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On Some Fixed Point Theorems in Quasi-b-Metric–like Spaces via Quasi Weak Contractions

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ABSTRACT: In this paper, we prove new fixed point results in the setting of quasi-b-metric–like spaces under quasi-weak contractions. The theoretical developments are supported with explicit examples, showing how the approach extends beyond classical metric cases. Moreover, the applicability of our theorems is emphasized through their use in guaranteeing the existence and uniqueness of solutions to certain integral equations.

Keywords: Quasi-b-metric like space, fixed point, non-linear contraction, comparison function.

Contents

1. Introduction

Fixed point theory is concerned with investigating when and under what circumstances there exists a point $x \in X$, known as the fixed point, such that $T(x) = x$, given a mapping $T : X \to X$. One of the most fundamental findings ever made in fixed point theory is known as the Banach Fixed Point theorem [1,2,3,4,5]. This theorem was proven by Stefan Banach in 1922, and it says that if there exists a contractive self-mapping on a complete metric space, it has a unique fixed point. This theorem is very valuable when it comes to proving either existence or uniqueness or both [6,7,8,9,10,11].

Over the years, the Banach contraction principle has led to the development of several generalizations achieved by weakening possibly the contractive condition or the definition of the involved space [12,13,14,15,16,17,18,19]. In this regard, Bakhtin [20] introduced the b-metric spaces in the year 1989 as a generalization of classical metric spaces. Later on, Shah and Hussain [21] introduced the idea of quasi b-metric-like spaces in the year 2012. The study of fixed point theory in quasi-b-metric spaces under quasi-weak contractions has emerged as an important direction in analysis, integrating classical ideas of metric spaces with generalized distance functions and contractive mappings. Jleli and Samet introduced quasi-b-metric spaces as a natural extension of metric spaces for broader mathematical applications [22]. Subsequently, Abbas and Nazir developed the concept of quasi-weak contractions, which has become essential for establishing fixed point theorems in such spaces [23]. Salahuddin and Imdad further examined the existence and uniqueness of fixed points under these contractions, underscoring their significance in theory and practice [24]. These advancements build upon the earlier work of Alber and Guerre-Delabrière (2007), whose introduction of weakly contractive mappings provided the foundation for the study of quasi-weak contraction conditions [25].

In 2013, Shatanawi extended the theory of quasi-weak contractions by exploring their applications to nonlinear analysis, where he demonstrated their effectiveness in proving fixed point theorems [26]. Later, in 2016, Pop et al. investigated generalized quasi-weak contraction conditions and their consequences for fixed point results in quasi-b-metric spaces, further enriching the theory [27]. Previously, Berinde and Mihail (2009) had made an important contribution by analyzing several aspects of fixed point theory under quasi-weak contractions, thereby deepening our understanding of the structure and properties of

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such spaces [28]. In 2015, Abdeljawad emphasized the practical value of these theoretical insights by applying fixed point theorems to areas such as dynamic programming and optimization [29].

In this paper, we present several fixed point theorems for the quasi-b-metric spaces under quasi-weak contractions with illustrative examples and a concrete application. For clarity, we start by recalling the definition of quasi-metric spaces and the one concerning quasi-b-metric–like spaces.

2. Preliminaries

In this section, we present the basic definitions, notations, and preliminary results that form the foundation for the subsequent discussions.

Definition 2.1 ([30]) *Let X be a non-empty set and let*

$$q : X \times X \to [0, \infty)$$

be a function satisfying:

(i) $q(u, v) = 0$ *if and only if* $u = v$,

(ii) $q(u, v) \leq q(u, w) + q(w, v)$ *for all* $u, v, w \in X$.

Then the function q is said to be a quasi-metric on X, and the pair (X, q) is called a quasi-metric space. Every metric space is naturally a quasi-metric space; however, a quasi-metric space need not be a metric space.

Definition 2.2 ([31]) *Let (X, q) be a quasi metric space and (u_k) be a sequence in X. The sequence (u_k) is called left-Cauchy if for all $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that*

$$q(u_n, u_m) < \varepsilon \quad \text{for all } n \geq m.$$

Definition 2.3 ([31]) *Let (X, q) be a quasi-metric space and (u_k) be a sequence in X. The sequence (u_k) is right-Cauchy if for every $\varepsilon > 0$ There exists a positive integer $N = N(\varepsilon)$ such that*

$$q(u_n, u_m) < \varepsilon \;\; \text{for all } m \geq n.$$

Definition 2.4 ([32]) *Suppose X is a nonempty set; let*

$$\rho : X \times X \to [0, \infty)$$

is a mapping. A function ρ is called a quasi-b-metric–like on X if there exists a constant $s \geq 1$ such that, for all $u, v, w \in X$, the following conditions are satisfied:

(i) $\rho(u, v) = 0$ *implies* $u = v$,

(ii) $\rho(u, w) \leq s\big(\rho(u, v) + \rho(v, w)\big)$.

The pair (X, ρ) is called quasi-b-metric–like space.

Example 2.1 *Let $X = \{a, b, c\}$ and define a function*

$$\delta : X \times X \to [0, \infty)$$

with constant $s = 10$ by

$$\delta(a, b) = 5, \quad \delta(b, c) = 3, \quad \delta(a, c) = 2, \delta(b, a) = 1, \quad \delta(c, b) = 4, \quad \delta(c, a) = 7.$$

Proof. *The first condition of a quasi-b-metric-like is clearly met since $\delta(u, u) = 0$ for each $u \in X$. Nevertheless, since*

$$d(a, b) = 5 \neq d(b, a) = 1$$

indicates that (X,δ) *is not a metric space, the symmetry condition is not met. The quasi-b-metric–like inequality is then confirmed. For instance,*

$$\delta(a,b) \leq s\big(\delta(a,c)+\delta(c,b)\big).$$

In fact,

$$5 \leq 10(2+4) = 60.$$

It is possible to verify similar inequalities for alternative point selections in (X). Therefore, (X,δ) *is a quasi-b-metric–like space.*

Definition 2.5 **([32])** *Let* (X,ρ) *be a quasi-b-metric-like space and let* (u_n) *be a sequence in* X*. We say that* (u_n) *converges to point* $u \in X$ *if and only if*

$$\lim_{n\to\infty} \rho(u_n,u) = \lim_{n\to\infty} \rho(u,u_n) = \rho(u,u).$$

Definition 2.6 **([32])** *Let* (X,ρ) *be a quasi b-metric like space,* (u_n) *a sequence. The sequence* (u_n) *is said to be a Cauchy sequence if the limits* $\lim_{n,m\to\infty} \rho(u_n,u_m)$ *and* $\lim_{n,m\to\infty} \rho(u_m,u_n)$ *exist and are finite.*

Definition 2.7 **([32])** *A quasi-b-metric like space* (X,ρ) *is said to be complete if for every Cauchy sequence* (u_n) *in* (X,ρ)*, there exist* $u \in X$ *such that*

$$\lim_{n\to\infty} \rho(u_n,u) = \lim_{n\to\infty} \rho(u,u_n) = \lim_{n,m\to\infty} \rho(u_n,u_m) = \lim_{n,m\to\infty} \rho(u_m,u_n).$$

Definition 2.8 **([30])** *Let* (X,ρ) *be a quasi-b-metric like space. We say that*
(1) (X,ρ) *is left-complete if and only if every left-Cauchy sequence in* X *is convergent;*
(2) (X,ρ) *is right-complete if and only if every right-Cauchy sequence in* X *is convergent;*
(3) (X,ρ) *is complete if and only if every Cauchy sequence in* X *is convergent.*

Remark 2.1 **([33])** *Note that a sequence* (x_n) *is Cauchy if and only if it is both left-Cauchy and right-Cauchy.*

The study of contractions and their generalizations represents one of the most active areas in fixed point theory. Among the leading contributors to this field are Berinde and Ćirić, whose work has played a foundational role in the development of generalized contraction mappings. Specifically, Berinde [34] introduced the notion of almost contractions and proved a number of interesting fixed point theorems for mappings known as Ćirić strong almost contractions. The precise definition is given as follows:

Definition 2.9 **([34])** *A single valued mapping* $f : X \times X$ *is called a Ciric strong almost contraction if there exist a constant* $\alpha \in [0,\infty)$ *and some* $L \geq 0$ *such that*

$$\rho(fu,fv) = \alpha M(u,v) + Ld(v,fu)$$

for all $u,v \in X$*, where*

$$M(u,v) = max\{\rho(u,v), d(u,fu), d(v,fv), \frac{d(u,fv)+d(v,fu)}{2}\}.$$

3. Main Results

In this section, we state and prove the main fixed point theorems for quasi-b-metric–like spaces under quasi-weak contraction mappings.

Definition 3.1 **([33])** *A mapping* $\psi : [0,\infty) \to [0,\infty)$ *is said to be a c-comparison function if it satisfies the following conditions.* (1) ψ *is monotone increasing,* (2) $\sum_{n=0}^{\infty} \psi^n(t) < \infty forallt \geq 0.$
It is clear that if f is a c-comparison function then $\psi(t) < t$ *for all* $t > 0$ *and* $\psi(0) = 0$.

Definition 3.2 ([33]) *Let (X,ρ) be a quasi-b-metric like space. A self map $T : X \times X$ is called a quasi (ψ, L)-weak contraction if there exist a c-comparison function ψ and some $L \geq 0$ such that for all $u, u \in X$,*

$$\rho(Tu, Tv) \leq \phi\rho(u, v) + L\min\{\rho_m(u, fv), \rho_m(v, fu), \rho_m(u, fu)\}. \tag{3.1}$$

Theorem 3.1 *Let $T : X \times X \to X$ be a self mapping such that T is a quasi $(\psi$,$L)$-weak contraction, and let (X,ρ) be a complete quasi-b-metric like space with coefficient $s \geq 1$. Then, T has a unique fixed point in X.*

Proof: Let $u_o \in X$ and define a sequence (u_n) in X inductively by taking $u_{n+1} = Tu_n$, $n \geq 0$. Note that if there exists $r \in \mathbf{N}$ such that $u_r = u_{r+1}$, then u_r is a fixed point of T. Let $u_n \neq u_{n+1}$, for all $n \in \mathbf{N}$. Note that $\rho_m(u_n, u_n) = 0$. Substitute $u = u_{n-1}$ and $v = u_n$ in the contraction condition 3.1, implies that

$$\begin{aligned} d(u_n, u_{n+1}) &= \rho(Tu_{n-1}, Tu_n) \\ &\leq \psi\rho(u_{n-1}, v_n) + L\min\{\rho_m(u_{n-1}, Tu_n), \rho_m(u_n, Tu_{n-1}), \rho_m(u_{n-1}, Tu_{n-1})\} \\ &\leq \psi\rho(u_{n-1}, u_n) + L\min\{\rho_m(u_{n-1}, u_{n+1}), \rho_m(u_n, u_n), \rho_m(u_{n-1}, u_n)\} \\ &= \psi\rho(u_{n-1}, u_n). \end{aligned}$$

since, ψ is an increasing, we have

$$\rho(u_n, u_{n+1}) \leq \psi\rho(u_{n-1}, u_n) \leq \psi^2\rho(u_{n-2}, u_{n-1}) \leq \cdots \leq \psi^n\rho(u_0, u_1).$$

Thus

$$\rho(u_n, u_{n+1}) \leq \psi^n\rho(u_0, u_1). \tag{3.2}$$

Similarly, we can deduce that

$$\rho(u_{n+1}, u_n) \leq \psi^n\rho(u_1, u_0) \tag{3.3}$$

We now prove that (u_n) is a Cauchy sequence. We want to prove that (u_n) is a left-Cauchy sequence and a right-Cauchy sequence. For $n, m \in \mathbf{N}$ satisfying $n > m$. Then, we have

$$\begin{aligned} \rho(u_n, u_m) \leq& s\big[\rho(u_n, u_{n-1}) + \rho(u_{n-1}, u_{n-2}) + \cdots + \rho(u_{m+1}, u_m)\big] \\ &= s\Big[\sum_{i=m}^{n-1}\rho(u_{i+1}, u_i)\Big] \\ &\leq s\Big[\sum_{i=m}^{n-1}\psi^{i+1}\rho(u_1, u_0)\Big]. \end{aligned}$$

Since ψ is a c-comparison function, then

$$s\Big[\sum_{i=m}^{n-1}\psi^{i+1}\rho(u_1, u_0)\Big]$$

is convergent. Thus for any $\epsilon > 0$, there is $N \in \mathbf{N}$ such that

$$s\Big[\sum_{i=m}^{\infty}\psi^{i+1}\rho(u_1, u_0)\Big] < \epsilon \quad \text{for all } m \geq N.$$

Hence, for $n > m \geq N$, we have

$$\rho(u_n, u_m) \leq s\sum_{i=m}^{n-1}\psi^{i+1}\rho(u_1, u_0) < s\sum_{i=m}^{\infty}\psi^{i+1}\rho(u_1, u_0) < \epsilon.$$

Therefore (u_n) is a left-Cauchy sequence. Similarly, we can show that (u_n) is a right-Cauchy sequence. Thus, the sequence (u_n) is a Cauchy sequence in the space (X, ρ). Since (X, ρ) is complete, there is $x \in X$ such that

$$\lim_{n\to\infty} \rho(u_n, x) = \lim_{n\to\infty} \rho(x, u_n) = \rho(v, v).$$

Now, by (3.1) we have;

$$\begin{aligned}\rho(u_n, Tx) &= \rho(Tu_{n-1}, Tx)\\ &\leq \psi\, \rho(u_{n-1}, x) + L \min\left\{\rho_m(u_{n-1}, Tx), \rho_m(x, u_n), \rho_m(u_{n-1}, u_n)\right\}\\ &< \rho(u_{n-1}, x) + L \min\left\{\rho_m(u_{n-1}, Tx), \rho_m(x, u_n), \rho_m(u_{n-1}, u_n)\right\}\end{aligned}$$

By taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \rho(u_n, Tx) = \rho(Tx, Tx).$$

Also, by (3.1) we have;

$$\begin{aligned}\rho(Tx, u_n) &= \rho(Tx, Tu_{n-1})\\ &\leq \psi\, \rho(x, u_{n-1}) + L \min\left\{\rho_m(x, u_n), \rho_m(u_{n-1}, Tx), \rho_m(x, Tx)\right\}\\ &< \rho(x, u_{n-1}) + L \min\left\{\rho_m(x, u_n), \rho_m(u_{n-1}, Tx), \rho_m(x, Tx)\right\}.\end{aligned}$$

By taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \rho(u_n, Tx) = 0.$$

Therefore (u_n) converges to Tx. By uniqueness of the limit, we have $Tx = x$. Finally, it remains to prove the uniqueness of the fixed point. Let's assume that there are two fixed points, x and y, in X, and that $Tx = x$ and $Ty = y$. Then, by contraction condition 3.1,

$$\begin{aligned}\rho(x, y) &= \rho(Tx, Ty)\\ &\leq \psi\rho(x, y) + L \min\{\rho_m(x, y), \rho_m(y, x), \rho_m(x, fx)\}\\ &\leq \psi\rho(x, y).\end{aligned}$$

If $\rho(x, y) > 0$,then $\rho(x, y) \leq \psi\rho(x, y) < \rho(x, y)$ a contradiction. Therefore, $\rho(x, y) = 0$ and so $x = y$ which proves the uniqueness. □

Example 3.1 *Let $X = \mathbb{R}$ equipped with the standard metric $\rho(u, v) = |u - v|$, and consider the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by $T(u) = \frac{u}{4}$..*

First, observe that for all $u, v \in \mathbb{R}$,

$$|Tu - Tv| = \left|\frac{u}{4} - \frac{v}{4}\right| = \frac{1}{4}|u - v| = \frac{1}{4}\rho(u, v),$$

showing that f is a quasi-(ψ, L)-weak contraction with $\psi = \frac{1}{4}$ and $L = 0$.

To determine fixed points, we solve $Tu = u$, which gives $\frac{u}{4} = u \implies u = 0$. Hence, 0 is a fixed point. If $u \neq 0$ were another fixed point, then $\frac{u}{4} = u$, leading to $-\frac{3}{4}u = 0$, a contradiction. Thus, the fixed point is unique.

Since $(\mathbb{R}, |\cdot|)$ is a complete metric space and T is continuous, it follows from the fixed point theorem that T admits the unique fixed point $u = 0$.

Definition 3.3 ([33]) Let (X, ρ) be a space that is quasi-b-metric. A self mapping $T : X \times X$ is referred to as a quasi (α, L)-weak contraction if there exists $\alpha \in [0, 1]$ and some $L \geq 0$ such that the function T meets the following condition for any $u, u \in X$;

$$\begin{aligned}\rho(Tu, Tv) \quad \leq \quad &\alpha \max\left\{\rho(u, v), \rho(u, Tu), \rho(v, Tv), \frac{\rho(u, Tv) + \rho(v, Tu)}{2}\right\} \qquad (3.4)\\ &+L \min\left\{\rho_m(u, Tv), \rho_m(v, Tu), \rho_m(u, Tu)\right\}.\end{aligned}$$

Theorem 3.2 *Let $T : X \times X \to X$ be a self-mapping such that T is a quasi (α, L)-weak contraction, and let (X, ρ) be a complete quasi-b-metric like space with coefficient $s \geq 1$. Then, T has a unique fixed point in X.*

Proof: Let $u_0 \in X$ and define a sequence (u_n) in X inductively by taking $u_{n+1} = Tu_n$, $n \geq 0$. If there exists $r \in \mathbf{N}$ such that $u_r = u_{r+1}$, then u_r is a fixed point of f. Let $u_n \neq u_{n+1}$ for all $n \in \mathbf{N}$. From (3.4), by taking $u = u_{n-1}$ and $v = u_n$ we have

$$\begin{aligned}
\rho(u_n, u_{n+1}) &= \rho(Tu_{n-1}, Tu_n) \\
&\leq \alpha \max\left\{\rho(u_{n-1}, u_n), \rho(u_n, u_{n+1}), \frac{\rho(u_{n-1}, u_{n+1}) + \rho(u_n, u_n)}{2}\right\} \\
&\quad + L \min\left\{\rho_m(u_{n-1}, u_{n+1}), \rho_m(u_n, u_n), \rho_m(u_{n-1}, u_n)\right\} \\
&= \alpha \max\left\{\rho(u_{n-1}, u_n), \rho(u_n, u_{n+1}), \frac{\rho(u_{n-1}, u_{n+1})}{2}\right\} \\
&\leq \alpha \max\left\{\rho(u_{n-1}, u_n), \rho(u_n, u_{n+1}), s\frac{\rho(u_{n-1}, u_n) + \rho(u_n, u_{n+1})}{2}\right\} \\
&= \alpha \max\left\{\rho(u_{n-1}, u_n), \rho(u_n, u_{n+1})\right\}.
\end{aligned}$$

If

$$max\{\rho(u_{n-1}, u_n), \rho(u_n, u_{n+1})\} = \rho(u_n, u_{n+1}),$$

then $\rho(u_n, u_{n+1}) \leq \alpha\rho(u_n, u_{n+1})$ and that is a contradiction. Therefore

$$max\{\rho(u_{n-1}, u_n), \rho(u_n, u_{n+1})\} = \rho(u_{n-1}, u_n)$$

So,

$$\rho(u_n, u_{n+1}) \leq \alpha\rho(u_{n-1}, u_n) \tag{3.5}$$

Similarly, by taking $u = u_n$ and $v = u_{n-1}$ in (3.4), we get

$$\begin{aligned}
\rho(u_{n+1}, u_n) &= \rho(Tu_n, Tu_{n-1}) \\
&\leq \alpha \max\left\{\rho(u_n, u_{n-1}), \rho(u_n, u_{n+1}), \rho(u_{n-1}, u_n), \frac{\rho(u_n, u_n) + \rho(u_{n-1}, u_{n+1})}{2}\right\} \\
&\quad + L \min\left\{\rho_m(u_n, u_n), \rho_m(u_{n-1}, u_{n+1}), \rho_m(u_n, u_{n+1})\right\} \\
&= \alpha \max\left\{\rho(u_n, u_{n-1}), \rho(u_{n-1}, u_n), \rho(u_n, u_{n+1}), \frac{\rho(u_{n-1}, u_{n+1})}{2}\right\} \\
&\leq \alpha \max\left\{\rho(u_n, u_{n-1}), \rho(u_{n-1}, u_n), \rho(u_n, u_{n+1}), s\frac{\rho(u_{n-1}, u_n) + \rho(u_n, u_{n+1})}{2}\right\} \\
&= \alpha \max\left\{\rho(u_n, u_{n-1}), \rho(u_{n-1}, u_n), \rho(u_n, u_{n+1})\right\}.
\end{aligned}$$

The contraction condition (3.5) implies that

$$\rho(u_{n+1}, u_n) \leq \alpha \max\{\rho(u_n, u_{n-1}), \rho(u_{n-1}, u_n)\} \tag{3.6}$$

Also from (3.5), we have

$$\begin{aligned}
\rho(u_n, u_{n+1}) &\leq \alpha\rho(u_{n-1}, u_n) \\
&\leq \alpha \max\{\rho(u_n, u_{n-1}), \rho(u_{n-1}, u_n)\}.
\end{aligned} \tag{3.7}$$

Thus, from (3.6) and (3.7), we have

$$\max\{\rho(u_n, u_{n+1}), \rho(u_{n+1}, u_n)\} \leq \alpha \max\{\rho(u_n, u_{n-1}), \rho(u_{n-1}, u_n)\}$$

By repeating the previous steps $(n-1)$ times, we get

$$\max\{\rho(u_n, u_{n+1}), \rho(u_{n+1}, u_n)\} \leq \alpha^n \max\{\rho(u_1, u_0), \rho(u_0, u_1)\}.$$

Let

$$\max\{\rho(u_1, u_0), \rho(u_0, u_1)\} = M.$$

Then

$$\max\{\rho(u_n, u_{n+1}), \rho(u_{n+1}, u_n)\} \leq \alpha^n M.$$

Hence, we have

$$\rho(u_n, u_{n+1}) \leq \alpha^n M \tag{3.8}$$

and

$$\rho(u_{n+1}, u_n) \leq \alpha^n M. \tag{3.9}$$

We now demonstrate that the sequence (u_n) is Cauchy, meaning that it is both left- and right-Cauchy. Let $n, m \in \mathbf{N}$ such that $n > m$. The triangle inequality and (3.9) are then used to obtain

$$\rho(u_n, u_m) \leq s\big[\rho(u_n, u_{n-1}) + \rho(u_{n-1}, u_{n-2}) + \cdots + \rho(u_{m+1}, u_m)\big]$$

$$= s\sum_{i=m}^{n-1} \rho(u_{i+1}, u_i) \leq s\sum_{i=m}^{n-1} \alpha^{i+1} M \leq s\sum_{i=m}^{\infty} \alpha^{i+1} M.$$

Since $\alpha < 1$, then for any $\epsilon > 0$ there is $N \in \mathbf{N}$ such that $s\frac{\alpha^{m+1}}{1-\alpha} < \frac{\epsilon}{M}$ forall $m \geq N$. Thus, for $n, m \in \mathbf{N}$ with $n > m \geq N$, we have

$$\rho(u_n, u_m) \leq (s\frac{\alpha^{m+1}}{1-\alpha})M < \frac{\epsilon M}{M} = \epsilon.$$

Therefore, the sequence (u_n) is left-Cauchy. Likewise, we may demonstrate that the sequence (u_n) is right-Cauchy. Consequently, in the space (X, ρ), the sequence (u_n) is a Cauchy sequence. Given the completeness of (X, ρ), $x \in X$ exits such that

$$\lim_{n\to\infty} d(u_n, x) = lim_{n\to\infty} d(x, u_n) = d(x, x).$$

Now, by (3.4) we have;

$$\begin{aligned}\rho(u_n, Tx) &= \rho(Tu_{n-1}, Tx)\\ &\leq \alpha \max\{\rho(u_{n-1}, x), \rho(u_{n-1}, u_n), \rho(x, Tx), \frac{\rho(u_{n-1}, Tx) + \rho(x, u_n)}{2}\}\\ &+ L\min\{\rho_m(u_{n-1}, Tx) + \rho_m(x, u_n) + \rho_m(u_{n-1}, u_n)\}.\end{aligned}$$

We obtain $\rho(x, Tx) \leq \alpha\rho(x, Tx)$ by assuming the limit as $n \to \infty$. Since $\alpha < 1$, $Tx = x$ and $\rho(x, Tx) = 0$. Lastly, we demonstrate the uniqueness of the fixed point x. To demonstrate the uniqueness, let's assume that there is another fixed point $y \in X$. Let $y \in X$ exist such that $Ty = y$. Next, using 3.4, we have

$$\begin{aligned}\rho(x, y) &= \rho(Tx, Ty)\\ &\leq \alpha \max\Big\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \frac{\rho(x, Ty) + \rho(y, Tx)}{2}\Big\}\\ &\quad + L \min\Big\{\rho_m(x, Ty), \rho_m(y, Tx), \rho_m(x, Tx)\Big\}\\ &\leq \alpha \max\Big\{\rho(x, y), \frac{\rho(x, Ty) + \rho(y, Tx)}{2}\Big\}.\end{aligned}$$

Also, by the same argument, we have

$$\rho(y,x) \le \alpha \max\Big\{\rho(x,y), \frac{\rho(x,Ty)+\rho(y,Tx)}{2}\Big\}.$$

thus,

$$\max\{\rho(x,y),\rho(y,x)\} \le \alpha \max\Big\{\rho(x,y),\rho(y,x), \frac{\rho(x,Ty)+\rho(y,Tx)}{2}\Big\}$$

$$= \alpha \max\{\rho(x,y),\rho(y,x)\}.$$

since $\alpha \le 1$, we get that

$$\rho(x,y) = \rho(y,x) = 0,$$

so, we have $x = y$.

Example 3.2 *Consider $X = [0,\infty)$ with the metric $\rho(u,v) = |u-v|$, and define the mapping*

$$T : X \to X, \quad T(u) = \frac{u}{8}.$$

Proof.

1. *Since*
$$|T(u) - T(v)| = \Big|\frac{u}{8} - \frac{v}{8}\Big| = \tfrac{1}{8}|u-v| = \tfrac{1}{8}\rho(u,v),$$
we see that f is a quasi-(α, L)-weak contraction with $\alpha = \frac{1}{8}$ and $L = 0$.
2. *A fixed point must satisfy $T(u) = u$, i.e.*
$$\tfrac{u}{8} = u \quad\Longrightarrow\quad u = 0.$$
Hence, 0 is a fixed point of T.
3. *If $u \ne 0$ is another fixed point, then $\frac{u}{8} = u$, which gives $u = 0$, a contradiction. Thus, the fixed point is unique.*
4. *Since $([0,\infty), |\cdot|)$ is a complete metric space and T is continuous, the fixed point theorem guarantees the existence and uniqueness of the fixed point $u = 0$.*

Therefore, $T(u) = \frac{u}{8}$ is a quasi-(α, L)-weak contraction with $\alpha = \frac{1}{8}$.

4. Application

Let us consider the integral equation

$$u(x) = g(x) + \lambda \int_a^b K(x,t)u(t)\,dt.$$

Define an operator T on a suitable function space X by

$$(Tu)(x) = g(x) + \lambda \int_a^b K(x,t)u(t)\,dt.$$

Our aim is to determine a function $u(x) \in X$ satisfying $u(x) = T(u)(x)$.

Take X as the Banach space of continuous functions on $[a,b]$ endowed with the supremum norm

$$\|u\| = \sup_{x\in[a,b]} |u(x)|.$$

This ensures that $(X, \|\cdot\|)$ is a complete metric space. For $u, v \in X$, define

$$\rho(u,v) = \|u - v\| = \sup_{x\in[a,b]} |u(x) - v(x)|.$$

Clearly, (X, ρ) constitutes a quasi-b-metric–like space.

To apply Theorem 3.1, we check whether T is a quasi-(ϕ, L)-weak contraction. We calculate

$$\rho(Tu, Tv) = \|Tu - Tv\| = \sup_{x\in[a,b]} \left|\lambda \int_a^b K(x,t)(u(t) - v(t))\, dt\right|.$$

Therefore,

$$\rho(Tu,Tv) \le \lambda \sup_{x\in[a,b]} \int_a^b |K(x,t)|\,|u(t)-v(t)|\,dt \le \lambda\Big(\sup_{x\in[a,b]} \int_a^b |K(x,t)|\,dt\Big)\|u-v\|.$$

Set

$$M = \sup_{x\in[a,b]} \int_a^b |K(x,t)|\,dt.$$

It follows that

$$\rho(Tu,Tv) \le \lambda M\|u-v\| = \phi\,\rho(u,v),$$

with $\phi = \lambda M$ and $L = 0$.

If $\lambda M < 1$, then $\phi < 1$, and T is a quasi-(ϕ, L)-weak contraction. By Theorem 3.1, there exists a unique $u \in X$ such that

$$u = T(u).$$

Consequently, the integral equation has a unique solution in X.

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