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Ulam Stability of Volterra Integral Equations on Time Scales via Fixed Point Approach

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ABSTRACT: This article is about the Ulam stability of Volterra integral equations on time scales. We present the Hyers–Ulam and Hyers–Ulam–Rassias stability by employing fixed point alternative on complete generalized metric spaces. An Example is provided to illustrate the effectiveness and benefit of the proven results.

Key Words: Fixed point theorem, generalized metric spaces, time scales, Ulam stability, Volterra integral equation.

Contents

1. Introduction

The concept of Ulam stability is originated from one of the questions raised by Ulam [32] in his famous talk at the Mathematics club of the University of Wisconsin in 1940. The question was "Under what conditions does there exists a homomorphism near an approximately homomorphism of a complete metric group?" More precisely: Let G_1 be a group and G_2 be a group endowed with a metric d. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h\colon G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H\colon G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$? When there is an affirmative answer to this question, the functional equation $h(xy) = h(x)h(y)$ is said to be Ulam stable. One year later, in 1941, Hyers [23] partially answered this question for approximately additive functions on Banach spaces. Later on, in 1978, Rassias [26] generalized Hyers' work by replacing the constant ε with a variable in Ulam's original problem. Since then this topic was attracted by several researchers and a remarkably large amount of work has been done for the stability of many algebraic, functional, differential, difference, integral, integro-differential, fractional equations. The researchers have employed various techniques and tools for investigating the stability of these equations. These techniques and tools include fixed point methods, successive approximation, inequality, analytical method, Laplace transform, etc.

On the other hand, the topic of dynamic equations on time scales (DETS) has emerged in late 90's. DETS has received great attention from the researchers around the globe. This is mainly due to the fact that the time scale theory basically unifies and extends the existing theory of continuous calculus and discrete calculus. Consequently, DETS found applicable in every discipline where continuous and discrete data present. The study of nonuniform systems, such as population dynamics [8,12,16], economics [9,10,19,20], and optimization [25], benefited from the unification of discrete and continuous processes through time scale analysis. For an excellent introduction to the calculus and dynamic equations on time scales and recent advances in this area, readers can refer to [11,13,14] and [1,18], respectively. As the

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development of DETS continues, in the near past, a significant amount of interest in the investigation of Ulam stability of DETS has been seen among the researchers.

At the outset, Anderson et al. [4], using analytical method, established the Hyers–Ulam stability of the second-order linear variable coefficients nonhomogeneous dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x^{\Delta}(t) + r(t)x(t) = f(t), \quad t \in [a,b]_{\mathbb{T}},$$

where $p, r, f \in \mathcal{C}_{\rm rd}([a,b]_{\mathbb{T}})$.

András and Mészáros [7] used both direct and operational methods and the theory of Picard operators to study the Ulam stability of some linear and nonlinear dynamic equations and integral equations on time scales. Shen [31] has established the Ulam stability of first-order linear dynamic equation

$$y^{\Delta}(t) = p(t)y(t) + f(t), \quad t \in \mathbb{T},$$

and its adjoint equation

$$x^{\Delta}(t) = -p(t)x^{\sigma}(t) + f(t), \quad t \in \mathbb{T},$$

where $p \in \mathcal{R}^+$ and $f \in \mathcal{C}_{\rm rd}(\mathbb{T})$, by using the integrating factor method. Based on the same technique, Shah and Zada [30] presented the Ulam stability of nonlinear Volterra integro-dynamic equation

$$y^{\Delta}(t) = p(t)y(t) + \int_{t_0}^{t} K(t,s,y(s))\Delta s, \quad t \in [t_0,\infty)_{\mathbb{T}},$$

and its adjoint equation

$$x^{\Delta}(t) = -p(t)x^{\sigma}(t) + \int_{t_0}^{t} K(t,s,x(s))\Delta s, \quad t \in [t_0,\infty)_{\mathbb{T}},$$

where $p \in \mathcal{R}^+$ and $K \in \mathcal{C}_{\rm rd}([t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}})$.

Hua et al. [22] employed the fixed point method to investigate the Ulam stability of integral equation on time scales

$$x(t) = f\left(t, x(t), \int_{t_0}^{t} g(t,s,x(s))\Delta s\right). \quad t \in [t_0,\infty)_{\mathbb{T}}.$$

Hamza and Ghallab [21], based on the successive approximation method, studied the Ulam stability of a class of Volterra integral equations on time scales

$$x(t) = f(t) + \int_{a}^{t} K(t,s,x(s))\Delta s, \quad t \in [a,b]_{\mathbb{T}},$$

where $f \in \mathcal{C}_{\rm rd}([a,b]_{\mathbb{T}})$ and $K \in \mathcal{C}_{\rm rd}([a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}})$.

Reinfelds and Christian [27], employing contraction mapping principle and Bielecki norm, investigated the Hyers–Ulam stability of a general Volterra-type integral equations on bounded and unbounded time scales

$$x(t) = f\left(t, x(t), x(\sigma(t)), \int_{a}^{t} K(t,s,x(s),x(\sigma(s))\Delta s)\right), \quad t \in \mathbb{T}.$$

Motivated by the above papers and the work done by the researchers in [2,3,5,6,15,24,28,29], in this paper, employing the technique of fixed point in generalized metric space, we explore the Hyers–Ulam and Hyers–Ulam–Rassias stability of dynamic Volterra integral equations (VIE)

$$x(t) = \zeta(t) + \int_{t_0}^{t} f(t,s,x(s))\Delta s, \quad t \in \mathbb{I} := [t_0,b]_{\mathbb{T}}, \tag{1.1}$$

where $t_0, b \in \mathbb{T}$ with $t_0 \le b$, $f \in \mathcal{C}_{\rm rd}(\mathbb{I} \times \mathbb{I} \times \mathbb{R})$, $\zeta \in \mathcal{C}_{\rm rd}(\mathbb{I})$ possibly nonlinear, and x is the unknown function. We believe that the results in the present paper complement and extend the corresponding results in the literature.

The set up of the paper is as follows: In Section 2, we present some definitions and results which are useful for our work. In Section 3, we prove our main results of Hyers–Ulam and Hyers–Ulam–Rassias stability of VIE (1.1). A suitable example that supports our theoretical findings is given in Section 4 and a concluding remark is given in Section 5.

2. Preliminary results

We provide some basic definitions and results that are useful throughout the paper. The fundamental theory of time scale calculus can be found in [13,14]. To dive into the recent development in the field, reader can refer [1,18].

A time scale, denoted by $\mathbb{T}$, is an arbitrary nonempty closed subset of $\mathbb{R}$. To describe the structure of a time scale $\mathbb{T}$, the concept of jump operators are required. This is given in the following definition.

Definition 2.1 *Let $\mathbb{T}$ be a given time scale. The forward jump operator $\sigma\colon \mathbb{T} \to \mathbb{T}$ is defined as*

$$\sigma(t) = \inf\{s \in t\colon s > t\},$$

and the backward jump operators $\rho\colon \mathbb{T} \to \mathbb{T}$ is defined as

$$\rho(t) = \sup\{s \in t\colon s < t\}.$$

For convenience, we set $\inf \emptyset = \sup \mathbb{T}$, i.e., $\sigma(M) = M$ provided $\mathbb{T}$ has a maximum M, and $\sup \emptyset = \inf \mathbb{T}$, i.e., $\rho(m) = m$ provided $\mathbb{T}$ has a minimum m, where $\emptyset$ denotes the empty set.
The elements of time scale $\mathbb{T}$ can be classified as follows: A point t is said to be right-scattered if $\sigma(t) > t$, t is said to be left-scattered if $\rho(t) < t$, t is said to be right-dense if $\sigma(t) = t < \sup \mathbb{T}$, t is said to be left-dense if $\rho(t) = t > \inf \mathbb{T}$. If $\rho(t) < t < \sigma(t)$, then t is said to be isolated point. If $\rho(t) = t = \sigma(t)$, then t is said to be dense point.

From a given time scale $\mathbb{T}$, we derive a set $\mathbb{T}^\kappa$ as $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$ if $\mathbb{T}$ has a left-scattered maximum M otherwise we define $\mathbb{T}^\kappa = \mathbb{T}$. The graininess function $\mu\colon \mathbb{T} \to \mathbb{R}^+$ of $\mathbb{T}$ is defined as $\mu(t) = \sigma(t) - t$.

Definition 2.2 *A function $f\colon \mathbb{T} \to \mathbb{R}$ is said to be delta differentiable at $t \in \mathbb{T}^\kappa$ provided there exists $f^\Delta(t) \in \mathbb{R}$ with the property that for every $\varepsilon > 0$, there is a neighborhood U of t such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \quad s \in U.$$

The number $f^\Delta(t) \in \mathbb{R}$ is known as the delta derivative of f at $t \in \mathbb{T}^\kappa$. We say that f is delta differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for every $t \in \mathbb{T}^\kappa$ and the function $f^\Delta\colon \mathbb{T} \to \mathbb{R}$ is known as the delta derivative of f on $\mathbb{T}^\kappa$.

Definition 2.3 *A function $f\colon \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limit exists finitely at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f\colon \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.*

Definition 2.4 *We write $f \in \mathcal{C}_{\mathrm{rd}}(\mathbb{T} \times \mathbb{T} \times \mathbb{R}, \mathbb{R})$, when $f\colon \mathbb{T} \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is rd-continuous in the first and second arguments.*

Definition 2.5 *Let $f \in \mathcal{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. Then the Cauchy delta integral of f is defined by*

$$\int_s^t f(\tau)\Delta\tau := F(t) - F(s), \quad s, t \in \mathbb{T},$$

where $F\colon \mathbb{T} \to \mathbb{R}$ is such that $F^\Delta(t) = f(t)$ for each $t \in \mathbb{T}^\kappa$.

Remark 2.1 *For $\mathbb{T} = \mathbb{R}$, the delta derivative and delta integral are the ordinary derivative and integral respectively, whereas for $\mathbb{T} = \mathbb{Z}$, the delta derivative and delta integral are the forward difference and ordinary summation respectively.*

Definition 2.6 *A function $p\colon \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all regressive rd-continuous real-valued functions defined on $\mathbb{T}$ is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R})$ or $\mathcal{R}$.*

Definition 2.7 *For $p \in \mathcal{R}$, the exponential function $e_p(t,s)$ on $\mathbb{T}$ is defined as*

$$e_p(t,s) := \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \quad \text{for} \quad s,t \in \mathbb{T},$$

with

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h} & \text{if} \quad h > 0 \\ z & \text{if} \quad h = 0, \end{cases}$$

where Log *is the principal logarithm function.*

It should be noted that $e_p(\cdot,s)$ is the unique solution of the dynamic initial value problem

$$x^{\Delta}(t) = p(t)x, \quad x(s) = 1,$$

for $s,t \in \mathbb{T}$, where p is regressive and rd-continuous function.

Remark 2.2 *For $p,q \in \mathcal{R}$, we define the following:*

$$p \oplus q = p + q + \mu pq, \quad \ominus p = -\frac{p}{1+\mu p}, \quad \text{and} \quad p \ominus q = p \oplus (\ominus q).$$

Below, we list some important properties of $e_p(t,s)$.

Theorem 2.1 *For $p,q \in \mathcal{R}(\mathbb{T},\mathbb{R})$, the following hold:*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;

(iii) $1/e_p(t,s) = e_{\ominus p}(t,s)$;

(iv) $e_p(t,s) = 1/e_p(s,t)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

(vii) $e_p(t,s)/e_q(t,s) = e_{p\ominus q}(t,s)$;

(viii) $[e_p(s,\cdot)]^{\Delta} = -p[e_p(s,\cdot)]^{\sigma}$.

Now, we present the concept of generalized metric space.

Definition 2.8 [17] *Let X be a nonempty set. A function $d\colon X \times X \to [0,\infty]$ is called a generalized metric on X if and only if it satisfies*

(GM_1) : $d(x,y) = 0$ *if and only if* $x = y$;

(GM_2) : $d(x,y) = d(y,x)$ *for* $x,y \in X$;

(GM_3) : $d(x,z) \leq d(x,y) + d(y,z)$ *for* $x,y,z \in X$.

With this d, (X,d) is called a generalized metric space. Further, (X,d) is called a complete generalized metric space whenever X is complete with respect to d.

The main tool of the present paper is the following fixed point theorem.

Theorem 2.2 [17] *Let (X,d) be a complete generalized metric space. Assume that $T\colon X \to X$ is a strictly contraction mapping with Lipschitz constant $L < 1$. If there is a nonnegative integer k such that $d(T^{k+1}[y], T^k[y]) < \infty$ for some $y \in X$, then the following are true:*

1. *The sequence $\{T^n[y]\}$ converges to a fixed point y^* of T,*

2. *y^* is the unique fixed point of T in $X^* = \{x \in X\colon d(T^k[y], x) < \infty\}$,*
3. *If $x \in X^*$, then*
$$d(x, y^*) \le \frac{1}{1-L}\, d(T[x], x).$$

The advantage of the use of generalized metric over the usual metric is that the generalized metric allows for distance to extend into unbounded intervals.

3. Stability Results

Before proving the stability results, we first present the following auxiliary lemma which is fundamental result in this paper.

Let $\mathcal{C}_{\mathrm{rd}}$ denotes the space of rd-continuous functions on $\mathbb{I}$. Define a metric $d\colon \mathcal{C}_{\mathrm{rd}} \times \mathcal{C}_{\mathrm{rd}} \to [0, \infty]$ by

$$d(f, g) := \inf\left\{C \in [0, \infty]\colon |f(t) - g(t)| e_{\ominus p}(t, t_0) \le C\Phi(t),\ t \in \mathbb{I}\right\}, \tag{3.1}$$

where $p > 0$ is a given constant and $\Phi \in \mathcal{C}_{\mathrm{rd}}(\mathbb{I})$ is a given rd-continuous function.

Lemma 3.1 *The metric space $(\mathcal{C}_{\mathrm{rd}}, d)$ is a complete generalized metric space, where $d(\cdot, \cdot)$ is defined in (3.1).*

Proof: We first show that d is a generalized metric on $\mathcal{C}_{\mathrm{rd}}$ according to the Definition 2.8. The conditions (GM_1) and (GM_2) are verified easily. We only check (GM_3). If possible, suppose $d(f, h) > d(f, g) + d(g, h)$ for some $f, g, h \in \mathcal{C}_{\mathrm{rd}}$. Then there exists $t_1 \in \mathbb{I}$ such that

$$\begin{aligned} |f(t_1) - h(t_1)| e_{\ominus p}(t_1, t_0) &> [d(f, g) + d(g, h)]\Phi(t_1) \\ &= d(f, g)\Phi(t_1) + d(g, h)\Phi(t_1) \\ &\ge |f(t_1) - g(t_1)| e_{\ominus p}(t_1, t_0) + |g(t_1) - h(t_1)| e_{\ominus p}(t_1, t_0), \end{aligned}$$

which is a contradiction. Therefore, (GM_3) holds and hence d is a generalized metric on $\mathcal{C}_{\mathrm{rd}}$. Next, we show that $(\mathcal{C}_{\mathrm{rd}}, d)$ is a complete metric space. Let $\{h_n\}$ be a Cauchy sequence in $(\mathcal{C}_{\mathrm{rd}}, d)$. Then, for given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(h_m, h_n) \le \varepsilon$ for $m, n \ge N$. That is,

$$|h_m(t) - h_n(t)| e_{\ominus p}(t, t_0) \le \varepsilon\Phi(t) \tag{3.2}$$

for $m, n \ge N$ and $t \in \mathbb{I}$. This means that for a fixed $t \in \mathbb{I}$, $h_n(t)$ is a Cauchy sequence in $\mathbb{R}$. Hence there is $h\colon \mathbb{I} \to \mathbb{R}$ such that $\{h_n(t)\}$ converges to $h(t)$ for each $t \in \mathbb{I}$. Now, letting $m \to \infty$ in (3.2) we obtain that for given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|h(t) - h_n(t)| e_{\ominus p}(t, t_0) \le \varepsilon\Phi(t) \tag{3.3}$$

for each $n \ge N$ and each $t \in \mathbb{I}$. That is, $d(h, h_n) \le \varepsilon$ for all $n > N$. Now, keeping in mind the boundedness of Φ on $\mathbb{I}$, from (3.3), we can conclude that h_n converges uniformly to h, which implies $h \in \mathcal{C}_{\mathrm{rd}}$. This completes the proof. □

Now, we establish the Hyers–Ulam stability of VIE (1.1) in the following theorem.

Theorem 3.1 *Suppose $f \in \mathcal{C}_{\mathrm{rd}}(\mathbb{I} \times \mathbb{I} \times \mathbb{R}, \mathbb{R})$ and satisfy the Lipschitz condition*

$$|f(t, s, x_1) - f(t, s, x_2)| \le L|x_1 - x_2|$$

for every $t, s \in \mathbb{I}$ and $x_1, x_2 \in \mathbb{R}$, where $L > 0$. If $x \in \mathcal{C}_{\mathrm{rd}}(\mathbb{I}, \mathbb{R})$ is such that

$$\left| x(t) - \zeta(t) - \int_{t_0}^{t} f(t, s, x(s))\Delta s \right| \le \varepsilon, \quad t \in \mathbb{I}, \tag{3.4}$$

for some $\varepsilon \ge 0$, then there exists a unique solution $y \in \mathcal{C}_{\mathrm{rd}}(\mathbb{I}, \mathbb{R})$ of VIE (1.1) such that

$$|x(t) - y(t)| \le (L + 1)\varepsilon, \quad t \in \mathbb{I}. \tag{3.5}$$

Proof: Define a function $d\colon \mathcal{C}_{\rm rd}\times\mathcal{C}_{\rm rd}\to[0,\infty]$ by

$$d(x_1,x_2):=\inf\left\{C\in[0,\infty]\colon |x_1(t)-x_2(t)|e_{\ominus(L+1)}(t,t_0)\le C,\quad t\in\mathbb{I}\right\}, \tag{3.6}$$

where $L>0$ is given. Then, by the virtue of Lemma 3.1, we see that $(\mathcal{C}_{\rm rd},d)$ is a complete generalized metric space. Next, define a mapping $T\colon\mathcal{C}_{\rm rd}\to\mathcal{C}_{\rm rd}$ by

$$T[x](t):=\zeta(t)+\int_{t_0}^{t}f(t,s,x(s))\Delta s \tag{3.7}$$

for $x\in\mathcal{C}_{\rm rd}$ and $t\in\mathbb{I}$. Note that T is well-defined on $\mathcal{C}_{\rm rd}$ and every fixed point of T is a solution of (1.1). The nature of ζ and the property of delta integral allow us to write $T[x]\in\mathcal{C}_{\rm rd}$ and thus, for $y\in\mathcal{C}_{\rm rd}$, we get

$$|T[y](t)-y(t))|\,e_{\ominus(L+1)}(t,t_0)<\infty,\quad t\in\mathbb{I},$$

that is, $d(T[y],y)<\infty$ for $y\in\mathcal{C}_{\rm rd}$. Also, on the similar arguments, we have

$$|y(t)-x(t)|e_{\ominus(L+1)}(t,t_0)<\infty,\quad x\in\mathcal{C}_{\rm rd},\quad t\in\mathbb{I},$$

that is, $d(y,x)<\infty$ for $x\in\mathcal{C}_{\rm rd}$. Thus, we can ensure that $\{x\in\mathcal{C}_{\rm rd}\colon d(y,x)<\infty\}=\mathcal{C}_{\rm rd}$. Next, we show that T defined in (3.7) is a contractive on $\mathcal{C}_{\rm rd}$. For $x_1,x_2\in\mathcal{C}_{\rm rd}$, we have

$$\begin{aligned}
|T[x_1](t)-T[x_2](t)| &= \left|\int_{t_0}^{t}[f(t,s,x_1(s))-f(t,s,x_2(s))]\Delta s\right|\\
&\le \int_{t_0}^{t}|f(t,s,x_1(s))-f(t,s,x_2(s))|\Delta s\\
&\le L\int_{t_0}^{t}|x_1(s)-x_2(s)|\Delta s\\
&\le L\int_{t_0}^{t}|x_1(s)-x_2(s)|e_{\ominus(L+1)}(s,t_0)e_{(L+1)}(s,t_0)\Delta s\\
&= L\ d(x_1,x_2)\int_{t_0}^{t}e_{(L+1)}(s,t_0)\Delta s\\
&\le \frac{L}{L+1}d(x_1,x_2)\ e_{(L+1)}(t,t_0),\quad t\in\mathbb{I}.
\end{aligned}$$

Hence, for any $x_1,x_2\in\mathcal{C}_{\rm rd}$ and $t\in\mathbb{I}$, we obtain

$$|T[x_1](t)-T[x_2](t)|e_{\ominus(L+1)}(t,t_0)\le\frac{L}{L+1}\ d(x_1,x_2).$$

This yields that

$$d(T[x_1],T[x_2])\le\frac{L}{L+1}\ d(x_1,x_2),\quad x_1,x_2\in\mathcal{C}_{\rm rd},$$

and hence T is contraction mapping on $\mathcal{C}_{\rm rd}$. In this way, we have verified all assumptions of Theorem 2.2 with $k=1$ and $\mathcal{C}^*_{\rm rd}=\mathcal{C}_{\rm rd}$. On the other hand, keeping in mind (3.4), we have $|x(t)-T[x](t)|\le\varepsilon$, $t\in\mathbb{I}$. Now, multiplying the last inequality by $e_{\ominus(L+1)}(t,t_0)$, we find

$$|T[x](t)-x(t)|e_{\ominus(L+1)}(t,t_0)\le\varepsilon e_{\ominus(L+1)}(t,t_0),\quad t\in\mathbb{I},$$

which means $d(T[x],x)\le\varepsilon e_{\ominus(L+1)}(t,t_0)$, $t\in\mathbb{I}$. Thus, by the virtue of Theorem 2.2, there exists a unique solution $y\in\mathcal{C}_{\rm rd}(\mathbb{I},\mathbb{R})$ of VIE (1.1) such that

$$d(x,y)\le\frac{d(T[x],x)}{1-\left(\frac{L}{L+1}\right)}\le(L+1)\varepsilon\ e_{\ominus(L+1)}(t,t_0),\quad t\in\mathbb{I}.$$

Next, from the definition of d given in (3.6), we get

$$|x(t)-y(t)|e_{\ominus(L+1)}(t,t_0) \ \le (L+1)\varepsilon\ e_{\ominus(L+1)}(t,t_0), \quad t\in\mathbb{I},$$

and thus $|x(t)-y(t)|\le (L+1)\varepsilon$, $t\in\mathbb{I}$. This completes the proof. □

Our next results presents the Hyers–Ulam–Rassias stability of VIE (1.1).

Theorem 3.2 *Suppose $f\in\mathcal{C}_{\rm rd}(\mathbb{I}\times\mathbb{I}\times\mathbb{R},\mathbb{R})$ and satisfy the Lipschitz condition*

$$|f(t,s,x_1)-f(t,s,x_2)|\le L|x_1-x_2|$$

for every $t,s\in\mathbb{I}$ and $x_1,x_2\in\mathbb{R}$, where $L>0$. If $x\in\mathcal{C}_{\rm rd}(\mathbb{I},\mathbb{R})$ is such that

$$\left|x(t)-\zeta(t)-\int_{t_0}^{t} f(t,s,x(s))\Delta s\right|\le \phi(t), \quad t\in\mathbb{I}, \tag{3.8}$$

where $\phi\colon\mathbb{I}\to(0,\infty)$ is a nondecreasing rd-continuous function. Then there is a unique solution $y\in\mathcal{C}_{\rm rd}(\mathbb{I},\mathbb{R})$ of VIE (1.1) such that

$$|x(t)-x_0(t)| \ \le (L+1)\phi(t), \quad t\in\mathbb{I}. \tag{3.9}$$

Proof: Define function $d\colon\mathcal{C}_{\rm rd}\times\mathcal{C}_{\rm rd}\to[0,\infty]$ as

$$d(x_1,x_2):=\inf\left\{C\in[0,\infty]\colon\ |x_1(t)-x_2(t)|\ e_{\ominus(L+1)}(t,t_0)\le C\phi(t), \quad t\in\mathbb{I}\right\}, \tag{3.10}$$

where $L>0$ is given and $\phi\in\mathcal{C}_{\rm rd}(\mathbb{I},\mathbb{R}^+)$. Then, by the virtue of Lemma 3.1, we see that $(\mathcal{C}_{\rm rd},d)$ is a complete generalized metric space. Now, define the mapping $T\colon\mathcal{C}_{\rm rd}\to\mathcal{C}_{\rm rd}$ by (3.7). Note that T is well-defined on $\mathcal{C}_{\rm rd}$ and every fixed point of T is a solution of VIE (1.1). Moreover, as in proof of Theorem 3.1, it can be demonstrated that

$$d(Ty,x)<\infty \quad \text{for} \quad x\in\mathcal{C}_{\rm rd},$$

and

$$\{x\in\mathcal{C}_{\rm rd}\colon d(y,x)<\infty\}=\mathcal{C}_{\rm rd}.$$

Under the given conditions, we prove that this T is a contraction on $\mathcal{C}_{\rm rd}$. First, by using integration by parts and monotonicity of ϕ, we note that

$$\begin{aligned}\int_{t_0}^{t}\phi(s)\ e_{(L+1)}(s,t_0)\Delta s &= \frac{\phi(s)e_{(L+1)}(s,t_0)}{L+1}-\int_{t_0}^{t}\frac{\phi^{\Delta}(s)e_{(L+1)}(\sigma(s),t_0)}{L+1}\Delta s\\ &\le \frac{1}{L+1}\phi(t)e_{(L+1)}(t,t_0)\end{aligned}$$

for all $t\in\mathbb{I}$. Now, for $x_1,x_2\in\mathcal{C}_{\rm rd}$, we define a constant $C_{x_1,x_2}\in[0,\infty]$ such that $d(x_1,x_2)\le C_{x_1,x_2}$. That is,

$$|x_1(t)-x_2(t)|e_{\ominus(L+1)}(t,t_0)\le C_{x_1,x_2}\phi(t), \quad t\in\mathbb{I}.$$

Then, keeping in mind Theorem 2.2, it follows, for $x_1,x_2\in\mathcal{C}_{\rm rd}$ and $t\in\mathbb{I}$, that

$$\begin{aligned}|T[x_1](t)-T[x_2](t)| &= \left|\int_{t_0}^{t}[f(s,x_1(s))-f(s,x_2(s))]\Delta s\right|\\ &\le \int_{t_0}^{t}|f(s,x_1(s))-f(s,x_2(s))|\,\Delta s\\ &\le L\int_{t_0}^{t}|(x_1)(s)-(x_2)(s)|\Delta s\\ &= LC_{x_1,x_2}\int_{t_0}^{t}\phi(s)e_{(L+1)}(s,t_0)\Delta s\end{aligned}$$

$$\leq \frac{L}{L+1}\, C_{x_1,x_2}\phi(t)e_{(L+1)}(t,t_0).$$

Thus, for any $x_1, x_2 \in \mathcal{C}_{\mathrm{rd}}$, we have

$$d(T[x_1], T[x_2]) \;\leq \frac{L}{L+1} d(x_1, x_2).$$

Since $\frac{L}{L+1} < 1$, we find that T is contraction mapping on $\mathcal{C}_{\mathrm{rd}}$. In this way, we verified all assumptions of Theorem 2.2 with $k = 1$ and $\mathcal{C}^*_{\mathrm{rd}} = \mathcal{C}_{\mathrm{rd}}$. On the other hand, by the virtue of (3.8), we get

$$-\phi(t) \leq x(t) - \zeta(t) - \int_{t_0}^{t} f(t,s,x(s))\Delta s \leq \phi(t), \quad t \in \mathbb{I},$$

i.e.,

$$-\phi(t) \leq x(t) - T[x](t) \leq \phi(t), \quad t \in \mathbb{I},$$

i.e.,

$$|x(t) - T[x](t)| \leq \phi(t), \quad t \in \mathbb{I}.$$

Multiplying the last inequality by $e_{\ominus(L+1)}(t,t_0)$, we obtain

$$|x(t) - T[x](t)|e_{\ominus(L+1)}(t,t_0) \leq \phi(t)e_{\ominus(L+1)}(t,t_0), \quad t \in \mathbb{I},$$

i.e.,

$$d(T[x], x) \leq \phi(t)e_{\ominus(L+1)}(t,t_0),$$

for all $t \in \mathbb{I}$. Thus, in the view of Theorem 2.2, VIE (1.1) admits a unique solution $y\colon \mathbb{I} \to \mathbb{R}$ satisfying

$$d(x,y) \leq \frac{1}{1-\frac{L}{L+1}} d(T[x], x) \leq \phi(t)e_{\ominus(L+1)}(t,t_0)(L+1).$$

Therefore

$$|x(t) - y(t)| \leq (L+1)\phi(t), \quad t \in \mathbb{I}.$$

This completes the proof. □

4. Illustrative Example

Now, we present a concrete example to illustrate our findings.

Example 4.1 *Let $\mathbb{T} = 2\mathbb{Z}$ and $\mathbb{I} := [0,6]_{\mathbb{T}}$. Consider the Volterra integral equation*

$$x(t) = \frac{t^4}{3} - \frac{t^3}{2} - \frac{t^2}{3} + t + 1 - t\int_0^t sx(s)\Delta s, \quad t \in \mathbb{I}. \tag{4.1}$$

Comparing (4.1) with (1.1), we have $\zeta(t) = \frac{t^4}{3} - \frac{t^3}{2} - \frac{t^2}{3} + t + 1$ and $f(t,s,x(s)) = tsx(s)$. Note that $\zeta \in \mathcal{C}_{\mathrm{rd}}(\mathbb{I}, \mathbb{R})$ and $f \in \mathcal{C}_{\mathrm{rd}}(\mathbb{I} \times \mathbb{I} \times \mathbb{R}, \mathbb{R})$ is bounded. Moreover, f satisfies the Lipschitz condition with Lipschitz constant $L = 36$, i.e.,

$$|f(t,s,x_1(s)) - f(t,s,x_2(s))| \leq 36|x_1 - x_2|.$$

We discuss the Hyers–Ulam stability of (4.1) by employing Theorem 3.1. For this, we find the exact solution of (4.1) and, corresponding to given values of ε, approximate solutions of (4.1) and verify the Inequality (3.5). First, we see that

$$x(t) = t + 1, \quad t \in \mathbb{I}, \tag{4.2}$$

is the exact solution of (4.1). In fact, for any $t \in \mathbb{I}$,

$$\int_0^t sx(s)\Delta s = \int_0^t s(s+1)\Delta t$$

$$
\begin{aligned}
&= \int_0^t \left(\left(\frac{s^3}{3} \right)^\Delta - (s^2)^\Delta + \frac{2}{3} + \left(\frac{s^2}{2} \right)^\Delta - 1 \right) \Delta s \\
&= \frac{1}{3} \int_0^t (s^3)^\Delta \Delta t - \frac{1}{2} \int_0^t (s^2)^\Delta \Delta s - \frac{1}{3} \int_0^t \Delta s \\
&= \frac{t^3}{3} - \frac{t^2}{2} - \frac{t}{3}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
x(t) &= \frac{t^4}{3} - \frac{t^3}{2} - \frac{t^2}{3} + t + 1 - t \int_0^t s x(s) \Delta s \\
&= \frac{t^4}{3} - \frac{t^3}{2} - \frac{t^2}{3} + t + 1 - t \left(\frac{t^3}{3} - \frac{t^2}{2} - \frac{t}{3} \right) \\
&= \frac{t^4}{3} - \frac{t^3}{2} - \frac{t^2}{3} + t + 1 - \frac{t^4}{3} + \frac{t^3}{2} + \frac{t^2}{3} \\
&= t + 1.
\end{aligned}
$$

This gives that the exact solution of (4.1) *is given by* (4.2). *Next, we shall take few approximate solutions of* (4.1) *and verify Theorem 3.1.*

(i) *Let* $x_1 \in \mathcal{C}_{\mathrm{rd}}(\mathbb{I}, \mathbb{R})$ *be defined* $x_1(t) = t + \frac{9}{10}$ *and take* $\varepsilon = 14.5$ *Then, for* $t \in \mathbb{I}$, *we have*

$$
\begin{aligned}
&\left| x_1(t) - \left(\frac{t^4}{3} - \frac{t^3}{2} - \frac{t^2}{3} + t + 1 - t \int_0^t s x_1(s) \Delta s \right) \right| \\
&= \left| t + \frac{9}{10} - \frac{t^4}{3} + \frac{t^3}{2} + \frac{t^2}{3} - t - 1 + t \int_0^t s \left(s + \frac{9}{10} \right) \Delta s \right| \\
&= \left| -\frac{1}{10} - \frac{t^4}{3} + \frac{t^3}{2} + \frac{t^2}{3} + t \left(\sum_{s=0}^{\frac{t}{2}-1} 8s^2 + \frac{36}{10} s \right) \right| \\
&= \left| -\frac{1}{10} - \frac{t^4}{3} + \frac{t^3}{2} + \frac{t^2}{3} + t \frac{4}{3} \left(\frac{t}{2} - 1 \right) \left(\frac{t}{2} \right) (t-1) + \frac{9}{5} \left(\frac{t}{2} - 1 \right) \left(\frac{t}{2} \right) \right| \\
&= \left| -\frac{1}{10} - \frac{t^3}{2} + \frac{t^2}{3} - t^3 + \frac{2t^2}{3} + \frac{9t^3}{20} - \frac{9t^2}{10} \right| \\
&= \left| -\frac{1}{10} - \frac{t^3}{20} + \frac{t^2}{10} \right| \\
&\leq \frac{1}{10} + \frac{|t^3|}{20} + \frac{|t^2|}{10} \leq 14.5 = \varepsilon.
\end{aligned}
$$

This shows that $x_1(t) = t + \frac{9}{10}$ *is an approximate solution of* (4.1) *with* $\varepsilon = 14.5$. *Next, for the exact solution* x *given in* (4.2), *we have*

$$
\begin{aligned}
|x_1(t) - x(t)| &= |t + \frac{9}{10} - t - 1| \\
&\leq \frac{1}{10} \\
&\leq (L+1)\varepsilon.
\end{aligned}
$$

Thus, $|x_1(t) - x(t)| \leq (L+1)\varepsilon$, $t \in \mathbb{I}$.

(ii) *Let* $x_2 \in \mathcal{C}_{\mathrm{rd}}(\mathbb{I}, \mathbb{R})$ *be defined by* $x_2(t) = t + \frac{3}{2}$ *and take* $\varepsilon = 24.5$. *Then, for* $t \in \mathbb{I}$, *we have*

$$
\left| x_2(t) - \left(\frac{t^4}{3} - \frac{t^3}{2} - \frac{t^2}{3} + t + 1 - t \int_0^t s x_2(s) \Delta s \right) \right|
$$

$$
\begin{aligned}
&= \left| t + \frac{3}{2} - \frac{t^4}{3} + \frac{t^3}{2} + \frac{t^2}{3} - t - 1 + t \int_0^t s\left(s + \frac{3}{2}\right)\Delta s \right| \\
&= \left| t + \frac{3}{2} - \frac{t^4}{3} + \frac{t^3}{2} + \frac{t^2}{3} - t - 1 + t\left(\sum_{s=0}^{\frac{t}{2}-1} 8s^2 + 6s\right)\right| \\
&= \left| \frac{1}{2} - \frac{t^4}{3} + \frac{t^3}{2} + \frac{t^2}{3} + t\left[\frac{4}{3}\left(\frac{t}{2} - 1\right)\left(\frac{t}{2}\right)(t-1) + 3\left(\frac{t}{2} - 1\right)\left(\frac{t}{2}\right)\right]\right| \\
&= \left| \frac{1}{2} - \frac{t^4}{3} + \frac{t^3}{2} + \frac{t^2}{3} + \frac{t^4}{3} - \frac{4t^3}{3} + \frac{4t^2}{3} + \frac{3t^3}{4} - \frac{3t^2}{2} \right| \\
&= \left| \frac{1}{2} - \frac{t^3}{12} + \frac{t^2}{6} \right| \\
&\leq \frac{1}{2} + \frac{|t^3|}{12} + \frac{|t^2|}{6} \leq 24.5 = \varepsilon.
\end{aligned}
$$

This shows that $x_2(t) = t + \frac{3}{2}$ *is an approximate solution of* (4.1) *with* $\varepsilon = 24.5$. *Next, for the exact solution* x *given in* (4.2), *we have*

$$
\begin{aligned}
|x_2(t) - x(t)| &= \left| t + \frac{3}{2} - t - 1 \right| \\
&\leq \frac{1}{2} \\
&\leq (L+1)\varepsilon.
\end{aligned}
$$

Thus, $|x_2(t) - x(t)| \leq (L+1)\varepsilon$, $t \in \mathbb{I}$.

(iii) *Let* $x_3 \in \mathcal{C}_{\mathrm{rd}}(\mathbb{I}, \mathbb{R})$ *be defined by* $x_3(t) = t + \frac{4}{3}$ *and take* $\varepsilon = 48.34$. *Then, for* $t \in \mathbb{I}$, *we have*

$$
\begin{aligned}
&\left| x_3(t) - \left(\frac{t^4}{3} - \frac{t^3}{2} - \frac{t^2}{3} + t + 1 - t \int_0^t s x_3(s)\Delta s \right) \right| \\
&= \left| t + \frac{4}{3} - \frac{t^4}{3} + \frac{t^3}{2} + \frac{t^2}{3} - t - 1 + t \int_0^t s\left(s + \frac{4}{3}\right)\Delta s \right| \\
&= \left| t + \frac{4}{3} - \frac{t^4}{3} + \frac{t^3}{2} + \frac{t^2}{3} - t - 1 + t\left(\sum_{s=0}^{\frac{t}{2}-1} 8s^2 + \frac{16}{3}s\right)\right| \\
&= \left| \frac{1}{3} - \frac{t^4}{3} + \frac{t^3}{2} + \frac{t^2}{3} + t\left[\frac{4}{3}\left(\frac{t}{2} - 1\right)\left(\frac{t}{2}\right)(t-1) + \frac{8}{3}\left(\frac{t}{2} - 1\right)\left(\frac{t}{2}\right)\right]\right| \\
&= \left| \frac{1}{3} - \frac{t^4}{3} + \frac{t^3}{2} + \frac{t^2}{3} + \frac{t^4}{3} - \frac{t^3}{3} + \frac{2t^2}{3} + \frac{2t^3}{3} - \frac{4t^2}{3} \right| \\
&= \left| \frac{1}{3} + \frac{t^3}{6} - \frac{t^2}{3} \right| \\
&\leq \frac{1}{3} + \frac{|t^3|}{6} + \frac{|t^2|}{3} \leq 48.34 = \varepsilon.
\end{aligned}
$$

This shows that $x_3(t) = t + \frac{4}{3}$ *is an approximate solution of* (4.1) *with* $\varepsilon = 48.34$. *Next, for the exact solution* x *given in* (4.2), *we have*

$$
\begin{aligned}
|x_3(t) - x(t)| &= \left| t + \frac{4}{3} - t - 1 \right| \\
&\leq \frac{1}{3} \\
&\leq (L+1)\varepsilon.
\end{aligned}
$$

Thus, $|x_3(t) - x(t)| \leq (L+1)\varepsilon$, $t \in \mathbb{I}$.

(iv) *Let* $x_4 \in \mathcal{C}_{\mathrm{rd}}(\mathbb{I}, \mathbb{R})$ *be defined by* $x_4(t) = t + \frac{1}{2}$ *and take* $\varepsilon = 72.5$. *Then, for* $t \in \mathbb{I}$, *we have*

$$\begin{aligned}
&\left| x_4(t) - \left(\frac{t^4}{3} - \frac{t^3}{2} - \frac{t^2}{3} + t + 1 - t \int_0^t s x_4(s) \Delta s \right) \right| \\
&= \left| t + \frac{1}{2} - \frac{t^4}{3} + \frac{t^3}{2} + \frac{t^2}{3} - t - 1 + t \int_0^t s \left(s + \frac{1}{2} \right) \Delta s \right| \\
&= \left| t + \frac{1}{2} - \frac{t^4}{3} + \frac{t^3}{2} + \frac{t^2}{3} - t - 1 + t \left(\sum_{s=0}^{\frac{t}{2}-1} 8s^2 + 2s \right) \right| \\
&= \left| -\frac{1}{2} - \frac{t^4}{3} + \frac{t^3}{2} + \frac{t^2}{3} + t \left[\frac{4}{3} \left(\frac{t}{2} - 1 \right) \left(\frac{t}{2} \right) (t-1) + \left(\frac{t}{2} - 1 \right) \left(\frac{t}{2} \right) \right] \right| \\
&= \left| -\frac{1}{2} - \frac{t^4}{3} + \frac{t^3}{2} + \frac{t^2}{3} + \frac{t^4}{3} - \frac{t^3}{3} + \frac{2t^2}{3} - \frac{2t^3}{3} - \frac{t^2}{2} + \frac{t^3}{4} \right| \\
&= \left| -\frac{1}{2} + \frac{t^3}{4} + \frac{t^2}{2} \right| \\
&\leq \frac{1}{2} + \frac{|t^3|}{4} + \frac{|t^2|}{2} \leq 72.5 = \varepsilon.
\end{aligned}$$

This shows that $x_4(t) = t + \frac{1}{2}$ *is an approximate solution of* (4.1) *with* $\varepsilon = 72.5$. *Next, for the exact solution* x *given in* (4.2), *we have*

$$\begin{aligned}
|x_4(t) - x(t)| &= \left| t + \frac{1}{2} - t - 1 \right| \\
&\leq \frac{1}{2} \\
&\leq (L+1)\varepsilon.
\end{aligned}$$

Thus, $|x_4(t) - x(t)| \leq (L+1)\varepsilon$, $t \in \mathbb{I}$.

5. Concluding Remark

In this paper, we have demonstrated the Ulam stability of a specific class of dynamic Volterra integral equations on time scales. We accomplished this by utilizing a fixed point theorem on generalized metric space. Nevertheless, it would be of great interest to investigate the Ulam stability in the case of other dynamic equations in near future, utilizing a fixed point theorem on generalized metric space.

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