A Trek on the Nonic Surface $x^2 - 4xy^4 = 2y^9 - y^8$

S. Devibala [1*], J. Shanthi [2], N. Thiruniraiselvi [3], M. A. Gopalan [4], SharadhaKumar [5]

ABSTRACT: The non-homogeneous polynomial equation of degree nine with two unknowns given by $x^2 - 4xy^4 = 2y^9 - y^8$ is studied to determine its distinct integer solutions. Some connections between the solutions are presented.

Key Words: Binary heptic equation, Non-homogeneous heptic equation, integer solutions.

Notations

$$t_{3,n} = \frac{n(n+1)}{2}$$
$$S_n = 6n(n+1) + 1$$
$$Th_n = 3 * 2^n - 1$$

Contents

1. Introduction

The theory of Diophantine equations is an ancient subject that typically involves solving, polynomial equation in two or more variables or a system of polynomial equations with the number of unknowns greater than the number of equations, in integers and occupies a pivotal role in the region of mathematics. The subject of Diophantine equations has fascinated and inspired both amateurs and mathematicians alike and so they merit special recognition. Solving higher degree Diophantine equations can be challenging as they involve finding integer solutions that satisfy the given polynomial equation. Learning about the various techniques to solve these higher power Diophantine equation in successfully deriving their solutions help us understand how numbers work and their significance in different areas of mathematics and science. For the sake of clear understanding by the readers, one may refer the varieties of Cubic, Quintic and Heptic Diophantine equations with multi variables [1-30] It seems that much work has not been done regarding polynomial Diophantine equations of degree nine with two unknowns. This paper aims at determining many integer solutions to non-homogeneous polynomial equation of degree nine with two unknowns given by $x^2 - 4xy^4 = 2y^9 - y^8$. A few relations between the solutions are presented.

---

2. Method of Analysis

The non-homogeneous nonic equation with two unknowns under consideration is

$$x^2 - 4xy^4 = 2y^9 - y^8 \tag{2.1}$$

Treating (2.1) as a quadratic in x and solving for the same, we have

$$x = y^4[2 \pm \sqrt{2y + 3}] \tag{2.2}$$

2.1. Method

Consider the positive sign before the square-root in (2.2).
After some calculations, it is seen that the square-root in (2.2) is removed when

$$y = y(k) = 2k^2 + 2k - 1 \tag{2.3}$$

and correspondingly we have from (2.2)

$$x = x(k) = (2k + 3)(y(k))^4 \tag{2.4}$$

Observe that (2.3) & (2.4) satisfy (2.1). A few numerical solutions to (2.1) are presented in Table 2.1 below:

Table 1: Numerical solutions

k	y = y(k)	x = x(k)
1	3	$5^*(3)^4$
2	11	$7^*(11)$
3	23	$9^*(23)^4$
4	39	$11^*(39)^4$
5	59	$13^*(59)^4$
6	83	$15^*(83)^4$

2.2. Relations observed

1. $y(k + 2) - 2y(k + 1) + y(k) = 2$

2. $3y(k) = [S_k - 4]$

3. $y(k) + 1 = 4t_{3,k}$

4. $[y(k)]^4[y(k + 1) - y(k)] + 2x(k) = 0$

5. $[y(k + 1) - y(k) + 1] = Th_n$ when $k = 3.2^{n-2}$

2.3. Note

Taking the negative sign before the square-root in (2.2) and repeating the process as above, the corresponding integer solutions to (2.1) are given below:

$$y = y(s) = (2k^2 + 2k - 1)$$
$$x = x(s) = (1 - 2k)(y(k))^4.$$

A few numerical solutions to (2.1) are presented in Table 2.2 below:

Table 2: Numerical solutions

k	$y = y(k)$	$x = x(k)$
1	3	$-1^*(3)^4$
2	11	$-3^*(11)^4$
3	23	$-5^*(23)^4$
4	39	$-7^*(39)^4$
5	59	$-9^*(59)^4$
6	83	$-11^*(83)^4$

2.4. Method

To remove the square-root in (2.2), let

$$\alpha^2 = 2y + 3 \tag{2.5}$$

which, after some algebra , is satisfied by

$$y_0 = \left(2k^2 + 2k - 1\right), \alpha_0 = 2k + 1 \tag{2.6}$$

Assume the second solution to (2.5) as

$$\alpha_1 = h - \alpha_0, y_1 = h + y_0 \tag{2.7}$$

where h is an unknown to be determined. Substituting (2.7) in (2.5) and simplifying, we have

$$h = 2\alpha_0 + 2$$

and in view of (2.7), it is seen that

$$\alpha_1 = \alpha_0 + 2 * 1, y_1 = y_0 + 2\alpha_0 + 2 * 1^2$$

Assume the third solution to (2.5) as

$$\alpha_2 = h - \alpha_1, y_2 = h + y_1 \tag{2.8}$$

where h is an unknown to be determined. Substituting (2.8) in (2.5) and simplifying, we have

$$h = 2\alpha_1 + 2$$

and in view of (2.8) , it is seen that

$$\alpha_2 = \alpha_0 + 2 * 2, y_2 = y_0 + 4\alpha_0 + 2 * 2^2$$

The repetition of the above process leads to the general solution to (2.5) as

$$\alpha_n = \alpha_0 + 2n = 2k + 1 + 2n$$

,

$$y_n = y_n(k) = y_0 + 2n\alpha_0 + 2n^2 = \left[2(k+n)^2 + 2k + 2n - 1\right] \tag{2.9}$$

Taking the positive sign before the square-root in (2.2), we get

$$x_n = x_n(k) = (y_n(k))^4 [5 + \alpha_n] \qquad\qquad = (2n + 2k + 3)(y_n(k)^4 \tag{2.10}$$

Thus, the integer solutions to (2.1) are represented by (2.9) & (2.10).

Some numerical solutions to $y_n(k), , x_n(k)$ are given in the Table-2.3 and Table-2.4 below:

Table 3:   Values of $y_n(k)$

k \ n	1	2	3	4	5
0	3	11	23	39	59
1	11	23	39	59	83
2	23	39	59	83	111

Table 4:   Values of $x_n(k)$

k \ n	1	2	3	4	5
0	$5^*(3)^4$	$7^*(11)^4$	$9^*(23)^4$	$11^*(39)^4$	$13^*(59)^4$
1	$7^*(11)^4$	$9^*(23)^4$	$11^*(39)^4$	$13^*(59)^4$	$15^*(83)^4$
2	$9^*(23)^4$	$11^*(39)^4$	$13^*(59)^4$	$15^*(83)^4$	$17^*(111)^4$

2.5. Relations observed

1. $\frac{x_n(k)}{(y_n(k))^4} = S_n$ when $k = 3n^2 + 2n - 1$

2. $\frac{x_n(k)}{(y_n(k))^4} - 4 == Th_n$ when $k = 3.2^{n-1} - n$

3. $6\left(\frac{x_n(k)}{(y_n(k))^4} - 2\right)$ represents area of Pythagorean triangle

4. $\frac{x_n(k)}{(y_n(k))^4} - 3 = 4t_{3,n}$ when $k = n^2$

2.6. Note

Taking the negative sign before the square-root in (2.2), we get

$$
\begin{aligned}
x_n &= (y_n(k))^4 \left[2 - \alpha_n\right] \\
&= (y_n(k))^4 \left[2 - \alpha_0 - 2n\right] \\
&= (y_n(k))^4 \left[1 - 2k - 2n\right]
\end{aligned}
\tag{2.11}
$$

Thus, the integer solutions to (2.1) are represented by (2.9) & (2.11)

3. Conclusion

The polynomial equation of degree nine with two unknowns given by $x^2 - 4xy^4 = 2y^9 - y^8$ has been studied to obtain non-zero integer solutions .The process of eliminating the square-root will be beneficial for the researchers. As nonic equations are plenty, one may attempt to determine the solutions in integers for other choices of nonic Diophantine equations.

Acknowledgments

References

1. M.A.Gopalan ,S.Vidhyalakshmi ,J.Shanthi , *On the Cubic Equation with Four Unknowns* $x^3+4z^3 = y^3+4w^3+6(x-y)^3$, International Journal of Mathematics Trends andTechnology,Vol 20(1),Pg 75-84,2015.

2. M.A.Gopalan,S.Vidhyalakshmi,J.Shanthi, *On ternary cubic Diophantine equation* $3\left(x^2 + y^2\right) - 5xy + x + y + 1 = 12z^3$, International Journal of Applied Research, vol1(8), 209-212,2015.

3. M.A.Gopalan ,S.Vidhyalakshmi,J.Shanthi , *On Cubic Equation with Four Unknowns* $x^3+y^3+2(x+y)(x+y+2) = 19zw^2$, International Journal for Mathematics , Vol 2(3), Pg 1-8,2016.

4. M.A.Gopalan,S.Vidhyalakshmi,J.Shanthi , *On The Non-homogeneous Cubic Equation with Five Unknowns* $9\left(x^3 - y^3\right) = z^3 - w^3 + 12p^2 + 16$, International Journal of Information Research and Review (IJIRR), Vol 3(6),Pg 2525-2528,2016.

5. M.A.Gopalan,J.Shanthi, *On The Non-homogeneous Cubic Equation with Five Unknowns* $(a + 1)^2\left(x^3 - y^3\right) = (2a + 1)\left(z^3 - w^3\right) + 6a^2p^2 + 2a^2$, International Journal of Modern Sciences and Engineering Technology (IJMSET), Vol 3(5),Pg 32-36,2016.

6. E.Premalatha, J.Shanthi, M.A.Gopalan, *On Non - Homogeneous Cubic Equation with Four Unknowns* $\left(x^2 + y^2\right) + 4\left(35z^2 - 4 - 35w^2\right) = 6xyz$, Vol.14, Issue 5, 126-129, March 2021.

7. J.Shanthi,M.A.Gopalan, *A search on Non -distinct Integer solutions to cubic Diophantine equation with four unknowns* $x^2 - xy + y^2 + 4w^2 = 8z^3$, International Research Journal of Education and Technology,(IRJEdT), Volume2,Issue01, 27-32, May 2021.

8. S.Vidhyalakshmi, J.Shanthi, M.A.Gopalan, *On Homogeneous Cubic equation with four Unknowns* $x^3 - y^3 = 4\left(w^3 - z^3\right) + 6(x - y)^3$, International Journalof Engineering Technology Research and Management, 5(7), 180-185, July 2021.

9. S.Vidhyalakshmi,J.Shanthi, M.A.Gopalan, T. Mahalakshmi, *On the non-homogeneous Ternary Cubic Diophantine equation* $w^2 - z^2 + 2wx - 2zx = x^3$, International Journal of Engineering Applied Science &Technology, Vol-7, Issue-3, 120-121, July-2022.

10. M.A. Gopalan, J. Shanthi, V.Anbuvalli, *Observation on The Paper Entitled Solutions of the Homogeneous Cubic Eqution with Six Unknowns* $\left(w^2 + p^2 - z^2\right)(w - p) = \left(k^2 + 2\right)(x + y)R^2$, International Journal of Research Publication & Reviews, Vol-4, Issue-2, 313-317, Feb-2023.

11. J.Shanthi, S.Vidhyalakshmi,M.A.Gopalan, *On Homogeneous Cubic Equation with Four Unknowns* $\left(x^3 + y^3\right) = 7zw^2$, Jananabha, Vol-53(1), 165-172, May-2023.

12. J. Shanthi, M.A. Gopalan, *Cubic Diophantine equation of the form* $Nxyz = w(xy + yz + zx)$, International Journal of Modernization in Engineering Tech & Science, Vol-5, Issue-9, 1462-1463, Sep-2023.

13. J. Shanthi, M.A. Gopalan, *A Search on Integral Solutions to the Non- Homogeneous Ternary Cubic Equation* $ax^2 + by^2 = (a + b)z^3, a, b > 0$, International Journal of Advanced Research in Science, Communication and Technology, Vol-4, Issue-1, 88-92, Nov-2024.

14. J.Shanthi,M.A.Gopalan, *On finding Integer Solutions to Binary Cubic Equation* $x^2 - xy = y^3$, International Journal of Multidisciplinary Research in Science, Engineering and Technology ,7(11), 16816-16820,2024.

15. J.Shanthi,M.A.Gopalan , *A Classification of Integer Solutions to Binary Cubic Equation* $x^2 - xy = 3\left(y^3 + y^2\right)$, International Journal of Progressive Research in Engineering Management and Science (IJPREMS) ,5(5),825-1828, 2025.

16. J.Shanthi,M.A.Gopalan, *Observations on Binary Cubic Equation* $x^2 - 3xy = 4\left(y^3 + y^2\right)$, IJARSCT, 5(1), 1-5, 2025.

17. J.Shanthi, M.A.Gopalan, *On Solving Binary Cubic Equation* $x^2 - 4xy = 5y^3 - 3y^2$, IRJEdT, 8(6), 139-144, 2025.

18. J.Shanthi, M.A.Gopalan, *On finding Integer Solutions to Binary Quintic Equation* $x^2 - x^2 = y^5$, International Research Journal of Education andTechnology, 6(10), 226231, 2024.

19. Sharadha Kumar, M.A.Gopalan, *On the Cubic Equation* $x^3 + y^3 + 6(x + y)z^2 = 4w^3$, JETIR, 6(1), 658-660, January 2019.]

20. M.A.Gopalan, Sharadha Kumar, *On the non-homogeneous ternary cubic equation* $3\left(x^2 + y^2\right) - 5xy + x + y + 1 = 111z^3$, International Journal of Engineering and Techniques, 4(5), 105-107, Sep-Oct (2018).

21. J.Shanthi, M.A.Gopalan, *On finding Integer Solutions to Binary Quintic Equation* $x^2 - xy^2 = y^4 + y^5$, International Journal of Advanced Research in Science, Engineering and Technology, 11(10), 22369-22372, 2024.

22. J.Shanthi, M.A.Gopalan, *Delineation of Integer Solutions to Non-Homogeneous QuinaryQuintic Diophantine Equation* $\left(x^3 - y^3\right) - \left(x^2 + y^2\right) + \left(z^3 - w^3\right) = 2 + 87\ T^5$, International Journal of Research Publication and Reviews, 4(9), 1454-1457, 2023.

23. J.Shanthi,M.A.Gopalan, *On the Ternary Non-homogeneous Quintic Equation* $x^2 + 5y^2 = 2z^5$, International Journal of Research Publication and Reviews, 4(8), 329332, 2023.

24. J.Shanthi, M.A.Gopalan, On the Non-homogeneous Quinary Quintic Equation $x^4 + y^4 - (x+y)w^3 = 14z^2T^3$, International Research Journal of Education and Technology, 05 (08),238-245, 2023.

25. N.Thiruniraiselvi, M.A.Gopalan, *The Non-Homogeneous Quintic Equation with Six Unknowns* $\left(x^4 - y^4\right) = 109(z + w)P^3Q$, International Journal for Research in Applied Science & Engineering Technology ,7(VI), 2527-2530, 2019.

26. A.Vijayasankar, Sharadha Kumar, M.A.Gopalan, *On The Non-Homogeneous Quintic Equation with Five Unknowns* $3(x+y)\left(x^3 - y^3\right) = 7\left(z^2 - w^2\right)p^3$, 10(8), 44-49,2020.

27. A.Vijayasankar, Sharadha Kumar, M.A.Gopalan, *On Non-Homogeneous Quinary Quintic Equation* $\left(\mathrm{x}^4 - \mathrm{y}^4\right) = 125\left(\mathrm{z}^2 - \mathrm{w}^2\right)\mathrm{p}^3$, South East Asian J. of Mathematics and Mathematical Sciences,18(1), 27-34, 2022.

28. N.Thiruniraiselvi, M.A.Gopalan, *A Search on Integer Solutions to Ternary Nonhomogeneous Nonic Diophantine Equation* $\alpha\left(\mathrm{x}^2 + \mathrm{y}^2\right) - (2\alpha - 1)\mathrm{xy} = 4\alpha\mathrm{z}^9$, Advances in Nonlinear Variational Inequalities, 27(2), 13-17, 2024.

29. M.A.Gopalan, Sharadha Kumar, *On Finding Integer Solutions on Binary Heptic Equation* $\mathrm{x}^2 - \mathrm{xy}^3 = 2\mathrm{y}^7$,IRJEdT,8(7),782-787,2025.

30. M.A.Gopalan,Sharadha Kumar, *Techniques on solving Binary Heptic Equation* $x^2 - x^3 = 4y^6 + 2y^7$,IJPREMS, 5(8), 91-95, 2025.

[1] *Associate Professor, Department of Mathematics,*

Sri Meenakshi Government Arts College for Women(A),

Affiliated to Madurai Kamaraj University,

Madurai-625 002, Tamil Nadu, India.

E-mail address: devibalasubburaman@gmail.com

and

[2,3,4] *Department of Mathematics,*

Shrimati Indira Gandhi College,

Affiliated to Bharathidasan University,

Tiruchirappalli-620002, Tamil Nadu, India.

E-mail address: shanthitharshi@gmail.com

E-mail address: mayilgopalan@gmail.com

E-mail address: drntsmaths@gmail.com

and

[5] *Department of Mathematics,*

National College (Autonomous),

Affiliated to Bharathidasan University,

Tiruchirappalli-620001, Tamil Nadu, India.

E-mail address: sharadhak12@gmail.com