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Analysis of T_1 Separation Axioms within Extended Fuzzy Topological Frameworks

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ABSTRACT: In this paper, we introduce new definitions of extended fuzzy T_1 spaces and establish relations between them and their counterparts. We show that these concepts have projective, productive, and hereditary characteristics. We also demonstrate that generalized bijective fuzzy continuous and generalized fuzzy open mappings preserve these spaces. Furthermore, these ideas are examined in the framework of initial and final extended fuzzy topological spaces.

Key Words: Fuzzy topological space, quasi-coincidence, generalized fuzzy open, extended fuzzy T_1-space, subspace, generalized lower semi continuous, sum of extended fuzzy, initial and final extended fuzzy topological spaces.

Contents

1. Introduction

Zadeh first initiated the theory of fuzzy sets in 1965 [25]. Fuzzy sets are a generalisation of abstract sets or crisp sets and have a wide scope in solving various real-world physical problems (see [2,3], [11], [18], [26]). In 1968, utilizing the theory of fuzzy sets, Chang [6] defined fuzzy topological spaces. Since then, numerous researchers have conducted in-depth research on fuzzy topological spaces, including [5], [9,10], [12], [15,16,17], and [21,22,23,24]. For instance, Császar introduced generalised topological spaces in [4], and Chetty later extended this to include generalised fuzzy topological spaces in [7]. Afterwards, the notion of generalized fuzzy topological spaces was developed and extended (see [1], [7,8], [14], and [20]). This framework relies on separation axioms, of which the generalized fuzzy T_1 axiom is a well-known example.

This paper aims to advance the study of extended fuzzy topological spaces, particularly extended fuzzy T_1 topological spaces. We introduce novel concepts related to these structures and investigate their relationships.

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The paper's structure is as follows: Section 2 presents preliminary results. Section 3 introduces and discusses extended fuzzy T_1-spaces, along with their relationships and a generalised lower semi-continuous function. Section 4 introduces the notion of a subspace in extended fuzzy topological spaces and shows that these new ideas have additive, projective, and hereditary properties. Section 5 delves into how extended bijective fuzzy continuous and extended fuzzy open mappings preserve our notions of extended fuzzy T_1-spaces. Section 6 concludes with initial and final extended fuzzy topological spaces, followed by a discussion and analysis of a generalised lower semi-continuous function.

2. Preliminaries

In this section, key ideas for the discussions that follow are introduced. In this work, X and Y are non-empty sets, and I is the closed unit interval $[0, 1]$.

Definition 2.1 *[25] A function from X to I is a fuzzy set in X. The fuzzy sets 0_X and 1_X are defined as follows:*

$$0_X(x) = 0, \text{ and } 1_X(x) = 1, \forall x \in X.$$

We denote the other fuzzy sets on X as U, H, V, and W. I^X is the set of all fuzzy sets on X.

Definition 2.2 *[16] For each $x \in X$, the complement of a fuzzy set U, denoted as U^c, is*

$$U^c(x) = 1_X(x) - U(x) = 1 - U(x).$$

Definition 2.3 *[16] Let $\{H_\ell \mid \ell \in J\}$ represent a family of fuzzy sets in X, where J is an indexing set. The union and intersection of these collections are specified by:*

$$\left(\bigcup_{\ell \in J} H_\ell\right)(x) = \bigvee \{H_\ell(x) : \ell \in J\}, \ \forall x \in X;$$
$$\left(\bigcap_{\ell \in J} H_\ell\right)(x) = \bigwedge \{H_\ell(x) : \ell \in J\}, \ \forall x \in X.$$

Definition 2.4 *[16] A fuzzy singleton in a set X is a fuzzy set that has a value of α (with $0 < \alpha \leq 1$) for only one element and is 0 for all other elements. It is denoted as x_α, where x is its support.*
We note that when $\alpha = 1$, the fuzzy singleton is referred to as a crisp fuzzy singleton.
The collection of all fuzzy singletons in a set X is denoted as $FS(X)$. Two fuzzy singletons, x_α and y_β, are considered distinct if either $x \neq y$ or $\alpha \neq \beta$.

Definition 2.5 *[16] The fuzzy singleton x_α is regarded as a member of a fuzzy set U, represented as $x_\alpha \in U$, if $\alpha \leq U(x)$.*

Definition 2.6 *[6] Let $f\colon X \longrightarrow Y$ be a mapping and $U \in I^X$. The image of U, denoted by $f(U) \in I^Y$, is defined as:*

$$f(U)(y) = \begin{cases} \bigvee U(x) & \text{if} \quad x \in f^{-1}(y) \neq \emptyset, x \in X; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.7 *[6] For a function $f\colon X \longrightarrow Y$ and a fuzzy subset V of Y, the preimage $f^{-1}(V)$ can be used to represent the fuzzy subset of X defined as:*

$$f^{-1}(V)(x) = V(f(x)), \text{ for each } x \in X.$$

Definition 2.8 *[6] A fuzzy topological space is defined as a pair (X, τ) if τ is a set of fuzzy sets on X that satisfy:*

1. $0_X, \ 1_X \in \tau$,
2. $U, V \in \tau \Longrightarrow U \wedge V \in \tau$,

3. $U_\ell \in \tau$ *for* $\ell \in J \Longrightarrow \bigvee_{\ell \in J} U_\ell \in \tau$.

Theorem 2.1 *[6] Let U and V be fuzzy sets in X and Y, respectively, and $f\colon X \to Y$ be a mapping. Then:*

1. . $(f(U))^c \subseteq f(U^c)$, $f^{-1}(V^c) = (f^{-1}(V))^c$;

2. $U \subseteq f^{-1}(f(U))$, $f(f^{-1}(V)) \subseteq V$;

3. *If f is one-to-one, then* $f^{-1}(f(U)) = U$.

4. *If f is onto, then* $f(f^{-1}(V)) = V$;

5. *If f is one-to-one and onto, then* $(f(U))^c = f(U^c)$.

Definition 2.9 *[16] Two fuzzy sets U and V are regarded as quasi-coincident (represented as UqV) if $\exists x \in X$ for which $U(x) + V(x) > 1$. For each $x \in X$, they are not quasi-coincident (designated $U\,\bar{q}\,V$) if $U(x) + V(x) \leq 1$. Furthermore, if $\alpha + U(x) > 1$, then a fuzzy singleton x_α is quasi-coincident with U.*

Theorem 2.2 *[24] Let $x_\alpha \in X$ be a fuzzy singleton and $f\colon X \to Y$ be a function. Then*

1. *if* $V \in I^Y$ *and* $f(x_\alpha)qV$, *then* $x_\alpha q f^{-1}(V)$;

2. *if* $U \in I^X$ *and* $x_\alpha qU$, *then* $f(x_\alpha)qf(U)$.

Proposition 2.1 *[16] Let x_α be a fuzzy singleton and $U, V \in X$ be fuzzy sets. Then*

$$U \subseteq V \iff U\bar{q}V^c;$$

particularly,

$$x_\alpha \in U \iff x_\alpha \bar{q} U^c.$$

Proposition 2.2 *[5] Assume that x_α, y_β are fuzzy singletons, and U, V, and W are fuzzy sets on X. Then the following statements are true:*

1. $U\bar{q}V \Leftrightarrow V\bar{q}U$;

2. $U \cap V = 0_X \Rightarrow U\bar{q}V$;

3. $U\bar{q}U^c$;

4. *If* $W \subseteq V$, *then* $U\bar{q}V \Rightarrow U\bar{q}W$;

5. $U \subseteq V \Leftrightarrow (x_\alpha qU \Rightarrow x_\alpha qV)$;

6. $x_\alpha q(\bigcup_{\ell \in J} U_\ell) \Leftrightarrow x_\alpha qU_\ell$, *for some* $\ell \in J$;

7. $x_\alpha q(U \cap V) \Leftrightarrow (x_\alpha qU$ *and* $x_\alpha qV)$;

8. $x_\alpha \bar{q} y_\beta \Leftrightarrow x \neq y$.

Definition 2.10 *[7] A generalized fuzzy topology (abbreviated GFT) is a sub-collection ς of I^X if $0_X \in \varsigma$ and ς is closed under arbitrary unions of its members.*

A generalized fuzzy topological space (abbreviated as GFTS) is a nonempty set X paired with a GFT ς, represented as (X, ς).

Generalized fuzzy open sets (abbreviated as GFO(X)) constitute the elements of ς, and generalized fuzzy closed sets (abbreviated as GFC(X)) are the complements of these elements.

Definition 2.11 *[19] A fuzzy set U is a ς-neighbourhood of a fuzzy singleton x_α in a GFTS (X, ς) if there is a set $V \in \varsigma$ such that $x_\alpha \in V \subseteq U$. The set of all these ς-neighborhoods of x_α is represented by the symbol $N_\varsigma(x_\alpha)$.*

Definition 2.12 *[14] A fuzzy set U in a GFTS (X, ς) is classified as a ς-Q-neighborhood of x_α if there is a $V \in \varsigma$ such that $x_\alpha \, q \, V \subseteq U$. All such ς-Q-neighborhoods of x_α are represented by the notation $N^Q_\varsigma(x_\alpha)$. A generalized fuzzy open set U is an open ς-Q-neighborhood of x_α if $x_\alpha \, q \, U$. The symbol $N^Q_{o\varsigma}(x_\alpha)$ represents the set of all such open ς-Q-neighborhoods of x_α.*

Definition 2.13 *[14] Let (X, ς) be a GFTS. The set $c_\varsigma(H)$, defined by*

$$c_\varsigma(H) = \bigcap \{W : H \subseteq W, \ W \in GFC(X)\},$$

is the ς-closure of any fuzzy set $H \in I^X$.
Likewise, the set $i_\varsigma(U)$, which is the ς-interior of U, is defined by

$$i_\varsigma(U) = \bigcup \{W : W \subseteq U, \ W \in \varsigma.\}$$

Proposition 2.3 *[14] A GFTS(X, ς) space has the following characteristics:*

1. *For any $U, V \in I^X$; $U \subseteq V \Rightarrow i_\varsigma(U) \subseteq i_\varsigma(V)$ and $c_\varsigma(U) \subseteq c_\varsigma(V)$;*
2. *For any $U \in I^X$, $i_\varsigma(U) \in GFO(X)$ with $i_\varsigma(U) \subseteq U$ and $c_\varsigma(U) \in GFC(X)$ with $U \subseteq c_\varsigma(U)$;*
3. *For any $U \in I^X$, $U \in GFO(X) \Leftrightarrow U = i_\varsigma(U)$ and $U \in GFC(X) \Leftrightarrow U = c_\varsigma(U)$;*
4. *For any $U \in I^X$, $i_\varsigma(i_\varsigma(U)) = i_\varsigma(U)$ and $c_\varsigma(c_\varsigma(U)) = c_\varsigma(U)$;*
5. *For any $U \in I^X$, $1 - c_\varsigma(U) = i_\varsigma(1 - U)$.*

Proposition 2.4 *[14] Let (X, ς) be a GFTS on X, x_α be a fuzzy singleton, and $U \in I^X$. Then $x_\alpha \in c_\varsigma(U)$ if and only if x_α is quasi-coincident with U in every open ς-Q-neighborhood.*

Definition 2.14 *[14] Let (X, ς) and $(Y, \acute{\varsigma})$ be two GFTSs. A mapping $f : (X, \varsigma) \longrightarrow (Y, \acute{\varsigma})$ is said to be*

(1) generalized fuzzy continuous, if

$$\text{for every } U \in \acute{\varsigma}, \ f^{-1}(U) \in \varsigma;$$

(2) generalized fuzzy open, if

$$\text{for every } U \in \varsigma, f(U) \in \acute{\varsigma}.$$

Definition 2.15 *[13] A generalized lower semi-continuous function is defined as a real-valued function f on a GTS if the set $\{x : f(x) > \beta\}$ is generalized open for every real β.*

Definition 2.16 *[13] Let $\emptyset \neq X$ be a set equipped with a generalised topology ς. Let $\omega(\varsigma)$ denote the collection of all generalised lower semi-continuous functions from (X, ς) to I. Thus*

$$\omega(\varsigma) = \{H \in I^X : H^{-1}(\beta, 1] \in \varsigma\}, \forall \beta \in [0, 1).$$

It can be shown that $\omega(\varsigma)$ on X forms a GFT.

Definition 2.17 *[13] Considering the family of GFTS $\{(X_\ell, \varsigma_\ell)\}_{\ell\in J}$ and the set of functions $\{f_\ell : X \longrightarrow (X_\ell, \varsigma_\ell)\}_{\ell\in J}$, the initial GFTS on a set X is defined as the least GFT that ensures each f_ℓ is generalised fuzzy continuous.*
This GFT is produced by the family $\{f_\ell^{-1}(H_\ell) : H_\ell \in \varsigma_\ell\}_{\ell\in J}$.

Definition 2.18 *[13] Given the family of GFTS $\{(X_\ell, \varsigma_\ell)\}_{\ell\in J}$ and the family of functions $\{f_\ell : X \longrightarrow (X_\ell, \varsigma_\ell)\}_{\ell\in J}$, the final GFTS on a set X is defined as the finest GFT on X that ensures the generalised fuzzy continuity for every f_ℓ.*

Corollary 2.1 *[13] Given two fuzzy sets, U and V, and a GFTS (X, ς), let y_β be a fuzzy singleton. Then*

$$U\bar{q}V \iff y_\beta \bar{q}\, c_\varsigma(U), \text{ when } y_\beta \in V \in \varsigma.$$

3. Extensions of Fuzzy T_1 Separation Axioms

This section introduces extended fuzzy T_1-spaces and explores their connections.

Definition 3.1 *A extended fuzzy T_1-space (eFT_1-space, for short) is a GFTS (X, ς) where every pair of distinct fuzzy singletons x_α and y_β with $x \neq y$ has a neighbourhood $H \in N_{o\varsigma}(x_\alpha)$ such that $y_\beta\bar{q}H$ and a neighbourhood $W \in N_{o\varsigma}(y_\beta)$ such that $x_\alpha\bar{q}W$.*

Example 3.1 *Let $X = \{x, y, z\}$ and the relation $\varsigma = \{0_X, U_1, U_2, U_3\}$ be considered. In this case, $U_1 = \{(x, 1)\}$, $U_2 = \{(y, 1), (z, 0.5)\}$ and $U_3 = \{(x, 1), (y, 1), (z, 0.5)\}$. If $\alpha, \beta \in (0, 1]$, then we know that $U_1 \in N_{o\varsigma}(x_\alpha)$ exists such that $y_\beta\bar{q}U_1$ and that $U_2 \in N_{o\varsigma}(y_\beta)$ exists such that $x_\alpha\bar{q}U_2$. The space (X, ς) is therefore a eFT_1-space.*

The following theorem presents several equivalent properties of the eFT_1-space.

Theorem 3.1 *Let (X, ς) be a GFTS. The statements that follow are equivalent:*

1. *(X, ς) is a eFT_1-space;*
2. *Every two different crisp fuzzy singletons x_α and $y_\beta \in I^X$, it holds that $x_\alpha \notin c_\varsigma(y_\beta)$ and $y_\beta \notin c_\varsigma(x_\alpha)$;*
3. *Every two different fuzzy singletons x_α and $y_\beta \in I^X$, it holds that $x_\alpha\bar{q}y_\beta$, $x_\alpha\bar{q}c_\varsigma(y_\beta)$ and $y_\beta\bar{q}c_\varsigma(x_\alpha)$;*
4. *For every pair $x_\alpha, y_\beta \in FS(X)$ where $x \neq y$, $\exists\, H, W \in \varsigma$ so that $x_\alpha \in H \subseteq (y_\beta)^c$ and $y_\beta \in W \subseteq (x_\alpha)^c$;*
5. *For every pair $x_\alpha, y_\beta \in FS(X)$ where $x \neq y$, $\exists\, H, W \in \varsigma$ such that $x_\alpha \in H$, $y_\beta\bar{q}H$ and $y_\beta \in W$, $x_\alpha\bar{q}W$.*
6. *Every crisp fuzzy singleton point is a generalized fuzzy closed set in X.*

Proof:

1 ⇒ 2. Consider (X, ς) is a eFT_1-space, with $x_1, y_1 \in X$ being two different crisp fuzzy singletons. Then there exists $H \in N_{o\varsigma}(x_1)$ such that $y_\beta\bar{q}H$ and there exists $W \in N_{o\varsigma}(y_1)$ such that $x_\alpha\bar{q}W$. Without loss of generality, let us say that $H \in N_{o\varsigma}(x_1)$ implies $H \in N^Q_{o\varsigma}(x_\alpha)$ such that $y_\beta\bar{q}H$. Therefore, $H \in N^Q_{o\varsigma}(x_\alpha)$ and $y_\beta\bar{q}H$. Hence $x_\alpha \notin c_\varsigma(y_\beta)$. Similarly, we can prove that $y_\beta \notin c_\varsigma(x_\alpha)$.

2 ⇒ 3. Consider two different crisp fuzzy singletons $x_\alpha, y_\beta \in X$. Since $x_\alpha \notin c_\varsigma(y_\beta)$ and $y_\beta \notin c_\varsigma(x_\alpha)$, we can assume without lossing generality that $x_\alpha \notin c_\varsigma(y_\beta)$. Next, based on Proposition 2.1, $x_\alpha q(c_\varsigma(y_\beta))^c$ and $(c_\varsigma(y_\beta))^c \in \varsigma$. Let $(c_\varsigma(y_\beta))^c = U$. Then $x_\alpha qU \subseteq (y_\beta)^c$, so, $x_\alpha \in U$ and $U \in \varsigma$. Similarly, $y_\beta\bar{q}U$ and $x_\alpha \in U \in \varsigma$. Therefore, by Corollary 2.1, $x_\alpha\bar{q}c_\varsigma(y_\beta)$, and $y_\beta\bar{q}c_\varsigma(x_\alpha)$.

$3 \Rightarrow 4$. Consider two separate fuzzy singletons, x_α and y_β such that $x_\alpha \bar{q} y_\beta$. Then $x_\alpha \bar{q} c_\varsigma(y_\beta)$ and $y_\beta \bar{q} c_\varsigma(x_\alpha)$. Without lossing generality, we can assume $x_\alpha \bar{q} c_\varsigma(y_\beta)$. On the basis of Proposition 2.1, we see that $x_\alpha \in (c_\varsigma(y_\beta))^c$ and $(c_\varsigma(y_\beta))^c \in \varsigma$. Additionally, we have $(c_\varsigma(y_\beta))^c \bar{q} y_\beta$. Let's say $(c_\varsigma(y_\beta))^c = H$. Then $H \bar{q} y_\beta$ implies $H \subseteq (y_\beta)^c$. Consequently, $x_\alpha \in H \subseteq (y_\beta)^c$. Similarly, we can demonstrate that $y_\beta \in W \subseteq (x_\alpha)^c$.

$4 \Rightarrow 1$. Consider every distinct pair $x_\alpha, y_\beta \in FS(X)$. For each pair, there exist $H, W \in \varsigma$ such that $x_\alpha \in H \subseteq (y_\beta)^c$ and $y_\beta \in W \subseteq (x_\alpha)^c$. Suppose, without loss of generality, that there exists $H \in \varsigma$ such that $x_\alpha \in H \subseteq (y_\beta)^c$. Since, $x_\alpha \in H$ and $H \in \varsigma$ we have $H \in N_{o\varsigma}(x_\alpha)$. Additionally, since $H \subseteq (y_\beta)^c$ we have $H \bar{q} y_\beta$, which implies $y_\beta \bar{q} H$ by Proposition 2.2. Similarly, we can show that there exists $W \in N_{o\varsigma}(y_\beta)$ such that $x_\alpha \bar{q} W$. Therefore, (X, ς) is a eFT_1-space.

$4 \Leftrightarrow 5$. Follows from Proposition 2.1 and Proposition 2.2(1).

$4 \Leftrightarrow 6$. Necessity. Consider an arbitrary fuzzy singleton, $x_1 \in X$. For any other fuzzy singleton, y_β such that $x_1 \neq y$, there exists $U_1, U_2 \in GFO(X)$ such that $x_1 \in U_1 \subseteq y_\beta^c$ and $y_\beta \in U_2 \subseteq x_1^c$. Every fuzzy set can be thought of as the union of all the fuzzy singletons that make it up. Therefore, $x_1^c = \bigcup\limits_{y_\beta \subseteq x_1^c} y_\beta$. We deduce that $x_1^c = \bigcup\limits_{y_\beta \subseteq x_1^c} U_2$ and thus $x_1^c \in GFO(X)$. Hence $x_1 \in GFC(X)$.

Sufficiency: Let x_α and y_β be any pair of fuzzy singletons with $x_1 \neq y_\beta$. Let further assume that, x_1 and y_1 are crisp fuzzy singletons. The fuzzy sets x_1^c and y_1^c are generalized fuzzy open sets and satisfy the conditions $x_\alpha \in y_1^c \subseteq y_\beta^c$ and $y_\beta \in x_1^c \subseteq x_\alpha^c$.

□

Definition 3.2 *A GFTS (X, ς) is referred to as*

1. $eFT_1^{(i)}$ *if for any pair of distinct elements $x_\alpha, y_\beta \in FS(X)$, there exists $H, W \in \varsigma$ such that $x_\alpha \in H$, $y_\beta \notin H$ and $y_\beta \in W$, while $x_\alpha \notin W$;*

2. $eFT_1^{(ii)}$ *if for any pair of distinct elements $x_\alpha, y_\beta \in FS(X)$, there exists $H, W \in \varsigma$ such that $x_\alpha q H$, $y_\beta \cap H = 0_X$ and $y_\beta q W$, $x_\alpha \cap W = 0_X$.*

3. $eFT_1^{(iii)}$ *if for any pair of distinct elements $x_\alpha, y_\beta \in FS(X)$, there exists $H, W \in \varsigma$ such that $x_\alpha q H$, $y_\beta \bar{q} H$ and $y_\beta q W$, $x_\alpha \bar{q} W$.*

Theorem 3.2 *For a GFTS (X, ς), the following statements are equivalent.*

1. *(X, ς) is a $eFT_1^{(i)}$-space;*

2. *for each $x, y \in X\,(x \neq y)$, $\exists\, H, W \in \varsigma$ such that $H(x) = 1$, $H(y) = 0$ and $W(y) = 1$, $W(x) = 0$;*

3. *for any pair of different elements $x_\alpha, y_\beta \in FS(X)$, there exists $H, W \in \varsigma$ such that $x_\alpha \in H$, $y_\beta \cap H = 0_X$ and $y_\beta \in W$, $x_\alpha \cap W = 0_X$.*

Proof:

$1 \Leftrightarrow 2$ Necessity. Let (X, ς) be a $eFT_1^{(i)}$-space, and let x_α and $y_\beta \in FS(X)$ such that $x \neq y$ and $x_\alpha(x) = y_\beta(y) = 1 - \frac{1}{n}$, where $n \in \mathbb{N}$. Then there exists $H_n, W_n \in \varsigma$ such that $x_\alpha \in H_n$ and $y_\beta \notin H_n$ and $y_\beta \in W_n$ and $x_\alpha \notin W_n$. Since $x_\alpha \in H_n$ we have $\alpha \leq H_n(x) \Longrightarrow H_n(x) > 1 - \frac{1}{n}$. Therefore, we define $H = \bigcup\limits_n H_n$. Since $H \in \varsigma$, we have $H(x) = 1$ and $H(y) = 0$. Similarly, we can prove that $W(y) = 1$ and $W(x) = 0$.

Sufficiency. Let $x_\alpha \in FS(X)$ and $y_\beta \in FS(X)$ such that, $x \neq y$ and $\alpha, \beta \in (0, 1]$. Then there exists $H, W \in \varsigma$ such that, $H(x) = 1$, $H(y) = 0$ and $W(y) = 1$, $W(x) = 0$. Since $H(x) = 1$, for any $\alpha \in (0, 1]$, $\alpha \leq H(x)$. Therefore, $x_\alpha \in H$. Similarly, since $\beta \in (0, 1]$, we have $\beta \nleq H(y)$ when $H(y) = 0$. In this case, $y_\beta \notin H$. The same is true for $y_\beta \in W$ and $x_\alpha \notin W$. This means that (X, ς) is a $eFT_1^{(i)}$.

$1 \Leftrightarrow 3$ Necessity. Let (X, ς) be a $eFT_1^{(i)}$-space and $x_\alpha \in FS(X)$ and $y_\beta \in FS(X)$ such that $x \neq y$. Then there exists $H, W \in \varsigma$ such that, $H(x) = 1$, $H(y) = 0$ and $W(y) = 1$, $W(x) = 0$. Since $H(x) = 1$ for any $\alpha \in (0, 1]$, $\alpha \leq H(x)$. Therefore, $x_\alpha \in H$. Similarly $H(y) = 0$ implies $y_\beta \cap H = 0_X$.
Sufficiency. Let x_α and y_β be elements of FS(X). We assume that $x_\alpha(x) = y_\beta(y) = 1 - \frac{1}{2n}$ for $n \in \mathbb{N}$. There exists $H_n, W_n \in \varsigma$ such that $x_\alpha \in H_n$, $y_\beta \cap H_n = 0_X$ and $y_\beta \in W_n$, $x_\alpha \cap W_n = 0_X$. Suppose that there is an infinite subsets $\mathbb{K}$ of $\mathbb{N}$ such that $x_\alpha \in H_n$, for all $n \in \mathbb{K}$. Then, $H_n(x) > 1 - \frac{1}{2n}$ for all $n \in \mathbb{K}$.. Similarly, if $y_\beta \cap H_n = 0_X$, then $H_n(y) = 0$ for all $n \in \mathbb{K}$. Therefore, we define $H = \bigcup_{n \in \mathbb{K}} H_n$. Since, $H \in \varsigma$, we have $H(x) = 1$ and $H(y) = \bigcup_{n \in \mathbb{K}} H_n(y) = 0$. We can also prove that exists a subset W of ς such that $H(x) = 1$. Therefore, (X, ς) is a $eFT_1^{(i)}$-space.

□

Theorem 3.3 *Given a GFTS (X, ς), the following implication are true:*

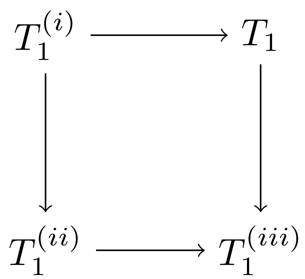

Figure 1:

But, in general, the converses are not true.

Proof:

1. Let (X, ς) be a $eFT_1^{(i)}$-space and $x_\alpha, y_\beta \in FS(X)$ such that $x \neq y$. Then there exists $H, W \in \varsigma$ such that $H(x) = 1$, $H(y) = 0$ and $W(y) = 1$, $W(x) = 0$. For $U \in \varsigma$ we have $H(x) = 1$ implies $\alpha \leq H(x)$, for any $\alpha \in (0, 1]$. Therefore $x_\alpha \in H$ for every $H \in \varsigma$ and $H(y) = 0$ implies $\beta + H(y) \leq 1$ for any $\beta \in (0, 1]$. Hence, $y_\beta \bar{q} H$. Similarly, we can prove that there exists $W \in \varsigma$ such that $y_\beta \in W$ and $x_\alpha \bar{q} W$. Therefore, (X, ς) is eFT_1.

2. Let (X, ς) be a $eFT_1^{(i)}$-space and $x_\alpha, y_\beta \in FS(X)$ and $(x \neq y)$ such that $x_\alpha(x) = y_\beta(y) = 1 - \frac{1}{n}, n \in \mathbb{N}$. Then there exists $U_n, V_n \in \varsigma$ such that $x_\alpha \in U_n$, $y_\beta \notin U_n$ and $y_\beta \in V_n$, $x_\alpha \notin V_n$. For $U_n \in \varsigma$, we have $x_\alpha \in U_n \Longrightarrow \alpha \leq U_n(x) \Rightarrow 1 - \frac{1}{n} < U_n(x)$. Define $U = \bigcup_n U_n$. Then $U \in \varsigma$ and $U(x) = 1$ and $U(y) = 0$. Now, $U(x) = 1$ implies $U(x) + \alpha > 1$ for any $\alpha \in (0, 1]$. Therefore, $x_\alpha q U$ and $U(y) = 0 \Rightarrow y_\beta \cap U = 0_X$. Similarly, we can show that there exists $V \in \varsigma$ such that $y_\beta q V$ and $x_\alpha \cap V = 0_X$. Hence (X, ς) is $eFT_1^{(iii)}$.

3. Obvious from Proposition 2.2(2).

4. Let (X, ς) be a eFT_1-space and $x_\alpha, y_\beta \in FS(X)$, where $(x \neq y)$ such that $x_\alpha(x) = y_\beta(y) = 1 - \frac{1}{n}$, for $n \in \mathbb{N}$. Then there exists $U_n, V_n \in \varsigma$ such that $x_\alpha \in U_n$, $y_\beta \bar{q} U_n$ and $y_\beta \in V_n$, $x_\alpha \bar{q} V_n$. Since,

$x_\alpha \in U_n$, we have $\alpha \le U_n(x) \Rightarrow 1 - \frac{1}{n} < U_n(x)$. Therefore, $U = \bigcup_n U_n$ is in ς, and $U(x) = 1$. Consequently, $U(x) + \alpha > 1$ for all $\alpha \in (0, 1]$. Hence $x_\alpha q U$ and $y_\beta \bar{q} U$. Similarly, we can show that there exists $V \in \varsigma$ such that $y_\beta q V$ and $x_\alpha \bar{q} V$. Therefore, (X, ς) is $eFT_1^{(iii)}$.

□

The following examples illustrate that the reverse of the above implications does not hold.

Example 3.2 *Consider the set* $X = \{x, y\}$ *and* $\varsigma = \{0_X, U_1, U_2\}$, *where* $U_1 = \{(x, 1)\}$ *and* $U_2 = \{(y, 0.1)\}$. *For any* $\alpha, \beta \in (0, 1]$, *we have* $U_1 \in \varsigma$ *with* $x_\alpha \in U_1$ *and* $y_\beta \bar{q} U_1$. *Additionally, for* $0 < \alpha \le 1$, *and* $0 < \beta \le 0.9$, *we get* $y_\beta \in U_2$ *and* $x_\alpha \bar{q} U_2$. *Therefore,* (X, ς) *is a* eFT_1*-space. However, it is not a* $eFT_1^{(i)}$*-space, as for any* $\alpha, \beta \in (0, 1]$, $U_2 \in \varsigma$ *with* $U_2(x) = 0$ *and* $U_2(y) = 0.1 \ne 1$.

Example 3.3 *Consider the set* $X = \{x, y\}$ *and* $\varsigma = \{0_X, U_1, U_2, U_3\}$ *where* $U_1 = \{(x, 1)\}$, *and* $U_2 = \{(y, 1 - \frac{\beta}{2})\}, \beta \in (0, 1]$ *and* $U_3 = \{(x, 1), (y, 1 - \frac{\beta}{2})\}$. *Then for any* $\alpha, \beta \in (0, 1]$, *we have* (X, ς) *is a* $eFT_1^{(iii)}$*-space. However, it is not a* eFT_1 *as* $U_2 \in \varsigma$ *with* $x_\alpha \bar{q} U_2$ *and* $y_\beta \notin U_2$ *for* $\beta = 1$.

Example 3.4 *Let* (X, ς) *be a GFTS as in Example 3.2. Assume that* (X, ς) *is a* $eFT_1^{(ii)}$ *but not* $eFT_1^{(i)}$, *because* $U_2 \in \varsigma$ *with* $U_2(x) = 0$ *and* $U_2(y) \ne 1$.

Example 3.5 *Consider the set* $X = \{x, y\}$ *and* $\varsigma = \{0_X, U_1, U_2\}$, *where* $U_1 = \{(x, 1), (y, 0.6)\}$ *and* $U_2 = \{(y, 0.6)\}$. *For* $0 < \alpha \le 1$ *and* $0 < \beta \le 0.4$, *we have* $x_\alpha q U_1, y_\beta \bar{q} U_1$ *and for* $0.4 < \beta \le 1$, $x_\alpha \bar{q} U_2$ *and* $y_\beta q U_2$. *Hence,* (X, ς) *is a* $eFT_1^{(iii)}$. *However, it is not a* $eFT_1^{(ii)}$ *as* $U_1 \in \varsigma$ *such that* $U_1(y) \ne 0 \Longrightarrow y_\beta \cap U_1 \ne 0_X$.

Definition 3.3 *A real-valued function* f *on a GTS is called a generalized lower semi-continuous function if the set* $\{x : f(x) > r\}$ *is generalized open for all real* r.

Definition 3.4 *The set* X *is nonempty, and its generalized topology is* ς. *The collection of all generalized lower semi-continuous functions from* (X, ς) *to* I *is represented by* $\omega(\varsigma)$. *Thus,* $\omega(\varsigma) = \{H \in I^X : H^{-1}(r, 1] \in \varsigma\}, \forall r \in [0, 1)$. *It can be shown that* $\omega(\varsigma)$ *forms a GFT on* X.

Theorem 3.4 *Let* (X, ς) *be a GTS. The following statements are equivalent:*

1. (X, ς) *is characterized as a* eT_1*-space;*
2. $(X, \omega(\varsigma))$ *is characterized as a* eFT_1*-space;*
3. $(X, \omega(\varsigma))$ *is characterized as a* $eFT_1^{(i)}$*-space;*
4. $(X, \omega(\varsigma))$ *is characterized as a* $eFT_1^{(ii)}$*-space;*
5. $(X, \omega(\varsigma))$ *is characterized as a* $eFT_1^{(iii)}$*-space.*

Proof:

$1 \Leftrightarrow 2$ Necessity: Suppose that (X, ς) equals eT_1. Our goal is to demonstrate that eFT_1 is $(X, \omega(\varsigma))$. Let $x \ne y$ in FS(X) be x_α and y_β. Since, (X, ς) is eT_1, there exists $H, W \in \varsigma$ such that $x \in H$, $y \notin H$ and $y \in W$, $x \notin W$. In accordance with the generalized lower semi-continuous function concept $1_H, 1_W \in \omega(\varsigma)$ and satisfy $1_H(x) = 1, 1_H(y) = 0$ and $1_W(y) = 1$, $1_W(x) = 0$. Thus:
• Given that $1_H(x) = 1$, $\alpha \le 1_H(x)$, so $x_\alpha \in 1_H$.
• Given that $1_H(y) = 0$, $1_H(y) + \beta \le 1$, meaning that $y_\beta \bar{q} 1_H$. Therefore, $1_H \bar{q} y_\beta$, which is equivalent to $1_H \subseteq (y_\beta)^c$. Consequently, $1_H \in \omega(\varsigma)$ and $x_\alpha \in 1_H \subseteq (y_\beta)^c$. The same can be said for $y_\beta \in 1_W \subseteq (x_\alpha)^c$. Hence, the space $(X, \omega(\varsigma))$ is therefore a eFT_1-space.
Sufficiency: Assume that $(X, \omega(\varsigma))$ is a eFT_1-space. We must demonstrate that (X, ς) is eT_1.

Let $x, y \in X$ with $x \neq y$. Since $(X, \omega(\varsigma))$ is a eFT_1-space, for all $x_\alpha, y_\alpha \in FS(X)$, there exists $H, W \in \omega(\varsigma)$ such that $x_\alpha \in H \subseteq (y_\alpha)^c$ and $y_\alpha \in W \subseteq (x_\alpha)^c$. Now, $x_\alpha \in H \Rightarrow \alpha < H(x) \Rightarrow 1-\alpha = m < H(x) \Rightarrow x \in H^{-1}(m, 1]$. Similarly, $H \subseteq (y_\alpha)^c \Rightarrow H\bar{q}y_\alpha \Rightarrow H(y) + \alpha \leq 1 \Rightarrow H(y) \leq 1-\alpha = m \Rightarrow y \notin H^{-1}(m, 1]$. It is also possible to show that $y \in W^{-1}(m, 1]$ and $x \notin W^{-1}(m, 1]$. Additionally, $H^{-1}(m, 1]$ and $W^{-1}(m, 1]$ are generalized open sets. Hence (X, ς) is a eT_1-space.

$1 \Leftrightarrow 3$ Necessity: Assume (X, ς) is eT_1. We will demonstrate that $(X, \omega(\varsigma))$ is $eFT_1^{(i)}$. Let $x \neq y$ and assume that $x_\alpha, y_\beta \in FS(X)$. Since (X, ς) is eT_1, there exists $H, W \in \varsigma$ such that $x \in H$, $y \notin H$ and $y \in W$, $x \notin W$. In accordance with the generalized lower semi-continuous function concept, $1_H, 1_W \in \omega(\varsigma)$ and satisfy $1_H(x) = 1, 1_H(y) = 0$ and $1_W(y) = 1$, $1_W(x) = 0$. Thus (X, ς) is a $eFT_1^{(i)}$-space.

Sufficiency: Suppose $(X, \omega(\varsigma))$ is a $eFT_1^{(i)}$-space. We need to demonstrate that (X, ς) is a eT_1-space. Assume $x, y \in X$ with $x \neq y$. Since $(X, \omega(\varsigma))$ is a $eFT_1^{(i)}$-space, for all $x_\alpha, y_\alpha \in FS(X)$, there exists $H, W \in \omega(\varsigma)$ such that $1_H(x) = 1, 1_H(y) = 0$ and $1_W(y) = 1$, $1_W(x) = 0$. Now, $1_H(x) = 1 \implies H(x) + \alpha > 1 \implies H(x) > 1-\alpha = m \implies x \in U^{-1}(m, 1]$. Similarly, $H(y) = 0 \Rightarrow H(y) + \alpha \leq 1 \Rightarrow H(y) \leq 1-\alpha = m \Rightarrow y \notin H^{-1}(m, 1]$. This can be similarly shown for the reverse case. Additionally, generalized open sets are $H^{-1}(m, 1]$ and $W^{-1}(m, 1]$. Thus, (X, ς) is a eT_1-space.

$1 \Leftrightarrow 4.$ Necessity: Suppose (X, ς) is eT_1. We shall demonstrate that $(X, \omega(\varsigma))$ is $eFT_1^{(ii)}$. Assume $x_\alpha, y_\beta \in FS(X)$ where $x \neq y$. Given that (X, ς) is gT_1, $\exists\, H, W \in \varsigma$ such that $x \in H$, $y \notin H$ and $y \in W$, $x \notin W$. According to the generalized lower semi continuous function definition, we are aware that $1_H, 1_W \in \omega(g)$. Thus,
- Considering that $1_H(x) = 1$, $1_H(x) + \alpha > 1$, so $x_\alpha q 1_H$.
- Since $1_H(y) = 0$, it follows that $y_\beta \cap 1_H(y) = 0_X$.The inverse case can be demonstrated in a similar manner. Therefore, $(X, \omega(\varsigma))$ is a $eFT_1^{(ii)}$-space.

Sufficiency: Suppose $(X, \omega(\varsigma))$ is a $eFT_1^{(ii)}$-space. We need to demonstrate that (X, ς) is eT_1. Assume $x, y \in X$ with $x \neq y$. Given that $(X, \omega(\varsigma))$ is $eFT_1^{(ii)}$, $\forall\, x_\alpha, y_\alpha \in FS(X)$, $\exists\, H, W \in \omega(\varsigma)$ such that $x_\alpha q H$, $y_\alpha \cap H = 0_X$ and $y_\alpha q W$, $x_\alpha \cap W = 0_X$. Now, $x_\alpha q H \Rightarrow H(x) + \alpha > 1 \Rightarrow H(x) > 1-\alpha = m \Rightarrow x \in H^{-1}(m, 1]$ and $y_\alpha \cap H = 0_X \Rightarrow H(y) = 0 \Rightarrow H(y) + \alpha \leq 1 \Rightarrow H(y) < 1-\alpha = m \Rightarrow y \notin H^{-1}(m, 1]$. The inverse case can be demonstrated in a similar manner. Additionally, generalized open sets are $H^{-1}(m, 1]$ and $W^{-1}(m, 1]$. Hence (X, ς) is a eT_1-space.

$1 \Leftrightarrow 5$ Necessity. Suppose (X, ς) is eT_1. We wish to show that $(X, \omega(\varsigma))$ is $eFT_1^{(iii)}$. Let x_α and y_β be $x \neq y$ in FS(X). Since (X, ς) is eT_1, there exists $H, W \in \varsigma$ such that $x \in H$, $y \notin H$ and $y \in W$, $x \notin W$. According to the generalized lower semi continuous function definition, we are aware that $1_H, 1_W \in \omega(\varsigma)$. Thus
$1_H(x) = 1$ implies $1_H(x) + \alpha > 1 \Rightarrow x_\alpha q 1_H$. Similarly, $1_H(y) = 0$ implies $1_H(y) + \beta \leq 1 \Rightarrow y_\beta \bar{q} 1_H$. It is also possible to demonstrate that $y_\beta q 1_W$ and $x_\alpha \bar{q} 1_W$. Hence, $(X, \omega(\varsigma))$ is a $eFT_1^{(iii)}$-space.

Sufficiency: Assume $(X, \omega(\varsigma))$ is $eFT_1^{(iii)}$. We need to demonstrate that (X, ς) is eT_1. Consider $x, y \in X$ with $x \neq y$. Sincet $(X, \omega(\varsigma))$ is $eFT_1^{(iii)}$, for all $x_\alpha, y_\alpha \in FS(X)$, there exists $H, W \in \omega(\varsigma)$ such that $x_\alpha q H$, $y_\beta \bar{q} 1_H$ and $y_\beta q W$, $x_\alpha \bar{q} 1_W$.
- $x_\alpha q H$ implies $H(x) + \alpha > 1$, so $H(x) > 1-\alpha = m$, meaning $x \in H^{-1}(m, 1]$.
- $y_\beta \bar{q} 1_H$ implies $\beta + H(y) \leq 1$, hence $H(y) < 1-\beta = m$ meaning $y \notin H^{-1}(m, 1]$. Since $H^{-1}(m, 1]$ and $W^{-1}(m, 1]$ are generalized open sets. Hence (X, ς) is a eT_1-space.

□

4. Extended Fuzzy Subspace, Product and Sum within Extended Fuzzy Topological Frameworks

In this section, we explore the hereditary property of extended fuzzy topology and introduce the concept of a subspace. We also delve into the additive, productive, and projective characteristics of extended fuzzy T_1-spaces.

Definition 4.1 *Let (X,ς) be a GFTS and $B \subseteq X$. The relative generalized fuzzy topology on B is defined as $\varsigma_B = \{U \cap B : U \in \varsigma\}$. The space (B, ς_B) is a generalized fuzzy subspace of (X,ς).*
Generalised fuzzy open sets on B (GFO(B)) are members of ς_B, and their complements are generalised fuzzy closed sets on B (GFC(B)).

Definition 4.2 *A property P of a GFTS is hereditary if every subspace of a GFTS with property P also has property P.*

Here, we shall demonstrate that our concepts of eFT_1-spaces adhere to the hereditary property.

Theorem 4.1 *Let (X, g) be a GFTS and $B \subseteq X$, then*

1. (X,ς) *is* $eFT_1 \Rightarrow (B,\varsigma_B)$ *is* eFT_1;
2. (X,ς) *is* $eFT_1^{(i)} \Rightarrow (B,\varsigma_B)$ *is* $eFT_1^{(i)}$;
3. (X,ς) *is* $eFT_1^{(ii)} \Rightarrow (B,\varsigma_B)$ *is* $eFT_1^{(ii)}$;
4. (X,ς) *is* $eFT_1^{(iii)} \Rightarrow (B,\varsigma_B)$ *is* $eFT_1^{(iii)}$.

Proof:

1. Suppose (X,ς) is eFT_1 and $x_\alpha, y_\beta \in FS(B)$ with $x \neq y$. Since $B \subseteq X$, we have $x_\alpha, y_\beta \in FS(X)$. Furthermore, since (X,ς) is a eFT_1-space, it follows that there exists $H \in N_\varsigma(x_\alpha)$ such that, $y_\beta \bar{q} H$ and $\exists\ W \in N_\varsigma(y_\beta)$ such that, $x_\alpha \bar{q} W$. For a subset B of X, both $H \cap B, W \cap B \in \varsigma_B$. Since $x_\alpha \in H$, we have $\alpha \leq H(x)$. Similarly, since $x \in X$, we have $\alpha \leq (H \cap B)(x)$. Since $x \in B \subseteq X$, we have $x_\alpha \in H \cap B$. Similarly, since $y_\beta \bar{q} H$, we have $\beta + H(y) \leq 1$. Similarly, since $y \in X$, we have $\beta + (H \cap B)(y) \leq 1$. Similarly, since $y \in B \subseteq X$, we have $y_\beta \bar{q}(H \cap B)$. Consequently, $H \cap B \in N_{\varsigma_B}(x_\alpha)$ and $y_\beta \bar{q}(H \cap B)$. Similarly, we can prove that $W \cap B \in N_{\varsigma_B}(y_\beta)$ and $x_\alpha \bar{q}(W \cap B)$. Hence (B,ς_B) is also eFT_1-space.

2. Suppose (X,ς) is $eFT_1^{(i)}$-space and $x_\alpha, y_\beta \in FS(B)$ with $x \neq y$. Since $B \subseteq X$, then $x_\alpha, y_\beta \in FS(X)$. Since (X,ς) is a $eFT_1^{(i)}$-space, there exists $H, W \in \varsigma$ such that, $x_\alpha \in H$, $y_\beta \notin H$ and $y_\beta \in W$, $x_\alpha \notin W$. For a subset B of X, both $H \cap B, W \cap B \in \varsigma_B$.
Now, $x_\alpha \in H \Rightarrow \alpha \leq H(x), x \in X \Rightarrow \alpha \leq (H \cap B)(x), x \in B \subseteq X \Rightarrow x_\alpha \in H \cap B$. Also, $y_\beta \notin H \Rightarrow \beta \nleq H(y), y \in X \Rightarrow \beta \nleq (H \cap B)(y), y \in B \subseteq X$ implies $y_\beta \notin H \cap B$. Similarly, we can prove that $y_\beta \in W \cap B$ and $x_\alpha \notin W \cap B$. Hence (B,ς_B) is also $eFT_1^{(i)}$-space.

3. Suppose (X,ς) is a $eFT_1^{(ii)}$-space and $x_\alpha, y_\beta \in FS(B)$ with $x \neq y$. Since B is a subset of X, then $x_\alpha, y_\beta \in FS(X)$. Given that (X,ς) is a $eFT_1^{(ii)}$-space, then there exists $H, W \in \varsigma$ such that, $x_\alpha q H$, $y_\beta \cap H = 0_X$ and $y_\beta q W$, $x_\alpha \cap W = 0_X$. For $B \subseteq X$, it follows that $H \cap B, W \cap B \in \varsigma_B$. Therefore, $x_\alpha q H \Rightarrow H(x) + \alpha > 1$, $x \in X \Rightarrow (H \cap B)(x) + \alpha > 1, x \in B \subseteq X \Rightarrow x_\alpha q(H \cap B)$ and $y_\beta \cap H = 0_X \Rightarrow H(y) = 0 \Rightarrow (H \cap B)(y) = 0, y \in B \subseteq X \Rightarrow y_\beta \cap (H \cap B) = 0_X$. In a similar manner, it can be demonstrated that. $y_\beta q(W \cap B), x_\alpha \cap (W \cap B) = 0_X$. Hence (B,ς_B) is also a $eFT_1^{(ii)}$-space.

4. Similarly, the proof for (3).

□

Definition 4.3 *Consider a collection of non empty sets $\{\ X_\ell, \ell \in J\}$. Define $X = \prod_{\ell \in J} X_\ell$ to represent the product of these sets. Assume that $\pi_\ell : X \to X_\ell$ is a projection mapping. Furthermore, assume that X_ℓ is a GFTS with GFT ς_ℓ. The GFT on X is generated by using $\{\pi_\ell^{-1}(b_\ell) : b_\ell \in \varsigma_\ell, \ell \in J\}$ as a sub-basis. This is referred to as X.*

Definition 4.4 *A GFTS property P is productive if each space in the given collection $\{(X_\ell, \varsigma_\ell) : \ell \in J\}$, has the property p. That is, if $(\prod X_\ell, \prod \varsigma_\ell)$ also possesses the property P.*

Definition 4.5 *A GFTS property P is projective if the product of the coordinate spaces (X_ℓ, ς_ℓ) and the product of the sign functions $(\prod X_\ell, \prod \varsigma_\ell)$ satisfies the property P. This implies that if P holds for a projective property, then each individual coordinate space (X_i, ς_i) also possesses the property P.*

Definition 4.6 *Consider two GFTS, (X, ς) and $(Y, \acute{\varsigma})$. A mapping $f : (X, \varsigma) \longrightarrow (Y, \acute{\varsigma})$ is a generalized fuzzy homeomorphism if it is bijective and both f and its inverse f^{-1} are generalized fuzzy continuous.*

We'll show that our notions of eFT_1 spaces satisfy the projective and productive properties.

Theorem 4.2 *Let $\{(X_\ell, \varsigma_\ell), \ell \in J\}$ be a collection of GFTSs. Let $X = \prod\limits_{\ell \in J} X_\ell$ and ς be the product generalised topology on X. Then, for all $\ell \in J$,*

1. *(X_ℓ, ς_ℓ) is eFT_1-space $\Leftrightarrow$ (X, ς) is eFT_1-space;*
2. *(X_ℓ, ς_ℓ) is $eFT_1^{(i)}$-space $\Leftrightarrow$ (X, ς) is $eFT_1^{(i)}$-space;*
3. *(X_ℓ, ς_ℓ) is $eFT_1^{(ii)}$-space $\Leftrightarrow$ (X, ς) is $eFT_1^{(ii)}$-space;*
4. *(X_ℓ, ς_ℓ) is $eFT_1^{(iii)}$-space $\Leftrightarrow$ (X, ς) is $eFT_1^{(iii)}$-space.*

Proof:

1. Necessity. Assume that for all $\ell \in J$, (X_ℓ, ς_ℓ) is a eFT_1-space. We need to prove that (X, ς) is also a eFT_1-space. Suppose $x_\alpha, y_\beta \in FS(X)$ with $x \neq y$. Then $(x_\ell)_\alpha, (y_\ell)_\beta \in FS(X_\ell)$ satisfy $x_\ell \neq y_\ell$ for some $\ell \in J$. Since (X_ℓ, ς_ℓ) is a eFT_1-space, there exists $H_\ell \in N_{\varsigma_\ell}((x_\ell)_\alpha)$ such that, $(y_\ell)_\beta \bar{q} H_\ell$ and $\exists\, V_\ell \in N_{\varsigma_\ell}((y_\ell)_\beta)$ such that $(x_\ell)_\alpha \bar{q} V_\ell$. Additionally, $\pi_\ell(x) = x_\ell$ and $\pi_\ell(y) = y_\ell$. Now, $H_\ell \in N_{\varsigma_\ell}((x_\ell)_\alpha) \Rightarrow (x_\ell)_\alpha \in H_\ell \Rightarrow \alpha \leq H_\ell(x_\ell) \Rightarrow \alpha \leq H_\ell(\pi_\ell(x)) \Rightarrow \alpha \leq (H_\ell \circ \pi_\ell)(x) \Rightarrow x_\alpha \in (H_\ell \circ \pi_\ell) \Rightarrow (H_\ell \circ \pi_\ell) \in N_\varsigma(x_\alpha)$. Similarly, $(y_\ell)_\beta \bar{q} H_\ell \Rightarrow H_\ell(y_\ell) + \beta \leq 1 \Rightarrow H_\ell(\pi_\ell(y)) + \beta \leq 1 \Rightarrow (H_\ell \circ \pi_\ell)(y) + \beta \leq 1 \Rightarrow y_\beta \bar{q} (H_\ell \circ \pi_\ell)$. We can also prove that $(V_\ell \circ \pi_\ell) \in N_\varsigma(y_\beta)$ and $x_\alpha \bar{q} (V_\ell \circ \pi_\ell)$. Hence (X, ς) is a eFT_1-space.
   Sufficiency. Let (X, ς) be a eFT_1-space. We need to prove that (X_ℓ, ς_ℓ) for $\ell \in J$ is also a eFT_1-space. Choose a constant element b_ℓ in X_ℓ. Define $B_\ell = \{x \in X = \prod\limits_{\ell \in J} X_\ell : x_j = b_j \text{ for some } \ell \neq j\}$. Since $B_\ell \subseteq X$. So $(B_\ell, \varsigma_{B_\ell})$ is a subspace of (X, ς). Since (X, ς) is a eFT_1-space, it follows that $(B_\ell, \varsigma_{B_\ell})$ is also a eFT_1. Furthermore, B_ℓ is homeomorphic to X_ℓ. Therefore, (X_ℓ, ς_ℓ) is a eFT_1-space for all $\ell \in J$.

2. Necessity. Assume that for all $\ell \in J$, (X_ℓ, ς_ℓ) is a $eFT_1^{(i)}$-space. We need to prove that (X, ς) is also a $eFT_1^{(i)}$. Suppose $x_\alpha, y_\beta \in FS(X)$ with $x \neq y$. Then $(x_\ell)_\alpha, (y_\ell)_\beta \in FS(X_\ell)$ satisfy $x_\ell \neq y_\ell$ for some $\ell \in J$. Since (X_ℓ, ς_ℓ) is a $gFT_1^{(i)}$-space, there exists $H_\ell, V_\ell \in \varsigma_\ell$ such that $(x_i)_\alpha \in H_\ell$, $(y_\ell)_\beta \notin H_\ell$ and $(y_\ell)_\beta \in V_\ell$, $(x_\ell)_\alpha \notin V_\ell$. However, we have $\pi_\ell(x) = x_\ell$ and $\pi_\ell(y) = y_\ell$. Now, if $(x_\ell)_\alpha \in H_\ell$, then $\alpha \leq H_\ell(x_\ell) \Rightarrow \alpha \leq H_\ell(\pi_\ell(x)) \Rightarrow \alpha \leq (H_\ell \circ \pi_\ell)(x)$ implies $x_\alpha \in (H_\ell \circ \pi_\ell)$. Similarly, if $(y_\ell)_\beta \notin H_\ell$, then $\beta \nleq H_\ell(y_\ell) \Rightarrow \beta \nleq H_\ell(\pi_\ell(y)) \Rightarrow \beta \nleq (H_\ell \circ \pi_\ell)(y)$ implies $y_\beta \notin (H_\ell \circ \pi_\ell)$. This means there exists $(H_\ell \circ \pi_\ell) \in \varsigma$ such that $x_\alpha \in (H_\ell \circ \pi_\ell)$ and $y_\beta \notin (H_\ell \circ \pi_\ell)$. Similarly, we can demonstrate that $y_\beta \in (V_\ell \circ \pi_\ell)$ and $x_\alpha \notin (V_\ell \circ \pi_\ell)$. Hence (X, ς) is also a $eFT_1^{(i)}$-space.
   Sufficiency. Let (X, ς) be a $eFT_1^{(i)}$-space. We need to prove that (X_ℓ, ς_ℓ) for $\ell \in J$ is also a $eFT_1^{(i)}$-space. Choose a constant element b_ℓ in X_ℓ. Define $B_\ell = \{x \in X = \prod\limits_{\ell \in J} X_\ell : x_j = b_j \text{ for some } \ell \neq j\}$. Since $B_\ell \subseteq X$, so $(B_\ell, \varsigma_{B_\ell})$ is a subspace of (X, ς). Since (X, ς) is a $eFT_1^{(i)}$-space, it follows that $(B_\ell, \varsigma_{B_\ell})$ is also a $eFT_1^{(i)}$-space. Furthermore, B_ℓ is homeomorphic to X_ℓ. Therefore, (X_ℓ, ς_ℓ) is a $eFT_1^{(i)}$-space for all $\ell \in J$.

3. Necessity. Assume that (X_ℓ, ς_ℓ) is a $eFT_1^{(ii)}$-space for every $\ell \in J$. It is necessary to demonstrate that (X, ς) is a $eFT_1^{(ii)}$-space. Let us say that $x_\alpha, y_\beta \in FS(X)$ where $x \neq y$. If $\ell \in J$, then $(x_\ell)_\alpha, (y_\ell)_\beta \in FS(X_\ell)$ with $x_\ell \neq y_\ell$. Given that (X_ℓ, ς_ℓ) is a $gFT_1^{(ii)}$-space, then there exists $H_\ell, W_\ell \in \varsigma_\ell$ such that, $(x_\ell)_\alpha q H_\ell$, $(y_\ell)_\beta \cap H_\ell = 0_X$ and $(y_\ell)_\beta q W_\ell$ and $(x_\ell)_\alpha \cap W_\ell = 0_X$. It should be noted that $\pi_\ell(x) = x_\ell$ and $\pi_\ell(y) = y_\ell$. This means that $(x_\ell)_\alpha q H_\ell \Rightarrow H_\ell(x_\ell) + \alpha > 1 \Rightarrow H_\ell(\pi_\ell(x)) + \alpha > 1 \Rightarrow (H_\ell \circ \pi_\ell)(x) + \alpha > 1 \Rightarrow x_\alpha q(H_\ell \circ \pi_\ell)$ and $(y_\ell)_\beta \cap H_\ell = 0_X \Rightarrow H_\ell(y_\ell) = 0 \Rightarrow H_\ell(\pi_\ell(y)) = 0 \Rightarrow (H_\ell \circ \pi_\ell)(y) = 0 \Rightarrow y_\beta \cap (H_\ell \circ \pi_\ell) = 0_X$. Consequently, $(H_\ell \circ \pi_\ell) \in \varsigma$ satisfies $x_\alpha q(H_\ell \circ \pi_\ell)$ and $y_\beta \cap (H_\ell \circ \pi_\ell) = 0_X$. Analogously, it can be demonstrated that $y_\beta q(W_\ell \circ \pi_\ell)$ and $x_\alpha \cap (W_\ell \circ \pi_\ell) = 0_X$. Therefore, (X, ς) is a $eFT_1^{(ii)}$-space.
   Sufficiency. Let (X, ς) be a $eFT_1^{(ii)}$-space. We need to demonstrate that (X_ℓ, ς_ℓ) is $eFT_1^{(ii)}$-space, for all $\ell \in J$. Consider the constant element b_ℓ in X_ℓ. Define $B_\ell = \{x \in X = \prod_{\ell \in J} X_\ell : x_j = b_j$ for some $\ell \neq j\}$. Then $B_\ell \subseteq X$, so $(B_\ell, \varsigma_{B_\ell})$ is a subspace of (X, ς). Since (X, ς) is a $eFT_1^{(ii)}$-space, then $(B_\ell, \varsigma_{B_\ell})$ is also $eFT_1^{(ii)}$-space. Furthermore, B_ℓ is homeomorphic to X_ℓ. Therefore (X_ℓ, ς_ℓ) is a $eFT_1^{(ii)}$-space for all $\ell \in J$.

4. The proof of (3) is similar.

□

Definition 4.7 *According to the above proposition, the GFT $\oplus_{i \in I} \varsigma_i$ is the sum of the GFT on X. For the family $\{(X_i, \varsigma_i) : i \in I\}$, the corresponding pair $(X, \oplus_{i \in I} \varsigma_i)$ is called the sum GFTS.*

Definition 4.8 *A family of GFTS $\{(X_i, \varsigma_i), i \in \Lambda\}$ is said to be additive if for any property P, the sum of this family $(X, \oplus_{i \in I} \varsigma_i)$ also has property P.*

We'll show that our notions of eFT_1-spaces satisfy the additive property.

Theorem 4.3 *The property of being a eFT_1 space is additive.*

Proof: For all $\ell \in J$, let (X_ℓ, ς_ℓ) be a eFT_1-space. We need to prove that $\oplus_{\ell \in J} X_\ell$ is a eFT_1-space. Consider two fuzzy singletons $x_\alpha, y_\beta \in X = \bigcup_{\ell \in J} X_\ell$ with $x \neq y$. If x and y belongs to different sets X_ℓ and X_j, then, $x_\alpha \in X_\ell \subseteq X_j - \{y_\beta\}$ and $y_\beta \in X_j \subseteq X_\ell - \{x_\alpha\}$. Since $X_\ell \subseteq X_j^c$, both X_ℓ and X_j are generalized fuzzy open sets in X under $\oplus_{\ell \in J}$. If x and y are in the same eFT_1-space $(X_{\ell_0}, \varsigma_{\ell_0})$, then $U_0, V_0 \in GFO(X_{\ell_0})$ such that $x_\alpha \in U_0 \subseteq X_{\ell_0} - \{y_\beta\}$ and $y_\beta \in V_0 \subseteq X_{\ell_0} - \{x_\alpha\}$. Since $X_{\ell_0} \in GFO(X), X = \oplus_{\ell \in J} X_\ell$, one thus $U_0, V_0 \in GFO(X)$ and proving the result. □

Theorem 4.4 *The property of being a $eFT_1^{(i)}$-space is additive.*

Proof: For every $\ell \in J$, let (X_ℓ, ς_ℓ) be a $eFT_1^{(i)}$-space. It is our responsibility to demonstrate that $\oplus_{\ell \in J} X_\ell$ is a eFT_1-space. We do this by considering two fuzzy singletons $x_\alpha, y_\beta \in X = \bigcup_{\ell \in J} X_\ell$ with $x \neq y$. If x and y belongs to different sets X_ℓ and X_j, then we can easily obtain $x_\alpha \in X_\ell$, $y_\beta \notin X_\ell$ and $y_\beta \in X_j$, $x_\alpha \notin X_j$. $X_\ell \subseteq X_j^c$, both X_ℓ and X_j are generalized fuzzy open sets in X under $\oplus_{\ell \in J}$. If x and y belong to the same $gFT_1^{(i)}$-space $(X_{\ell_0}, \varsigma_{\ell_0})$, then there exists $U_0, V_0 \in GFO(X_{\ell_0})$ such that $x_\alpha \in U_0$, $y_\beta \notin U_0$ and $y_\beta \in V_0$, $x_\alpha \notin V_0$. Since $X_{\ell_0} \in GFO(X), X = \oplus_{\ell \in J} X_\ell$, we find $U_0, V_0 \in GFO(X)$ and thus the result. □

Theorem 4.5 *The property of being a $eFT_1^{(ii)}$-space is additive.*

Proof: For every $\ell \in J$, let (X_ℓ, ς_ℓ) be a $eFT_1^{(ii)}$-space. Our objective is to demonstrate that $\oplus_{\ell\in J} X_\ell$ is a $eFT_1^{(ii)}$-space. This is accomplished by consideration two fuzzy singletons $x_\alpha, y_\beta \in X = \bigcup_{\ell\in J} X_\ell$ with distinct supports x and y. If x and y belongs to distinct sets X_ℓ and X_j one can easily obtain $x_\alpha(x)qX_\ell$, $y_\beta \cap X_\ell = 0_X$ and $y_\beta q X_j$, $x_\alpha \cap X_j = 0_X$. $X_\ell \subseteq X_j^c$, both X_ℓ and X_j are generalized fuzzy open sets in X under $\oplus_{\ell\in J}$. If x and y are part of the same $gFT_1^{(ii)}$-space $(X_{\ell_0}, \varsigma_{\ell_0})$, then $U_0, V_0 \in GFO(X_{\ell_0})$ exist and $x_\alpha q U_0$, $y_\beta \cap U_0 = 0_X$ and $y_\beta q V_0$, $x_\alpha \cap V_0 = 0_X$. Since $X_{\ell_0} \in GFO(X), X = \oplus_{\ell\in J} X_\ell$, one finds $U_0, V_0 \in GFO(X)$ and thus the result. □

Theorem 4.6 *The property of being a $eFT_1^{(iii)}$-space is additive.*

Proof: Similarly, the proof of Theorem 4.5. □

5. Mapping Structures in gFT_1-Spaces

In this section, we demonstrate that our concepts of generalised fuzzy T_1 spaces remain unchanged under generalised bijective fuzzy continuous and generalised fuzzy open mappings.

Theorem 5.1 *Assume that (X, ς) and $(Y, \acute{\varsigma})$ are two GFTSs, and let $f\colon X \longrightarrow Y$ be a bijective generalised fuzzy open map. Then*

1. *(X, ς) is eFT_1-space $\Rightarrow$ $(Y, \acute{\varsigma})$ is eFT_1-space;*
2. *(X, ς) is $eFT_1^{(i)}$-space $\Rightarrow$ $(Y, \acute{\varsigma})$ is $eFT_1^{(i)}$-space;*
3. *(X, ς) is $eFT_1^{(ii)}$-space $\Rightarrow$ $(Y, \acute{\varsigma})$ is $eFT_1^{(ii)}$-space;*
4. *(X, ς) is $eFT_1^{(iii)}$-space $\Rightarrow$ $(Y, \acute{\varsigma})$ is $eFT_1^{(iii)}$-space.*

Proof:

1. Consider (X, ς) as a eFT_1-space, and let $\acute{x}_\alpha$ and $\acute{y}_\beta$ be elements of $FS(Y)$ such that $\acute{x} \neq \acute{y}$. Since f is surjective, there exist elements $x, y \in X$ such that $f(x) = \acute{x}$ and $f(y) = \acute{y}$. Consequently, we have $x_\alpha, y_\beta \in FS(X)$ with $x \neq y$ due to the injectivity of f. Given that (X, ς) is a gFT_1-space, there exists $H \in N_\varsigma(x_\alpha)$ such that, $y_\beta \bar{q} H$ and there exists $W \in N_\varsigma(y_\beta)$ such that, $x_\alpha \bar{q} W$. Since, $H \in N_\varsigma(x_\alpha)$ implies that there exists $U \in \varsigma$ such that, $x_\alpha \in U \subseteq H$ and $y_\beta \bar{q} H \Rightarrow H(y) + \beta \leq 1$.

   Furthermore, we find that $f(H)(\acute{x}) = \sup\{H(x) : f(x) = \acute{x}\} = H(x)$, for some x, and similarly, $f(H)(\acute{y}) = H(y)$, for some y. Since both H and U belong to $GFO(X)$ and given that f is a generalised fuzzy open map, we conclude that $f(H)$ and $f(U)$ belong to $GFO(Y)$. Once more, if $x_\alpha \in U$, then $\alpha \leq U(x)$, which implies $\alpha \leq f(U)(\acute{x})$. Consequently, $\acute{x}_\alpha \in f(U)$. Given that $U \subseteq H$, it follows that $f(U) \subseteq f(H)$. Therefore, $\acute{x}_\alpha \in f(U) \subseteq f(H)$. Additionally, since $H(y) + \beta \leq 1$, we can deduce $f(H)(\acute{y}) + \beta \leq 1$, leading to $\acute{y}_\beta \bar{q} f(H)$. Thus, there exists $f(H) \in N_{\acute{\varsigma}}(\acute{x}_\alpha)$ and $\acute{y}_\beta \bar{q} f(H)$. Similarly, we can demonstrate that $f(W) \in N_{\acute{\varsigma}}(\acute{y}_\beta)$ and $\acute{x}_\alpha \bar{q} f(W)$. Hence, $(Y, \acute{\varsigma})$ is a eFT_1-space.

2. Consider (X, ς) as a $eFT_1^{(i)}$-space. Let $\acute{x}_\alpha$ and $\acute{y}_\beta$ be elements of $FS(Y)$ where $\acute{x} \neq \acute{y}$. Since f is surjective, there exist elements $x, y \in X$ such that $f(x) = \acute{x}$ and $f(y) = \acute{y}$. Consequently, we have $x_\alpha, y_\beta \in FS(X)$, and since f is injective, it follows that $x \neq y$.

   Given that (X, ς) is a $eFT_1^{(i)}$-space, there exist sets $H, W \in \varsigma$ such that $x_\alpha \in H$ and $y_\beta \notin H$, while $y_\beta \in W$ and $x_\alpha \notin W$. From $x_\alpha \in H$, we deduce $\alpha \leq H(x)$, and from $y_\beta \notin H$, we obtain $\beta \nleq H(y)$.

   It follows that $f(H)(\acute{x}) = \sup\{H(x) : f(x) = \acute{x}\} = H(x)$ for some x, and similarly, $f(H)(\acute{y}) = H(y)$ for some y. Since $H \in GFO(X)$ and f is a generalized fuzzy open map, we conclude that $f(H) \in GFO(Y)$.

   Thus, since $\alpha \leq H(x)$, we derive that $f(H)(\acute{x})$ implies $\acute{x}_\alpha \in f(H)$. Furthermore, since $\beta \nleq H(y)$, we find that $\beta \nleq f(H)(\acute{y})$, leading to $\acute{y}_\beta \notin f(H)$.

In a similar manner, we can show that $\acute{y}_\beta \in f(W)$ while $\acute{x}_\alpha \notin f(W)$. Consequently, we conclude that $(Y, \acute{\varsigma})$ is also a $eFT_1^{(i)}$-space.

3. Let (X, ς) be a space that satisfies the condition $eFT_1^{(ii)}$. Consider two elements $\acute{x}_\alpha$ and $\acute{y}_\beta$ in $FS(Y)$ such that $\acute{x} \neq \acute{y}$. Since the function f is surjective, there exist elements $x, y \in X$ such that $f(x) = \acute{x}$ and $f(y) = \acute{y}$. Furthermore, since f is injective, we have $x_\alpha, y_\beta \in FS(X)$ with $x \neq y$.

   Given that (X, ς) fulfills the $eFT_1^{(ii)}$ property, there exist sets H and W in ς such that $x_\alpha q H$ and $y_\beta \cap H = 0_X$, as well as $y_\beta q W$ and $x_\alpha \cap W = 0_X$. The condition $x_\alpha q H$ implies that $H(x) + \alpha > 1$, while $y_\beta \cap H = 0_X$ indicates that $H(y) = 0$.

   Since $f(H)(\acute{x}) = \sup\{H(x) : f(x) = \acute{x}\} = H(x)$ for some x, and similarly, $f(H)(\acute{y}) = H(y)$ for some y, we can ascertain that $H \in GFO(X)$, and that f is a generalized fuzzy open mapping. Therefore, it follows that $f(H) \in GFO(Y)$.

   Moreover, since $H(x) + \alpha > 1$, it leads to $f(H)(\acute{x}) + \alpha > 1$, which implies $\acute{x}_\alpha q f(H)$. Additionally, given that $H(y) = 0$, we conclude that $f(H)(\acute{y}) = 0$, resulting in $\acute{y}_\beta \cap f(H) = 0_X$.

   Similarly, it can be demonstrated that $\acute{y}_\beta q f(W)$ and $\acute{x}_\alpha \cap f(W) = 0_X$. Consequently, we establish that $(Y, \acute{\varsigma})$ also satisfies the condition $eFT_1^{(ii)}$.

4. Similarly, the proof of (3).

□

Theorem 5.2 . *Let (X, ς) and $(Y, \acute{\varsigma})$ be two GFTSs, and let $f : X \to Y$ be an injective and generalised fuzzy continuous map. Then*

1. *$(Y, \acute{\varsigma})$ is $eFT_1 \Rightarrow (X, \varsigma)$ is eFT_1;*

2. *$(Y, \acute{\varsigma})$ is $eFT_1^{(i)} \Rightarrow (X, \varsigma)$ is $eFT_1^{(i)}$;*

3. *$(Y, \acute{\varsigma})$ is $eF_1^{(ii)} \Rightarrow (X, \varsigma)$ is $eFT_1^{(ii)}$;*

4. *$(Y, \acute{\varsigma})$ is $eFT_1^{(iii)} \Rightarrow (X, \varsigma)$ is $eFT_1^{(iii)}$.*

Proof:

1. Let $(Y, \acute{\varsigma})$ be a eFT_1 and $x_\alpha, y_\beta \in FS(X)$ with $x \neq y$. Obviously, $(f(x))_\alpha$, $(f(y))_\beta \in FS(Y)$ with $f(x) \neq f(y)$ due to f being injective. Given that $(Y, \acute{\varsigma})$ is a eFT_1-space, there exists $\acute{H} \in N_{\acute{\varsigma}}((f(x))_\alpha)$ such that, $(f(y))_\beta \, \bar{q} \, \acute{H}$ and $\exists \ \acute{W} \in N_{\acute{\varsigma}}((f(y))_\beta)$ such that, $(f(x))_\alpha \, \bar{q} \, \acute{W}$.
   Now, $\acute{H} \in N_{\acute{\varsigma}}((f(x))_\alpha)$ implies $\exists \ \acute{U} \in \acute{\varsigma}$ such that, $(f(x))_\alpha \in \acute{U} \subseteq \acute{H}$. This implies that $f^{-1}((f(x))_\alpha) \in f^{-1}(\acute{U}) \subseteq f^{-1}(\acute{H})$. Therefore, $x_\alpha \in f^{-1}(\acute{U}) \subseteq f^{-1}(\acute{H})$ and $(f(y))_\beta \, \bar{q} \, \acute{H} \Rightarrow \acute{H}(f(y)) + \beta \leq 1 \Rightarrow f^{-1}(\acute{H})(y) + \beta \leq 1$. Therefore $y_\beta \, \bar{q} \, f^{-1}(\acute{H})$. Since f is a generalized fuzzy continuous map and $\acute{U}, \acute{H} \in GFO(Y)$, it follows that $f^{-1}(\acute{U}), f^{-1}(\acute{H}) \in GFO(X)$. Thus, there exists $f^{-1}(\acute{H}) \in N_\varsigma(x_\alpha)$ such that, $f^{-1}(\acute{H}) \, \bar{q} \, y_\beta$. Similarly, it can be shown that $\exists \ f^{-1}(\acute{W}) \in N_\varsigma(y_\beta)$ such that $f^{-1}(\acute{W}) \, \bar{q} \, x_\alpha$. Hence (X, ς) is a eFT_1.

2. Suppose $(Y, \acute{\varsigma})$ is a $eFT_1^{(i)}$ and let $x_\alpha, y_\beta \in FS(X)$ with $x \neq y$. Consequently, $(f(x))_\alpha$, $(f(y))_\beta \in FS(Y)$ and $f(x) \neq f(y)$ due to f being injective. Given that $(Y, \acute{\varsigma})$ is a $eFT_1^{(i)}$-space, then there exists $\acute{U}, \acute{V} \in \acute{\varsigma}$ such that $(f(x))_\alpha \in \acute{U}$, $(f(y))_\beta \notin \acute{U}$ and $(f(y))_\beta \in \acute{V}$, $(f(x))_\alpha \notin \acute{V}$. Now, $(f(x))_\alpha \in \acute{U} \Longrightarrow \alpha \leq \acute{U}(f(x)) \Longrightarrow \alpha \leq f^{-1}(\acute{U})(x) \Longrightarrow x_\alpha \in f^{-1}(\acute{U})$ and $(f(y))_\beta \notin \acute{U} \Longrightarrow \beta \nleq \acute{U}(f(y)) \Longrightarrow \beta \nleq f^{-1}(\acute{U})(y) \Longrightarrow y_\beta \notin f^{-1}(\acute{U})$. Since f is a generalized fuzzy continous map and $\acute{U} \in GFO(Y)$, it follows that $f^{-1}(\acute{U}) \in GFO(X)$. In the same way, we can show that there exists $f^{-1}(\acute{V}) \in GFO(X)$ such that $y_\beta \in f^{-1}(\acute{V})$ and $x_\alpha \notin f^{-1}(\acute{V})$. Therefore, (X, ς) is a $eFT_1^{(i)}$.

3. Let $(Y, \acute{\varsigma})$ be a $eFT_1^{(ii)}$-space and $x_\alpha, y_\beta \in FS(X)$ with different supports. Consequently, $(f(x))_\alpha$, $(f(y))_\beta \in FS(Y)$ with $f(x) \neq f(y)$ since f is injective. Since, $(Y, \acute{\varsigma})$ is a $gFT_1^{(ii)}$, there exists $\acute{H}, \acute{W} \in \acute{\varsigma}$ such that $(f(x))_\alpha q\acute{H}$, $(f(y))_\beta \cap \acute{H} = 0_X$ and $(f(y))_\beta q\acute{W}$, and $(f(x))_\alpha \cap \acute{W} = 0_X$. Now, $(f(x))_\alpha q\acute{H} \Rightarrow \acute{H}(f(x)) + \alpha > 1 \Rightarrow f^{-1}(\acute{H}(x)) + \alpha > 1 \Rightarrow (f^{-1}(\acute{H}))(x) + \alpha > 1 \Rightarrow x_\alpha q f^{-1}(\acute{H})$. Similarly, $(f(y))_\beta \cap \acute{H} = 0_X \Rightarrow \acute{H}(f(y)) = 0 \Rightarrow f^{-1}(\acute{H}(y)) = 0 \Rightarrow (f^{-1}(\acute{H}))(y) = 0 \Rightarrow y_\beta \cap f^{-1}(\acute{H}) = 0_X$.
   Since f is a generalised fuzzy continuous map and $\acute{H} \in \acute{\varsigma}$, it follows that $f^{-1}(\acute{H}) \in \varsigma$. Similarly, we can demonstrate that $y_\beta q f^{-1}(\acute{W})$ and $x_\alpha \cap f^{-1}(\acute{W}) = 0_X$. Therefore, (X, ς) is a $eFT_1^{(ii)}$-space.

4. In the same way, the proof of (3).

□

Theorem 5.3 *A GFTS (X, ς) is a eFT_1-space if and only if for all $x_\alpha, y_\beta \in FS(X)$ with $x \neq y$, there exists a generalised fuzzy continuous mapping f from X to a eFT_1-space $(Y, \acute{\varsigma})$ such that $f(x) \neq f(y)$.*

Proof: Necessity. Assume that (X, ς) is a eFT_1-space.
Let $(Y, \acute{\varsigma}) = (X, \varsigma)$ and f be the identity mapping id_X. It is clear that $(Y, \acute{\varsigma})$ and f have the required properties.
Sufficiency. Let x_α and y_β be fuzzy singletons in $FS(X)$ such that $x \neq y$. By the hypothesis, there is a generalised fuzzy continuous mapping $f\colon (X, \varsigma) \to (Y, \acute{\varsigma})$ with $f(x) \neq f(y)$. Since $(Y, \acute{\varsigma})$ is a eFT_1-space and $(f(x))_\alpha, (f(y))_\beta \in FS(Y)$ such that $f(x) \neq f(y)$, there is $H \in N_{\acute{\varsigma}}^Y((f(x))_\alpha)$ such that $(f(y))_\beta \bar{q} H$ and $W \in N_{\acute{\varsigma}}^Y((f(y))_\beta)$ such that $(f(x))_\alpha \bar{q} W$. By generalised fuzzy continuity of f, either $f^{-1}(H) \in N_\varsigma^X(x_\alpha)$ such that $y_\beta \bar{q} f^{-1}(H)$ and $f^{-1}(W) \in N_\varsigma^X(y_\beta)$ such that $x_\alpha \bar{q} f^{-1}(W)$ or $f^{-1}(H) \in N_\varsigma^X(y_\beta)$ such that $y_\beta \bar{q} f^{-1}(H)$ and $f^{-1}(W) \in N_\varsigma^X(x_\alpha)$ such that $x_\alpha \bar{q} f^{-1}(W)$. Therefore, (X, ς) is a eFT_1-space. □

Theorem 5.4 *A GFTS (X, ς) is a $eFT_1^{(i)}$-space if and only if for all $x_\alpha, y_\beta \in FS(X)$ with $x \neq y$, there exists a generalised fuzzy continuous mapping $f\colon X \to eFT_1^{(i)}(Y, \acute{\varsigma})$ such that $f(x) \neq f(y)$.*

Proof: Similarly, the proof of Theorem 5.3. □

Theorem 5.5 *A GFTS (X, ς) is a $eFT_1^{(ii)}$-space if and only if for all $x_\alpha, y_\beta \in FS(X)$ with $x \neq y$, there exists a generalised fuzzy continuous mapping f from X to a $eFT_1^{(ii)}$-space $(Y, \acute{\varsigma})$ such that $f(x) \neq f(y)$.*

Proof: Necessity. Assume that (X, Σ) is a $eFT_1^{(ii)}$-space. Consider $(Y, \acute{\Sigma}) = (X, \Sigma)$ with the identity mapping id_X. Clearly, $(Y, \acute{\Sigma})$ and f satisfy the requisite properties.
Sufficiency. Let x_α and y_β be fuzzy singletons in $FS(X)$ with $x \neq y$. Assume that there exists a generalised fuzzy continuous mapping f from (X, ς) to $gFT_1^{(ii)}(Y, \acute{\varsigma})$ such that $f(x) \neq f(y)$. Since $(Y, \acute{\varsigma})$ is $eFT_1^{(ii)}$ and $(f(x))_\alpha, (f(y))_\beta \in FS(Y)$ with $f(x) \neq f(y)$, there exists $H \in \acute{\varsigma}$ such that $(f(x))_\alpha q H$ and $(f(y))_\beta \cap H = 0_X$ or there exists $W \in \acute{\varsigma}$ such that $(f(y))_\beta q W$ and $(f(x))_\alpha \cap W = 0_X$. By the generalised fuzzy continuity of f, $f^{-1}(H) \in \varsigma$ such that $x_\alpha q f^{-1}(H)$ and $y_\beta \cap f^{-1}(H) = 0_X$ or $f^{-1}(W) \in \varsigma$ such that $y_\beta q f^{-1}(W)$ and $x_\alpha \cap f^{-1}(W) = 0_X$. Consequently, (X, ς) is a $eFT_1^{(ii)}$-space. □

Theorem 5.6 *A GFTS (X, ς) is a $eFT_1^{(iii)}$-space if and only if for all $x_\alpha, y_\beta \in FS(X)$ with $x \neq y$, there exists a generalised fuzzy continuous mapping f from X to a $eFT_1^{(iii)}$-space $(Y, \acute{\varsigma})$ such that $f(x) \neq f(y)$.*

Proof: Necessity. Assume (X, ς) is $eFT_1^{(iii)}$-space. Consider $(Y, \acute{\varsigma}) = (X, \varsigma)$ with the identity mapping id_X. Clearly, $(Y, \acute{\varsigma})$ and f have the required properties.
Sufficiency. Let x_α and y_β be fuzzy singletons in $FS(X)$ with $x \neq y$. Assume that there exists a

generalised fuzzy continuous mapping f from (X,ς) to $eFT_1^{(iii)}(Y,\acute{\varsigma})$ such that $f(x)\neq f(y)$. Since $(Y,\acute{\varsigma})$ is $eFT_1^{(iii)}$ and $(f(x))_\alpha,(f(y))_\beta\in FS(Y)$, we have $f(x)\neq f(y)$. Therefore, there exists $H\in\acute{\varsigma}$ such that $(f(x))_\alpha qH$ and $(f(y))_\beta\bar{q}H$. Similarly, there exists $W\in\acute{\varsigma}$ such that $(f(y))_\beta qW$ and $(f(x))_\alpha\bar{q}W$. Since f is generalised fuzzy continuous, $f^{-1}(H)\in g$ such that $x_\alpha qf^{-1}(H)$ and $y_\beta\bar{q}f^{-1}(H)$. Alternatively, $f^{-1}(W)\in g$ such that $y_\beta qf^{-1}(W)$ and $x_\alpha\bar{q}f^{-1}(W)$. Hence, (X,ς) is a $eFT_1^{(iii)}$-space. □

6. Generalized Lower Semi-Continuous Functions and the Initial & Final Fuzzy Topological Frameworks

This section provides a comprehensive analysis of a generalised lower semi-continuous function, encompassing both the initial and final generalised fuzzy topologies.

Theorem 6.1 *Consider (X,ς) be a GTS. The subsequent statements are equivalent:*

1. (X,ς) *is a eT_1-space;*
2. $(X,\omega(\varsigma))$ *is a eFT_1-space;*
3. $(X,\omega(\varsigma))$ *is a $eFT_1^{(i)}$-space;*
4. $(X,\omega(\varsigma))$ *is a $eFT_1^{(ii)}$-space;*
5. $(X,\omega(\varsigma))$ *is a $eFT_1^{(iii)}$-space.*

Proof:

$1\Leftrightarrow 2$. Necessity: Let (X,ς) be a eT_1 topological space. We will demonstrate that $(X,\omega(\varsigma))$ is a eFT_1 topological space. Assume that $x_\alpha,y_\beta\in FS(X)$ in which $x\neq y$. Since (X,ς) is a eT_1 topological space, there exists $H,V\in\varsigma$ such that $x\in H$, $y\notin H$ and $y\in V$, $x\notin V$. In accordance with the generalized lower semi-continuous function concept, $1_H,1_V\in\omega(\varsigma)$ and satisfies $1_H(x)=1$ and $1_H(y)=0$, as well as $1_V(y)=1$ and $1_V(x)=0$ Therefore:
• Since $1_H(x)=1$, we can concluded that $\alpha\leq 1_H(x)$, so $x_\alpha\in 1_H$.
• Since $1_H(y)=0$, we can concluded that $1_H(y)+\beta\leq 1$, so $y_\beta\bar{q}1_H$.
Therefore, $1_H\subseteq(y_\beta)^c$, so $1_H\in\omega(g)$ and $x_\alpha\in 1_H\subseteq(y_\beta)^c$. Similarly, $y_\beta\in 1_V\subseteq(x_\alpha)^c$. Hence, $(X,\omega(\varsigma))$ is a eFT_1-space.
Sufficiency: Since $(X,\omega(\varsigma))$ is a eFT_1-space, we must demonstrate that (X,ς) is a eT_1-space. Assume $x,y\in X$ with $x\neq y$. Since $(X,\omega(\varsigma))$ is a eFT_1-space, there exist $H,V\in\omega(\varsigma)$ such that $x_\alpha\in H\subseteq(y_\alpha)^c$ and $y_\alpha\in V\subseteq(x_\alpha)^c$. Without loss of generality, let $x_\alpha\in H\subseteq(y_\alpha)c$. Then $\alpha<H(x)\Rightarrow 1-\alpha=m<H(x)\Rightarrow x\in H{-}1(m,1]$ and $H\subseteq(y_\alpha)^c\Rightarrow H\bar{q}y_\alpha\Rightarrow H(y)+\alpha\leq 1\Rightarrow H(y)\leq 1-\alpha=m\Rightarrow y\notin H^{-1}(m,1]$. Similarly, $y\in V^{-1}(m,1]$ and $x\notin V^{-1}(m,1]$. Since $H^{-1}(m,1]$ and $V^{-1}(m,1]$ are generalised open sets, (X,ς) is a eT_1-space.

$1\Leftrightarrow 3$. Necessity: Let (X,ς) be a eT_1-space. We aim to demonstrate that $(X,\omega(\varsigma))$ is a $eFT_1^{(i)}$ space. Let $x_\alpha,y_\beta\in FS(X)$ with $x\neq y$. Since (X,ς) is eT_1, there exist $H,V\in\varsigma$ such that $x\in H$, $y\notin H$, and $y\in V$, $x\notin V$. By the definition of a generalised lower semi-continuous function, we know that $1_H,1_V\in\omega(\varsigma)$. Therefore, $1_H(x)=1$, $1_H(y)=0$, and $1_V(y)=1$, $1_V(x)=0$. Consequently, $(X,\omega(\varsigma))$ is a $eFT_1^{(i)}$ space.
Sufficiency: Given that $(X,\omega(\varsigma))$ is a $eFT_1^{(i)}$-space. We need to demonstrate that (X,ς) is a eT_1-space. Let $x,y\in X$ with $x\neq y$. Since $(X,\omega(\varsigma))$ is a $eFT_1(i)$-space, there exist $H,V\in\omega(\varsigma)$ such that $H(x)=1$, $H(y)=0$, $V(y)=1$, and $V(x)=0$. Suppose, without loss of generality, that $H(x)=1$ and $H(y)=0$. Then $H(x)+\alpha>1\Rightarrow H(x)>1-\alpha=m\Rightarrow x\in H^{-1}(m,1]$ and $H(y)+\alpha\leq 1\Rightarrow H(y)\leq 1-\alpha=m\Rightarrow y\notin H^{-1}(m,1]$. Similarly, $y\in V^{-1}(m,1]$ and $x\notin V^{-1}(m,1]$. Since $H^{-1}(m,1],V^{-1}(m,1]$ are generalised open sets, (X,ς) is a eT_1-space.

$1 \Leftrightarrow 4$. Necessity: Assuming (X, ς) is eT_1-space. We shall demonstrate that $(X, \omega(\varsigma))$ is $eFT_1^{(ii)}$-space. Assume $x_\alpha, y_\beta \in FS(X)$ where $x \neq y$. Given that (X, ς) is eT_1, $\exists H, V \in \varsigma$ such that $x \in H$, $y \notin H$ and $y \in V$, $x \notin V$. According to the concept of the generalized lower semi continuous function, we see that $1_H, 1_V \in \omega(\varsigma)$. Thus:

• Since $1_H(x) = 1$, it can be concluded that $1_H(x) + \alpha > 1$, so $x_\alpha q 1_H$.

• Since $1_H(y) = 0$, it can be concluded that $y_\beta \cap 1_H(y) = 0_X$. similarly, $y_\beta q 1_V$ and $x_\alpha \cap 1_V(y) = 0_X$. Therefore, $(X, \omega(\varsigma))$ is a $eFT_1^{(ii)}$-space.

Sufficiency: Let $(X, \omega(\varsigma))$ be a $eFT_1^{(ii)}$-space. We need to demonstrate that (X, ς) is a eT_1-space. Assume $x, y \in X$ where $x \neq y$. Given that $(X, \omega(\varsigma))$ is $eFT_1^{(ii)}$, for all $x_\alpha, y_\alpha \in FS(X)$, then there exists $H, V \in \omega(\varsigma)$ such that $x_\alpha q H$, $y_\alpha \cap H = 0_X$ and $y_\alpha q V$, $x_\alpha \cap V = 0_X$. Suppose, without loss of generality, there is $H \in \omega(\varsigma)$ such that $x_\alpha q H$ and $y_\alpha \cap H = 0_X$.

Now, $x_\alpha q H \Rightarrow H(x) + \alpha > 1 \Rightarrow H(x) > 1 - \alpha = m \Rightarrow x \in H^{-1}(m, 1]$ and $y_\alpha \cap H = 0_X \Rightarrow H(y) = 0 \Rightarrow H(y) + \alpha \leq 1 \Rightarrow H(y) < 1 - \alpha = m \Rightarrow y \notin H^{-1}(m, 1]$. Similarly, $y \in V^{-1}(m, 1]$ and $x \notin V^{-1}(m, 1]$. Additionally, $H^{-1}(m, 1]$ and $V^{-1}(r, 1]$ are generalized open sets. Hence (X, ς) is a gT_1-space.

$1 \Leftrightarrow 5$. Necessity: Let (X, ς) be a gT_1-space. We will demonstrate that $(X, \omega(\varsigma))$ is a $gFT_1^{(iii)}$-space. Let x_α and y_β be elements of FS(X) such that $x \neq y$. Given that (X, ς) is a gT_1, there exists $H, V \in \varsigma$ such that $x \in H$, $y \notin H$ and $y \in V$, $x \notin V$. According to the concept of the generalized lower semi continuous function, we get $1_H, 1_V \in \omega(\varsigma)$. Consequently:

• Since $1_H(x) = 1$, it can be concluded that $1_H(x) + \alpha > 1$, implying that $x_\alpha q 1_H$.

• Since $1_H(y) = 0$, it can be concluded that $y_\beta \cap 1_H(y) = 0_X$ implying that $y_\beta \bar{q} 1_H$. Similarly, $y_\beta q 1_V$ and $x_\alpha \bar{q} 1_V$. Therefore, $(X, \omega(\varsigma))$-space is a $eFT_1^{(iii)}$-space.

Sufficiency: Suppose $(X, \omega(\varsigma))$ is a $eFT_1^{(iii)}$-space. We must demonstrate that (X, ς) is a eT_1. Let $x, y \in X$ with $x \neq y$. Since $(X, \omega(\varsigma))$ is a $eFT_1^{(iii)}$, for all $x_\alpha, y_\alpha \in FS(X)$, there exists $H, V \in \omega(\varsigma)$ such that $x_\alpha q H$, $y_\alpha \bar{q} H$ and $y_\alpha q V$, $x_\alpha \bar{q} V$. Suppose, without loss of generality, there exists $H \in \omega(\varsigma)$ such that $x_\alpha q H$ and $y_\alpha \bar{q} H$.

Now, $x_\alpha q H \Rightarrow H(x) + \alpha > 1 \Rightarrow H(x) > 1 - \alpha = m \Rightarrow x \in H^{-1}(m, 1]$ and $y_\alpha \bar{q} H \Rightarrow H(y) + \alpha \leq 1 \Rightarrow H(y) < 1 - \alpha = m \Rightarrow y \notin H^{-1}(m, 1]$. Similarly, $y \in V^{-1}(m, 1]$ and $x \notin V^{-1}(m, 1]$. Additionally, $H^{-1}(m, 1]$ and $V^{-1}(r, 1]$ are generalized open sets. Therefore, (X, ς) is a eT_1-space.

□

Theorem 6.2 *If $\{(X_\ell, \varsigma_\ell)\}$ reflects a set of eFT_1-spaces and $\{f_\ell : X \to (X_\ell, \varsigma_\ell)\}$ indicates a set of injective and generalized fuzzy continuous functions, then the initial GFT induced by the set $\{f_\ell\}_{\ell \in J}$ is also a eFT_1-space.*

Proof: Assume ς is the initial GFT on X for the family $\{f_\ell\}_{\ell \in J}$. Consider $x_\alpha, y_\beta \in FS(X)$ where $x \neq y$. Since f_ℓ is one-to-one, $f_\ell(x)$ and $f_\ell(y)$ are distinct in X_ℓ. Given that (X_ℓ, ς_ℓ) is a eFT_1-space, for every different fuzzy singletons $(f_\ell(x))_\alpha$ and $(f_\ell(y))_\beta$, there exist $H_\ell, V_\ell \in \varsigma_\ell$ such that $(f_\ell(x))_\alpha \in H_\ell \subseteq ((f_\ell(y))_\beta)^c$ and $(f_\ell(y))_\beta \in V_k \subseteq ((f_\ell(x))_\alpha)^c$. For simplicity, assume there exists $H_\ell \in \varsigma_\ell$ such that $(f_\ell(x))_\alpha \in H_\ell \subseteq ((f_\ell(y))_\beta)^c$. Given that $(f_\ell(x))_\alpha \in H_\ell \Rightarrow \alpha \leq H_\ell(f_\ell(x)) \leq f_\ell^{-1}(H_\ell)(x)$, this condition holds for all $\ell \in J$, so $\alpha \leq \bigvee_{\ell \in J} f_\ell^{-1}(H_\ell)(x)$.

Also, $H_\ell \subseteq ((f_\ell(y))_\beta)^c \Rightarrow H_\ell \bar{q} (f_\ell(y))_\beta \Rightarrow H_\ell(f_\ell(y)) + \beta \leq 1 \Rightarrow f_\ell^{-1}(H_\ell)(y) + \beta \leq 1$ for every $\ell \in J$. Therefore, $\bigvee_{\ell \in J} f_\ell^{-1}(H_\ell)(y) + \beta \leq 1$.

Let $H = \bigvee_{\ell \in J} f_\ell^{-1}(H_\ell)$. Since f_ℓ is a generalised fuzzy continuous function, $H \in \varsigma$. Hence, $\alpha \leq H(x)$ and $H(y) + \beta \leq 1$. Therefore, $x_\alpha \in H$ and $H \bar{q} y_\beta \Rightarrow H \subseteq (y_\beta)^c$. Similarly, $y_\beta \in V \subseteq (x_\alpha)^c$. Hence, (X, ς) is a eFT_1-space. □

Theorem 6.3 *If $\{X_\ell, \varsigma_\ell\}$ is a collection of $eFT_1^{(i)}$-spaces and $\{f_\ell\colon X \to (X_\ell, \varsigma_\ell)\}$ is a collection of injective and generalised fuzzy continuous functions, then the initial GFT induced by $\{f_\ell\}_{\ell\in J}$ is also a $eFT_1^{(i)}$-space.*

Proof: Similarly, the proof of Theorem 6.2. □

Theorem 6.4 *If $\{X_\ell, \varsigma_\ell\}$ is a collection of $eFT_1^{(ii)}$-spaces and $\{f_\ell\colon X \to (X_\ell, \varsigma_\ell)\}$ is a collection of injective and generalised fuzzy continuous functions, then the initial GFT on X induced by $\{f_\ell\}_{\ell\in J}$ is also a $eFT_1^{(ii)}$-space.*

Proof: For the family $\{f_\ell\}_{\ell\in J}$, let ς represent the initial GFT on X. Let x_α and y_β in $FS(X)$ be considered, where $x \neq y$. Given that f_ℓ is a one-to-one function, $f_\ell(x)$ and $f_\ell(y)$ are separate components of X_ℓ instead. Given that (X_ℓ, ς_ℓ) is a $eFT_1^{(ii)}$-space, for every different fuzzy singletons $(f_\ell(x))_\alpha$ and $(f_\ell(y))_\beta$, there exists $H_\ell, V_\ell \in \varsigma_\ell$ such that $(f_\ell(x))_\alpha q H_\ell$, $(f_\ell(y))_\beta \cap H_\ell = 0_X$ and $(f_\ell(y))_\beta q V_\ell$, $(f_\ell(x))_\alpha \cap V_\ell = 0_X$. Assume there exists $H_\ell \in \varsigma_\ell$ such that $(f_\ell(x))_\alpha q H_\ell$ and $(f_\ell(y))_\beta \cap H_\ell = 0_X$.
Given that $(f_\ell(x))_\alpha q H_\ell \Rightarrow H_\ell(f_\ell(x)) + \alpha > 1 \Rightarrow f_\ell^{-1}(H_\ell)(x) + \alpha > 1$, this condition holds for all $\ell \in J$. Hence, $\bigvee_{\ell\in J} f_\ell^{-1}(H_\ell)(x) + \alpha > 1$. Also, $(f_\ell(y))_\beta \cap H_\ell = 0_X \Rightarrow H_\ell(f_\ell(y)) = 0_X \Rightarrow f_\ell^{-1}(H_\ell)(y) = 0_X \Rightarrow \bigvee_{\ell\in J} f_\ell^{-1}(H_\ell)(y) = 0_X$. Suppose $H = \bigvee_{\ell\in J} f_\ell^{-1}(H_\ell)$. Since f_ℓ is a generalised fuzzy continuous, $H \in \varsigma$. Therefore, $H(x) + \alpha > 1$ and $H(y) = 0_X$. Hence, $x_\alpha q H$ and $y_\beta \cap H = 0_X$. Similarly, $y_\beta q V$ and $x_\alpha \cap V = 0_X$. Hence, (X, ς) is a $eFT_1^{(ii)}$-space. □

Theorem 6.5 *If $\{X_\ell, \varsigma_\ell\}$ is a collection of $eFT_1^{(iii)}$-spaces and $\{f_\ell : X \to (X_\ell, \varsigma_\ell)\}$ is a collection of injective and generalised fuzzy continuous functions, then the initial GFT on X induced by $\{f_\ell\}_{\ell\in J}$ is also a $eFT_1^{(iii)}$-space.*

Proof: Similarly, the proof of Theorem 6.4. □

Theorem 6.6 *If (X_ℓ, ς_ℓ) is a collection of eFT_1-spaces and $\{f_\ell : (X_\ell, \varsigma_\ell) \to X\}$ is a set of bijective and generalised fuzzy open functions, then the final eFT_1-space corresponding to $\{f_\ell\}_{\ell\in J}$ will be a eFT_1-space.*

Proof: For the collection $\{f_\ell\}_{\ell\in J}$, let σ be the final GFT. Let $x_\alpha, y_\beta \in FS(X)$ be assumed, where $x \neq y$. For every ℓ, $f_\ell^{-1}(x)$ and $f_\ell^{-1}(y)$ are elements of X_ℓ. Therefore, since f_i is bijective, $f_\ell^{-1}(x) \neq f_\ell^{-1}(y)$. Given that (X_ℓ, σ_ℓ) is a eFT_1-space, then for all $(f_\ell^{-1}(x))_\alpha, (f_\ell^{-1}(y))_\beta \in FS(X_\ell)$ with $f_\ell^{-1}(x) \neq f_\ell^{-1}(y)$, there exists $H_\ell, V_\ell \in \sigma_\ell$ such that $(f_\ell^{-1}(x))_\alpha \in H_\ell \subseteq ((f_\ell^{-1}(y))_\beta)^c$ and $(f_\ell^{-1}(y))_\beta \in V_\ell \subseteq ((f_\ell^{-1}(x))_\alpha)^c$. It is assumed that, without sacrificing generality, there exists $H_\ell \in \sigma_\ell$ such that $(f_\ell^{-1}(x))_\alpha \in H_\ell \subseteq ((f_\ell^{-1}(y))_\beta)^c$. Consider the following: $(f_\ell^{-1}(x))_\alpha \in H_\ell \Rightarrow \alpha \leq H_\ell(f_\ell^{-1}(x)) \leq f_\ell(H_\ell)(x)$. Additionally, $H_\ell \subseteq ((f_\ell^{-1}(y))_\beta)^c \Rightarrow H_\ell \bar{q} (f_\ell^{-1}(y))_\beta \Rightarrow \beta + H_\ell(f_\ell^{-1}(y)) \leq 1 \Rightarrow \beta + f_\ell(H_\ell)(y) \leq 1$. This holds for all $\ell \in J$, implying that $\alpha \leq \bigvee_{\ell\in J} f_\ell(H_\ell)(x)$ and $\beta + \bigvee_{\ell\in J} f_\ell(H_\ell)(y) \leq 1$. Define $H = \bigvee_{\ell\in J} f_\ell(H_\ell)$. Since f_ℓ is a generalised fuzzy open function, $H \in \varsigma$. Consequently, $\alpha \leq H(x)$ and $\beta + H(y) \leq 1$. Hence, $x_\alpha \in H$ and $H \bar{q} y_\beta \Rightarrow H \subseteq (y_\beta)^c$. Similarly, $V \in \varsigma$ such that $y_\beta \in V \subseteq (x_\alpha)^c$. Therefore, (X, ς) is a eFT_1-space. □

Theorem 6.7 *If (X_ℓ, ς_ℓ) denotes a collection of $eFT_1^{(i)}$-spaces and $\{f_\ell : (X_\ell, \varsigma_\ell) \to X\}$ denotes a set of bijective and generalised fuzzy open functions, then the final GFT corresponding to $\{f_\ell\}_{\ell\in J}$ will be a $eFT_1^{(i)}$-space.*

Proof: In the same way, Theorem 6.6's proof. □

Theorem 6.8 *If (X_ℓ, ς_ℓ) is a collection of $eFT_1^{(ii)}$-spaces, and $\{f_\ell : (X_\ell, \varsigma_\ell) \to X\}$ is a set of bijective and generalised fuzzy open functions, then the final GFT corresponding to $\{f_\ell\}_{\ell \in J}$ will be a $eFT_1^{(ii)}$-space.*

Proof: Assume that ς is the collection's final GFT for $\{f_\ell\}_{\ell \in J}$. Let $x_\alpha, y_\beta \in FS(X)$ be distinct elements, and $x \neq y$. The elements of X_ℓ for each ℓ are $f_\ell^{-1}(x)$ and $f_\ell^{-1}(y)$. Since f_i is bijective, $f_\ell^{-1}(x) eq f_\ell^{-1}(y)$. Given that (X_ℓ, ς_ℓ) is a $eFT_1^{(ii)}$-space, then for all $(f_\ell^{-1}(x))_\alpha, (f_\ell^{-1}(y))_\beta \in FS(X_\ell)$ with $f_\ell^{-1}(x) \neq f_\ell^{-1}(y)$, there exist $H_\ell, V_\ell \in \varsigma_\ell$ such that $(f_\ell^{-1}(x))_\alpha q H_\ell$, $(f_\ell^{-1}(y))_\beta \cap H_\ell = 0_X$, and $(f_\ell^{-1}(y))_\beta q V_\ell$, $(f_\ell^{-1}(x))_\alpha \cap V_\ell = 0_X$. Without loss of generality, let $H_\ell \in \varsigma_\ell$ be such that $(f_\ell^{-1}(x))_\alpha q H_\ell$ and $(f_\ell^{-1}(y))_\beta \cap H_\ell = 0_X$.
Now, we have: $(f_\ell^{-1}(x))_\alpha q H_\ell \Rightarrow H_\ell(f_\ell^{-1}(x)) + \alpha > 1 \Rightarrow f_\ell(H_\ell)(x) + \alpha > 1 \Rightarrow \bigvee_{\ell \in J} (f_\ell(H_\ell))(x) + \alpha > 1$. Also, $(f_\ell^{-1}(y))_\beta \cap H_\ell = 0_X \Rightarrow H_\ell(f_\ell^{-1}(y)) = 0_X \Rightarrow f_\ell(H_\ell)(y) = 0_X \Rightarrow \bigvee_{\ell \in J} (f_\ell(H_\ell))(y) = 0_X$. Define $H = \bigvee_{\ell \in J} f_\ell(H_\ell)$. Since f_ℓ is a generalised fuzzy open function, $H \in \varsigma$. Therefore, $x_\alpha q H$ and $y_\beta \cap H = 0_X$. Similarly, $y_\beta q V$ and $x_\alpha \cap V = 0_X$. Hence, (X, ς) is a $eFT_1^{(ii)}$-space. □

Theorem 6.9 *If (X_ℓ, ς_ℓ) is a collection of $eFT_1^{(iii)}$-spaces and $\{f_\ell : (X_\ell, \varsigma_\ell) \to X\}$ is a set of bijective and generalised fuzzy open functions, then the final GFT corresponding to $\{f_\ell\}_{\ell \in J}$ will also be a $eFT_1^{(iii)}$-space.*

Proof: Similarly, the proof of Theorem 6.8. □

7. Concluding Remarks

In this work, we present new definitions of generalized fuzzy T_1 spaces and investigate their connections with other ideas in a methodical manner. We show that these spaces have important topological features, such as projective, productive, and hereditary properties. We also prove the preservation of these spaces by bijective generalized fuzzy continuous and generalized fuzzy open mappings. We then examine the behaviour of these spaces in the context of initial and final generalized fuzzy topological spaces to deepen our research. The study's conclusions offer a better comprehension of the structural characteristics of generalized fuzzy topological spaces and pave the way for further investigation into fuzzy topology and its various applications.

8. Declaration Statements

Availability of data and material: No data was used for the research described in the article.

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9. Author's Biography

Dr. Salahuddin, a professor in the Department of Mathematics at Jazan University (Jazan-45142, Saudi Arabia), specialises in variational inequalities. His research interests include inequalities, fixed point theory and topological structures, optimisation and optimal control theory, with a particular focus on inequality theory. Dr. Salahuddin has authored over 300 research papers in reputable journals and conferences, actively contributing to scholarly discourse in his field. He is passionate about exploring the broader impact of mathematical research on society and advancing knowledge through collaborative academic engagement.

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