0, 1)$,

$M(x,y) = \max\{d(fx,fy), D(fx,Tx), D(fy,Ty), \frac{D(fx,Ty)+D(fy,Tx)}{2s}\}$ *and* $\alpha, \beta : X \to \mathbb{R}^+$.
(iii) There exist $x_0 \in X$ *and* $fx_1 \in Tx_0$ *such that* $\alpha(fx_0) \geq 1$ *and* $\beta(fx_0) \geq 1$.
(iv) If $\{x_n\}$ *is a sequence in* X *such that* $x_n \to x$ *and* $\beta(x_n) \geq 1$ *for all* n*, then* $\beta(x) \geq 1$.
Then f *and* T *have a coincidence point.*

Proof: Take $\xi(t) = \psi(t) = t$ and $\phi(t) = (1-k)t$ for all $t \geq 0$ in Theorem 2.2. □

Corollary 2.6 *Let* (X,d) *be a complete b-metric space with parameter* $s \geq 1$ *and* $T : X \to \mathcal{CB}(X)$ *be a multivalued mapping. Suppose that the following conditions hold:*
(i) T *is a cyclic* (α_*, β_*)*-admissible mapping.*
(ii) T *satisfies* $\xi(sH(Tx,Ty)) \leq \psi(\xi(M(x,y))) - \phi(M(x,y))$ *whenever* $\alpha(x)\beta(y) \geq 1$*, where* $M(x,y) = \max\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty)+D(y,Tx)}{2s}\}$ *and* $\alpha, \beta : X \to \mathbb{R}^+$, $\xi \in \Xi$, $\psi \in \Psi_b$ *and* $\phi \in \Phi$.
(iii) There exist $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0) \geq 1$ *and* $\beta(x_0) \geq 1$.
(iv) If $\{x_n\}$ *is a sequence in* X *such that* $x_n \to x$ *and* $\beta(x_n) \geq 1$ *for all* n*, then* $\beta(x) \geq 1$.
Then T *has a fixed point in* X.

Proof: Take $f(x) = x$ for all x in X in Theorem 2.2. □

In the following theorem, we prove the existence of unique point of coincidence for a pair of self mappings satisfying certain conditions. We also prove the existence of unique common fixed point for a pair of weakly compatible self mappings.

Theorem 2.3 *Let* (X,d) *be a complete b-metric space with parameter* $s \geq 1$ *and* $f : X \to X$ *and* $T : X \to X$ *be mappings such that* $T(X) \subseteq f(X)$ *and* $f(X)$ *is a closed subset of* X. *Suppose that the following conditions hold:*
(i) T *is a cyclic* (α, β)*-admissible mapping with respect to* f.
(ii) T *and* f *satisfies* $\psi(s^3 d(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y))$ *whenever* $\alpha(fx)\beta(fy) \geq 1$*,where* $M(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2s}\}$, $\alpha, \beta : X \to [0,\infty)$ *are functions and* $\psi, \phi : [0,\infty) \to [0,\infty)$ *are altering distance functions.*
(iii) There exist $x_0 \in X$ *such that* $\alpha(fx_0) \geq 1$ *or* $\beta(fx_0) \geq 1$.
(iv) If $\{x_n\}$ *is a sequence in* X *such that* $x_n \to x$ *and* $\beta(x_{2n}) \geq 1$ *or* $\beta(x_{2n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$*, then* $\beta(x) \geq 1$.
(v) $\alpha(fu) \geq 1$ *and* $\beta(fu) \geq 1$ *whenever* $fu = Tu$.
Then f *and* T *have a unique point of coincidence in* X. *Moreover, if* f *and* T *are weakly compatible , then* f *and* T *have a unique commmon fixed point.*

Proof: Case I: Let $x_0 \in X$ be such that $\alpha(fx_0) \geq 1$. Define a sequence $\{x_n\}$ in X as $y_n = fx_n = Tx_{n-1}$ for all $n \geq 1$(This is possible as $T(X) \subseteq f(X)$). If $y_n = y_{n+1}$, that is, $fx_n = Tx_{n-1} = fx_{n+1} = Tx_n$, then x_n is a coincidence point of f and T. Suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Now, since T is a cyclic (α, β)-admissible mapping with respect to f, we conclude that $\alpha(fx_0) \geq 1$ implies $\beta(Tx_0) = \beta(fx_1) \geq 1$ and $\beta(fx_1) \geq 1$ implies $\alpha(Tx_1) = \alpha(fx_2) \geq 1$.
Continuing like this, we get $\alpha(fx_{2k}) \geq 1$ and $\beta(fx_{2k+1}) \geq 1$ for all $k \in \mathbb{N} \cup \{0\}$.
Since $\alpha(fx_0)\beta(fx_1) \geq 1$, we obtain

$$\psi(s^3 d(fx_1, fx_2)) = \psi(s^3 d(Tx_0, Tx_1)) \leq \psi(M(x_0, x_1)) - \phi(M(x_0, x_1)).$$

Also, since $\alpha(fx_2)\beta(fx_1) \geq 1$, we have

$$\begin{aligned}\psi(s^3 d(fx_2, fx_3)) = \psi(s^3 d(Tx_1, Tx_2)) = \psi(s^3 d(Tx_2, Tx_1)) &\leq \psi(M(x_2, x_1)) - \phi(M(x_2, x_1)) \\ &= \psi(M(x_1, x_2)) - \phi(M(x_1, x_2)).\end{aligned}$$

Proceeding in the same manner, we obtain

$$\psi(s^3 d(fx_{n+1}, fx_{n+2})) = \psi(s^3 d(Tx_n, Tx_{n+1})) \leq \psi(M(x_n, x_{n+1})) - \phi(M(x_n, x_{n+1}))$$

where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), \\ &\qquad \frac{d(fx_n, Tx_{n+1}) + d(fx_{n+1}, Tx_n)}{2s}\} \\ &= \max\{d(fx_n, fx_{n+1}), d(fx_{n+1}, fx_{n+2}), \frac{d(fx_n, Tx_{n+2})}{2s}\} \\ &\leq \max\{d(fx_n, fx_{n+1}), d(fx_{n+1}, fx_{n+2}), \frac{s[d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2})]}{2s}\} \\ &= \max\{d(fx_n, fx_{n+1}), d(fx_{n+1}, fx_{n+2})\}. \end{aligned}$$

If $M(x_n, x_{n+1}) = d(fx_{n+1}, fx_{n+2})$, then

$$\psi(s^3 d(fx_{n+1}, fx_{n+2})) \leq \psi(d(fx_{n+1}, fx_{n+2})) - \phi(d(fx_{n+1}, fx_{n+2})) < \psi(d(fx_{n+1}, fx_{n+2})),$$

which is a contradiction. Therefore, $M(x_n, x_{n+1}) = d(fx_n, fx_{n+1})$. Thus, we have

$$\begin{aligned} \psi(d(fx_{n+1}, fx_{n+2})) &\leq \psi(s^3 d(fx_{n+1}, fx_{n+2})) = \psi(s^3 d(Tx_n, Tx_{n+1})) \\ &\leq \psi(d(fx_n, fx_{n+1})) - \phi(d(fx_n, fx_{n+1})) < \psi(d(fx_n, fx_{n+1})) \end{aligned} \tag{2.3}$$

Since ψ is non-decreasing, we get that $\{d(fx_n, fx_{n+1})\}$ is a non-increasing sequence of non-negative terms. Therefore, $\lim d(fx_n, fx_{n+1})$ exists (say a) where $a \geq 0$. Let $n \to \infty$ in (2.3) and using continuity of ψ and ϕ we have $\psi(a) \leq \psi(a) - \phi(a)$. Therefore, $\phi(a) = 0$. That is, $a = 0$. Thus,

$$\lim d(fx_n, fx_{n+1}) = 0. \tag{2.4}$$

Now, we show that $\{fx_n\}$ is a Cauchy sequence in (X, d).
On the contrary, let us assume that $\{fx_n\}$ is not Cauchy. Then there exists $\epsilon > 0$ and sequence $\{n_k\}$ and $\{m_k\}$ such that for all $k > 0$,

$$n_k > m_k > k, n_k \quad is \quad odd, \quad m_k \quad is \quad even, \quad d(fx_{n_k}, fx_{m_k}) \geq \epsilon \tag{2.5}$$

and

$$d(fx_{n_k - 1}, fx_{m_k}) < \epsilon \tag{2.6}$$

where n_k is the smallest such number. Now,

$$\begin{aligned} \epsilon \leq d(fx_{m_k}, fx_{n_k}) &\leq s[d(fx_{m_k}, fx_{n_k - 1}) + d(fx_{n_k - 1}, fx_{n_k})] \\ &< s[\epsilon + d(fx_{n_k - 1}, fx_{n_k})] \end{aligned} \tag{2.7}$$

Taking lim sup as $k \to \infty$ in (2.7) and using (2.4), we get

$$\epsilon \leq \limsup d(fx_{m_k}, fx_{n_k}) \leq s\epsilon \tag{2.8}$$

Again,

$$d(fx_{m_k}, fx_{n_k}) \leq s[d(fx_{m_k}, fx_{n_k + 1}) + d(fx_{n_k + 1}, fx_{n_k})]$$

and

$$d(fx_{m_k}, fx_{n_k + 1}) \leq s[d(fx_{m_k}, fx_{n_k}) + d(fx_{n_k}, fx_{n_k + 1})]$$

Taking lim sup as $k \to \infty$, we obtain that $\epsilon \leq s \limsup d(fx_{m_k}, fx_{n_k + 1})$ and $\limsup d(fx_{m_k}, fx_{n_k + 1}) \leq s^2 \epsilon$. That is,

$$\frac{\epsilon}{s} \leq \limsup d(fx_{m_k}, fx_{n_k + 1}) \leq s^2 \epsilon. \tag{2.9}$$

Similarly, we can show that

$$\frac{\epsilon}{s} \leq \limsup d(fx_{n_k}, fx_{m_k+1}) \leq s^2\epsilon. \tag{2.10}$$

Again, using triangle's inequality,

$$d(fx_{m_k}, fx_{n_k+1}) \leq s[d(fx_{m_k}, fx_{m_k+1}) + d(fx_{m_k+1}, fx_{n_k+1})].$$

Taking lim sup as $k \to \infty$, we get $\frac{\epsilon}{s^2} \leq \limsup d(fx_{m_k+1}, fx_{n_k+1})$.
Similarly, we obtain $\limsup d(fx_{m_k+1}, fx_{n_k+1}) \leq s^3\epsilon$.
Combining, we have

$$\frac{\epsilon}{s^2} \leq \limsup d(fx_{m_k+1}, fx_{n_k+1}) \leq s^3\epsilon. \tag{2.11}$$

Now, $\alpha(x_{m_k})\beta(x_{n_k}) \geq 1$. Therefore from (ii), we get

$$\psi(s^3 d(fx_{m_k+1}, fx_{n_k+1})) = \psi(s^3 d(Tx_{m_k}, Tx_{n_k})) \leq \psi(M(x_{m_k}, x_{n_k})) - \phi(M(x_{m_k}, x_{n_k})) \tag{2.12}$$

where

$$\begin{aligned} M(x_{m_k}, x_{n_k}) &= \max\{d(fx_{m_k}, fx_{n_k}), d(fx_{m_k}, Tx_{m_k}), d(fx_{n_k}, Tx_{n_k}), \\ &\qquad \frac{d(fx_{m_k}, Tx_{n_k}) + d(fx_{n_k}, Tx_{m_k})}{2s}\} \\ &= \max\{d(fx_{m_k}, fx_{n_k}), d(fx_{m_k}, fx_{m_k+1}), d(fx_{n_k}, fx_{n_k+1}), \\ &\qquad \frac{d(fx_{m_k}, fx_{n_k+1}) + d(fx_{n_k}, fx_{m_k+1})}{2s}\} \end{aligned} \tag{2.13}$$

Taking lim sup as $k \to \infty$ in (2.13) and using (2.9)-(2.11), we get

$$\max\{\epsilon, \frac{\epsilon/s + \epsilon/s}{2s}\} \leq \limsup M(x_{m_k}, x_{n_k}) \leq \max\{s\epsilon, \frac{s^2\epsilon + s^2\epsilon}{2s}\}.$$

That is, $\epsilon \leq \limsup M(x_{m_k}, x_{n_k}) \leq s\epsilon$. Also, we can show that $\epsilon \leq \liminf M(x_{m_k}, x_{n_k}) \leq s\epsilon$.
Taking lim sup as $k \to \infty$ in (2.12), it follows that

$$\begin{aligned} \psi(s\epsilon) = \psi(s^3.\frac{\epsilon}{s^2}) &\leq \psi(s^3 \limsup d(fx_{m_k+1}, fx_{n_k+1})) \\ &\leq \psi(\limsup M(x_{m_k}, x_{n_k})) - \phi(\liminf M(x_{m_k}, x_{n_k})) \leq \psi(s\epsilon) - \phi(\epsilon). \end{aligned}$$

That is, $\phi(\epsilon) = 0$. Thus, $\epsilon = 0$, which is a contradiction.
Hence $\{fx_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $\lim fx_n = x^*$. Also, since $f(X)$ is a closed subset of X, there exists $x' \in X$ such that $x^* = f(x')$. Now as $y_{2n+1} = fx_{2n+1} \to x^*$ and $\beta(y_{2n+1}) = \beta(fx_{2n+1}) \geq 1$ for all n, by (iv), we conclude that $\beta(x^*) = \beta(fx') \geq 1$. Thus, $\alpha(fx_{2n})\beta(fx') \geq 1$ for all n. Therefore by (ii), we get

$$\psi(s^3 d(Tx_{2n}, Tx')) \leq \psi(M(x_{2n}, x')) - \phi(M(x_{2n}, x')) \tag{2.14}$$

where

$$M(x_n, x') = \max\{d(fx_{2n}, fx'), d(fx_{2n}, Tx_{2n}), d(fx', Tx'), \frac{d(fx_{2n}, Tx') + d(fx', Tx_{2n})}{2s}\}.$$

Taking lim sup as $n \to \infty$, we have

$$\limsup M(x_{2n}, x') \leq \max\{0, 0, d(fx', Tx'), d(fx', Tx')/2\} = d(fx', Tx').$$

Thus, taking lim sup as $n \to \infty$ in (2.14), we get

$$\psi(s^2 d(fx', Tx')) \leq \psi(d(fx', Tx').$$

This is possible if $d(fx',Tx')=0$. That is, $fx'=Tx'=x^*$. Thus, x^* is a point of coincidence and x' is a coincidence point of f and T.
Next, we show uniqueness of point of coincidence. Let $y=fu=Tu$ and $y'=fv=Tv$.
Then, by (v), $\alpha(fu)\beta(fv)\geq 1$. Therefore,

$$\psi(s^3d(Tu,Tv)\leq \psi(M(u,v))-\phi(M(u,v))$$

where
$M(u,v)=\max\{d(fu,fv),d(fu,Tu),d(fv,Tv),\frac{d(fu,Tv)+d(fv,Tu)}{2s}\}=d(fu,fv)$.
Therefore,

$$\psi(s^3d(fu,fv)\leq \psi(d(fu,fv))-\phi(d(fu,fv))\leq \psi(d(fu,fv)).$$

This is possible if $d(fu,fv)=0$. That is, $fu=fv$, i.e., $y=y'$. Thus, f and T have unique point of coincidence. Moreover, if f and T are weakly compatible, then $fx^*=fTx'=Tfx'=Tx^*$. Now, uniqueness of point of coincidence implies that $Tx^*=fx^*=fx'=x^*$. Hence, x^* is a unique common fixed point of f and T.
Case II: Suppose there exists $x_0\in X$ such that $\beta(x_0)\geq 1$. Proceeding in a similar way as above, we obtain the desired result. □

Corollary 2.7 *Let (X,d) be a complete b-metric space with parameter $s\geq 1$ and $f:X\to X$ and $T:X\to X$ be mappings such that $T(X)\subseteq f(X)$ and $f(X)$ is a closed subset of X. Suppose that the following conditions hold:*
(i) T is a cyclic (α,β)-admissible mapping with respect to f.
(ii) T and f satisfies $\alpha(x)\beta(y)\psi(s^3d(Tx,Ty))\leq\psi(M(x,y))-\phi(M(x,y))$ where $M(x,y)=\max\{d(fx,fy),d(fx,Tx),d(fy,Ty),\frac{d(fx,Ty)+d(fy,Tx)}{2s}\}$, $\alpha,\beta:X\to[0,\infty)$ are functions and $\psi,\phi:[0,\infty)\to[0,\infty)$ are altering distance functions.
(iii) There exist $x_0\in X$ such that $\alpha(fx_0)\geq 1$ or $\beta(fx_0)\geq 1$.
(iv) If $\{x_n\}$ is a sequence in X such that $x_n\to x$ and $\beta(x_{2n})\geq 1$ or $\beta(x_{2n+1})\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$, then $\beta(x)\geq 1$.
(v) $\alpha(fu)\geq 1$ and $\beta(fu)\geq 1$ whenever $fu=Tu$.
Then f and T have a unique point of coincidence in X. Moreover, if f and T are weakly compatible , then f and T have a unique commmon fixed point.

Proof: Let $\alpha(x)\beta(y)\geq 1$ for all $x,y\in X$,. Then

$$\psi(s^3d(Tx,Ty))\leq\alpha(x)\beta(y)\psi(s^3d(Tx,Ty))\leq\psi(M(x,y))-\phi(M(x,y)).$$

This shows that T and f satisfies condition (ii) of Theorem 2.3. Thus, by Theorem 2.3, we have the required result. □

Corollary 2.8 *Let (X,d) be a complete b-metric space with parameter $s\geq 1$ and $T:X\to X$ be a mapping. Suppose that the following conditions hold:*
(i) T is a cyclic (α,β)-admissible mapping.
(ii) T satisfies $\psi(s^3d(Tx,Ty))\leq\psi(M(x,y))-\phi(M(x,y))$ whenever $\alpha(x)\beta(y)\geq 1$, where $M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2s}\}$, $\alpha,\beta:X\to[0,\infty)$ are functions and $\psi,\phi:[0,\infty)\to[0,\infty)$ are altering distance functions.
(iii) There exist $x_0\in X$ such that $\alpha(x_0)\geq 1$ or $\beta(x_0)\geq 1$.
(iv) If $\{x_n\}$ is a sequence in X such that $x_n\to x$ and $\beta(x_{2n})\geq 1$ or $\beta(x_{2n+1})\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$, then $\beta(x)\geq 1$.
(v) $\alpha(u)\geq 1$ and $\beta(u)\geq 1$ whenever $Tu=u$.
Then T has a unique fixed point in X.

Proof: Take $f(x)=x$ for all x in X in Theorem 2.3. □

Corollary 2.9 *Let (X, d) be a complete b-metric space with parameter $s \geq 1$ and $f : X \to X$ and $T : X \to X$ be mappings such that $T(X) \subseteq f(X)$ and $f(X)$ is a closed subset of X. Suppose that the following conditions hold:*
(i) T is a cyclic (α, β)-admissible mapping with respect to f.
(ii) T and f satisfies $s^3 d(Tx, Ty) \leq kM(x, y)$ whenever $\alpha(fx)\beta(fy) \geq 1$, where $\alpha, \beta : X \to [0, \infty)$ are functions, $M(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2s}\}$ and $k \in [0, 1)$.
(iii) There exist $x_0 \in X$ such that $\alpha(fx_0) \geq 1$ or $\beta(fx_0) \geq 1$.
(iv) If $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_{2n}) \geq 1$ or $\beta(x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then $\beta(x) \geq 1$.
(v) $\alpha(fu) \geq 1$ and $\beta(fu) \geq 1$ whenever $fu = Tu$.
Then f and T have a unique point of coincidence in X. Moreover, if f and T are weakly compatible , then f and T have a unique commmon fixed point.

Proof: Take $\psi(t) = t$ and $\phi(t) = (1 - k)t$ for all $t \geq 0$ in Theorem 2.3. □

Corollary 2.10 *Let (X, d) be a complete b-metric space with parameter $s \geq 1$ and $T : X \to X$ be a mapping. Suppose that the following conditions hold:*
(i) T is a cyclic (α, β)-admissible mapping.
(ii) T satisfies $s^3 d(Tx, Ty) \leq kM(x, y)$ whenever $\alpha(x)\beta(y) \geq 1$, where $\alpha, \beta : X \to [0, \infty)$ are functions, $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\}$ and $k \in [0, 1)$.
(iii) There exist $x_0 \in X$ such that $\alpha(x_0) \geq 1$ or $\beta(x_0) \geq 1$.
(iv) If $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_{2n}) \geq 1$ or $\beta(x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then $\beta(x) \geq 1$.
(v) $\alpha(u) \geq 1$ and $\beta(u) \geq 1$ whenever $Tu = u$.
Then T has a unique fixed point in X.

Proof: Take $f(x) = x$ for all x in X, $\psi(t) = t$ and $\phi(t) = (1 - k)t$ for all $t \geq 0$ in Theorem 2.3. □

In the following theorem, we establish the existence of coincidence point and common fixed point for continuous and compatible pair of self mappings satisfying contractive type conditions in b-metric spaces.

Theorem 2.4 *Let (X, d) be a complete b-metric space with parameter $s \geq 1$ and $f : X \to X$ and $T : X \to X$ be continuous and compatible mappings such that $T(X) \subseteq f(X)$. Suppose that the following conditions hold:*
(i) T is a cyclic (α, β)-admissible mapping with respect to f.
(ii) T and f satisfies $\psi(s^3 d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y))$ whenever $\alpha(fx)\beta(fy) \geq 1$, where $M(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2s}\}$, $\alpha, \beta : X \to [0, \infty)$ are functions and $\psi, \phi : [0, \infty) \to [0, \infty)$ are altering distance functions.
(iii) There exist $x_0 \in X$ such that $\alpha(fx_0) \geq 1$ or $\beta(fx_0) \geq 1$.
Then f and T have a coincidence point. Further if $ffa = fa$ for some $a \in C(f, T)$, then f and T have a common fixed point.

Proof: Proceeding as in Theorem 2.3, we can define a sequence $\{x_n\}$ such that $\lim fx_n = x^* = \lim Tx_n$. Now since f and T are compatible, we have $d(fTx_n, Tfx_n) \to 0$ as $n \to \infty$. Also, since f and T are continuous, we get $\lim d(Tfx_n, Tx^*) = 0 = \lim d(fTx_n, fx^*)$. Consider

$$\begin{aligned} d(fx^*, Tx^*) &\leq s[d(fx^*, fTx_n) + d(fTx_n, Tx^*)] \\ &\leq s[d(fx^*, fTx_n) + sd(fTx_n, Tfx_n) + sd(Tfx_n, Tx^*)] \end{aligned}$$

$\to 0$ as $n \to \infty$.
This implies $d(fx^*, Tx^*) = 0$. That is, $fx^* = Tx^*$. Thus, x^* is a coincidence point of f and T. Further compatibility implies that f and T commute at their coincidence point, i.e. , $fTx^* = Tfx^*$. So, if $ffx^* = fx^*$, then $y = fx^* = ffx^* = fTx^* = Tfx^* = Ty$. That is, $fy = y = Ty$. Thus, f and T have a common fixed point. □

Corollary 2.11 *Let (X,d) be a complete b-metric space with constant $s \geq 1$ and $f: X \to X$ and $T: X \to X$ be continuous and compatible mappings such that $T(X) \subseteq f(X)$. Suppose that the following conditions hold:*
(i) T is a cyclic (α,β)-admissible mapping with respect to f.
(ii) T and f satisfies $\alpha(fx)\beta(fy)\psi(s^3 d(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y))$ where $M(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2s}\}$, $\alpha,\beta: X \to [0,\infty)$ are functions and $\psi,\phi: [0,\infty) \to [0,\infty)$ are altering distance functions.
(iii) There exist $x_0 \in X$ such that $\alpha(fx_0) \geq 1$ or $\beta(fx_0) \geq 1$.
Then f and T have a coincidence point. Further if $ffa = fa$ for some $a \in C(f,T)$ then f and T have a common fixed point.

Proof: Let $\alpha(x)\beta(y) \geq 1$ for all $x,y \in X$. Then

$$\psi(s^3 d(Tx,Ty)) \leq \alpha(x)\beta(y)\psi(s^3 d(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y)).$$

This shows that T and f satisfies condition (ii) of Theorem 2.4. Thus by Theorem 2.4, we obtain the required result. □

Corollary 2.12 *Let (X,d) be a complete b-metric space with parameter $s \geq 1$ and $f: X \to X$ and $T: X \to X$ be continuous and compatible mappings such that $T(X) \subseteq f(X)$. Suppose that the following conditions hold:*
(i) T is a cyclic (α,β)-admissible mapping with respect to f.
(ii) T and f satisfies $s^3 d(Tx,Ty) \leq kM(x,y)$ whenever $\alpha(fx)\beta(fy) \geq 1$, where $M(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2s}\}$, $\alpha,\beta: X \to [0,\infty)$ are functions and $k \in [0,1)$.
(iii) There exist $x_0 \in X$ such that $\alpha(fx_0) \geq 1$ or $\beta(fx_0) \geq 1$.
Then f and T have a coincidence point. Further if $ffa = fa$ for some $a \in C(f,T)$ then f and T have a common fixed point.

Proof: Take $\psi(t) = t$ and $\phi(t) = (1-k)t$ for all $t \geq 0$ in Theorem 2.4. □

Corollary 2.13 *([10],Theorem 3.2) Let (X,d) be a complete b-metric space with parameter $s \geq 1$ and $T: X \to X$ be a continuous mapping. Suppose that the following conditions hold:*
(i) T is a cyclic (α,β)-admissible mapping.
(ii) T satisfies $\psi(s^3 d(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y))$ whenever $\alpha(x)\beta(y) \geq 1$, where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\}$, $\alpha,\beta: X \to [0,\infty)$ are functions and $\psi,\phi: [0,\infty) \to [0,\infty)$ are altering distance functions.
(iii) There exist $x_0 \in X$ such that $\alpha(x_0) \geq 1$ or $\beta(x_0) \geq 1$.
Then T has a fixed point.

Proof: Let $f(x) = x$ for all x in X in Theorem 2.4. □

Corollary 2.14 *([10],Corollary 3.4) Let (X,d) be a complete b-metric space with parameter $s \geq 1$ and $T: X \to X$ be continuous mapping. Suppose that the following conditions hold:*
(i) T is a cyclic (α,β)-admissible mapping.
(ii) T satisfies $s^3 d(Tx,Ty) \leq kM(x,y)$ whenever $\alpha(x)\beta(y) \geq 1$, where $\alpha,\beta: X \to 0,\infty)$ are functions, $M(x,y) = \max\{d(x,fy), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\}$ and $k \in [0,1)$.
(iii) There exist $x_0 \in X$ such that $\alpha(x_0) \geq 1$ or $\beta(x_0) \geq 1$.
Then T has a fixed point.

Proof: Take $f(x) = x$ for all x in X, $\psi(t) = t$ and $\phi(t) = (1-k)t$ for all $t \geq 0$ in Theorem 2.4. □

We now give examples to illustrate the results.

Example 2.2 *Let $X = [0,\infty)$ and $d : X \times X \to [0,\infty)$ be given by $d(x,y) = (x-y)^2$ for all $x, y \in X$. Then (X,d) is a complete b-metric space with $s = 2$.*
Define $f : X \to X$ as $f(x) = x^2$ $\forall x \in X$ and $T : X \to \mathcal{CB}(X)$ as $Tx = [0, \frac{x^2}{4}]$ $\forall x \geq 0$.
Clearly $f(X)$ is closed and $Tx \subseteq f(X)$ $\forall x \in X$. Let $\alpha, \beta : X \to [0,\infty)$ be given by $\alpha(x) = \beta(x) = e^x$ $\forall x \in X$. It is clear that T is cyclic (α_*, β_*)-admissible mapping with respect to f.
Now for $x_0 = 1$ and $f(x_1) = f(\frac{1}{4}) = \frac{1}{16} \in Tx_0$ such that $\alpha(fx_0) \geq 1$ and $\beta(fx_0) \geq 1$.
Let $\xi(t) = t$, $\psi(t) = \frac{t}{4}$, $\phi(t) = \frac{t}{8}$, $t \geq 0$.
Now, let $x, y \in X$ such that $\alpha(fx)\beta(fy) \geq 1$. Then

$$\begin{aligned}\xi(sH(Tx,Ty)) &= 2\frac{(x^2-y^2)^2}{16} = \frac{1}{8}d(fx,fy) \leq \frac{1}{8}M(x,y) = \frac{1}{4}M(x,y) - \frac{1}{8}M(x,y) \\ &= \psi(\xi(M(x,y))) - \phi(M(x,y)).\end{aligned}$$

Thus all the conditions of Theorem 2.2 are satisfied. Therefore f and T have a coincidence point. Here 0 is a coincidence point.

Example 2.3 *Let $X = \mathbb{R}$ be endowed with the metric $d(x,y)) = (x-y)^2$ for all $x, y \in X$. Then (X,d) is a complete b-metric space with $s = 2$. Let $\psi, \phi : [0,\infty) \to [0,\infty)$ given by $\psi(t) = t$ for all $t \geq 0$ and $\phi(t) = \frac{17}{25}t$ for all $t \geq 0$ be altering distance functions.*
Now, define mappings $T : X \to X$ and $f : X \to X$ as

$$Tx = \begin{cases} \frac{-x}{5} & , x \in [0,1] \\ \frac{x}{25} & , x \in \mathbb{R} \sim [0,1] \end{cases}$$

$$fx = \begin{cases} \frac{x}{5} & , x \in [-1,0] \\ \frac{x}{6} & , x \in \mathbb{R} \sim [-1,0] \end{cases}$$

Then, clearly $T(X) \subseteq f(X)$. and $f(X)$ is a closed subset of X.
Also, let $\alpha, \beta. : X \to [0,\infty)$ be defined by

$$\alpha(x) = \begin{cases} e^x & , x \in (-\infty, \frac{1}{5}) \\ e^{-x} & , x \in [-\frac{1}{5}, 0] \\ 0 & , x \in (0,\infty) \end{cases}$$

$$\beta(x) = \begin{cases} e^{-x} & , x \in [-\frac{1}{25}, 0] \\ 0 & , x \in \mathbb{R} \sim [-\frac{1}{25}, 0] \end{cases}$$

Let $x \in X$ such that $\alpha(fx) \geq 1$ so that $fx \in [-\frac{1}{5}, 0]$ and hence $x \in [-1,0]$. This implies that $Tx \in [-\frac{1}{25}, 0]$ and thus $\beta(Tx) = e^{-Tx} \geq 1$.
Similarly we can show that if $\beta(fx) \geq 1$, then $\alpha(Tx) \geq 1$. Thus, T is a cyclic (α,β)-admissible mapping with respect to f. Also, if $x_0 = -\frac{1}{5}$, then $\alpha(fx_0) \geq 1$ and $\beta(fx_0) \geq 1$.
Further, if $\{x_n\}$ is a sequence in X such that $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. Then, by the definition of β, we have $x_n \in [-\frac{1}{25}, 0]$ for all $n \in \mathbb{N}$ and so $x \in [-\frac{1}{25}, 0]$,that is, $\beta(x) \geq 1$.
Now, let $x, y \in X$ such that $\alpha(fx)\beta(fy) \geq 1$. Then $fx \in [-\frac{1}{5}, 0]$,$fy \in [-\frac{1}{25}, 0]$ and so $x \in [-1,0]$, $y \in [-\frac{1}{5}, 0]$. Thus, we get $\psi(s^3 d(Tx,Ty)) = 8(Tx-Ty)^2 = \frac{8}{625}(x-y)^2 = \frac{8}{25}(fx-fy)^2 \leq \frac{8}{25}M(x,y)$ $= M(x,y) - \frac{17}{25}M(x,y) = \psi(M(x,y)) - \phi(M(x,y))$. This shows that condition (iii) of theorem is satisfied.
Now, $fu = Tu$ for $u = 0$. This implies, $\alpha(fu) \geq 1$ and $\beta(fu) \geq 1$, that is, condition (v) is satisfied. Also, f and T are weakly compatible. Thus, all the conditions of theorem are satisfied and and hence f and T have a unique common fixed point. Here 0 is the common fixed point of f and T.

3. Application to cyclic mappings

In this section, we apply our main results to prove fixed point theorems for cyclic mapping .

Definition 3.1 *Let A and B be nonempty subsets of a set X and let $f : A \cup B \to A \cup B$ be a mapping . A mapping $T : A \cup B \to A \cup B$ is called cyclic with respect to f if $T(A) \subseteq f(B)$ and $T(B) \subseteq f(A)$.*

Theorem 3.1 *Let A and B be closed subsets of complete b-metric space (X,d) such that $A \cap B \neq \phi$ and $f, T : A \cup B \to A \cup B$ be mappings such that $T(A) \subseteq f(B)$ and $T(B) \subseteq f(A)$. Assume that f is one to*

one and $f(A\cup B)$ is a closed subset of X such that $\psi(s^3d(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y))$ for all $x\in A$ and $y \in B$ where
$M(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2s}\}$, *$\alpha,\beta : X \to [0,\infty)$ are functions and $\psi,\phi : [0,\infty) \to [0,\infty)$ are altering distance functions. Then there exists $x' \in A\cap B$ such that $fx' = Tx'$. Further, if f and T are weakly compatible, then f and T have a unique common fixed point in $A \cap B$.*

Proof: Define $\alpha,\beta : X \to [0,\infty)$ by

$$\alpha(x) = \begin{cases} 1 & ,if x \in fA \\ 0 & ,\text{otherwise} \end{cases}$$

$$\beta(x) = \begin{cases} 1 & ,if x \in fB \\ 0 & ,\text{otherwise} \end{cases}$$

Let $\alpha(fx)\beta(fy) \geq 1$. Then $fx \in fA$ and $fy \in fB$. Since f is one to one, we have $x \in A$ and $y \in B$. Thus, we have $\psi(s^3d(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y))$. Now, let $\alpha(fx) \geq 1$ for some $x \in X$, so $fx \in fA$ and thus $x \in A$. Therefore, $Tx \in fB$ and so $\beta(Tx) \geq 1$. Again, let $\beta(fx) \geq 1$ for some $x \in X$. Then $fx \in fB$ and thus $x \in B$. Therefore, $Tx \in fA$ and so $\alpha(Tx) \geq 1$. Therefore, T is a cyclic (α,β)-admissible mapping with respect to f. Since $A\cap B$ is non-empty, there exists $x_0 \in A\cap B$. This implies that $fx_0 \in fA$ and $fx_0 \in fB$ and therefore $\alpha(fx_0) \geq 1$ and $\beta(fx_0) \geq 1$. Further, let $\{x_n\}$ be a sequence in X such that $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. Then $x_n \in fB$ for all $n \in \mathbb{N}$ and thus $x \in fB$. Therefore we get that $\beta(x) \geq 1$. Thus all the conditions of Theorem 2.3 hold. So there exist $x^*, x' \in A\cup B$ such that $x^* = fx' = Tx'$. Also, there exist $x_1 \in A$,$x_2 \in B$ such that $fx_1 = fx_2 = x^*$. Since f is one-one, this implies that $x_1 = x_2 = x'$. Therefore, $x^* = fx' = Tx'$ for $x' \in A\cap B$. If f and T are weakly compatible, then proceeding as in the proof of Theorem 2.3, we have $x^* = fx^* = Tx^*$, where x^* is the unique common fixed point of f and T. □

Theorem 3.2 *Let A and B be closed subsets of complete b-metric space (X,d) such that $A\cap B \neq \phi$ and $f,T : A\cup B \to A\cup B$ be continuous and compatible mappings such that $T(A) \subseteq f(B)$ and $T(B) \subseteq f(A)$. Assume that f is one to one and f and T satisfy $\psi(s^3d(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y))$ for all $x \in A$ and $y \in B$ where*
$M(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2s}\}$, *$\alpha,\beta : X \to [0,\infty)$ are functions and $\psi,\phi : [0,\infty) \to [0,\infty)$ are altering distance functions. Then there exists $x' \in A\cap B$ such that $fx' = Tx'$. Further if $ffa = fa$ for some $a \in C(f,T)$, then f and T have a common fixed point in $A\cap B$.*

Proof: Define $\alpha,\beta : X \to [0,\infty)$ by

$$\alpha(x) = \begin{cases} 1 & ,if x \in fA \\ 0 & ,\text{otherwise} \end{cases}$$

$$\beta(x) = \begin{cases} 1 & ,if x \in fB \\ 0 & ,\text{otherwise} \end{cases}$$

Let $\alpha(fx)\beta(fy) \geq 1$. Then $fx \in fA$ and $fy \in fB$. Since f is one to one, we have $x \in A$ and $y \in B$. Thus, we have $\psi(s^3d(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y))$.
Now, let $\alpha(fx) \geq 1$ for some $x \in X$, so $fx \in fA$ and thus $x \in A$. Therefore, $Tx \in fB$ and so $\beta(Tx) \geq 1$. Again, let $\beta(fx) \geq 1$ for some $x \in X$. Then $fx \in fB$ and thus $x \in B$. Therefore, $Tx \in fA$ and so $\alpha(Tx) \geq 1$. Hence, T is a cyclic (α,β)-admissible mapping with respect to f. Since $A\cap B$ is non-empty, there exists $x_0 \in A\cap B$. This implies that $fx_0 \in fA$ and $fx_0 \in fB$ and therefore $\alpha(fx_0) \geq 1$ and $\beta(fx_0) \geq 1$. Thus all the conditions of Theorem 2.4 hold. So there exist $x^*, x' \in A\cup B$ such that $x^* = fx' = Tx'$. Also, there exist $x_1 \in A$, $x_2 \in B$ such that $fx_1 = fx_2 = x^*$. Since f is one-one, this implies that $x_1 = x_2 = x'$. Therefore, $x^* = fx' = Tx'$ for $x' \in A\cap B$.
Now, if $ffx' = fx'$, then proceeding as in the proof of Theorem 2.4, we have $x' = fx' = Tx'$, i.e, x' is a common fixed point of f and T. □

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