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A New Perspective for Investigation of Rough Upper and Lower Approximation Operations Based on Hypersemigroups and Hyperideals

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ABSTRACT: This paper focuses on integrating rough approximations into hypersemigroups. In this context, the definition and properties of rough approximations are given on a hypersemigroup. Later, the concepts of subhypersemigroups and hyperideals are explained and some important properties are proved. Finally, rough sets in a quotient hypersemigroup presented and their structural properties are investigated.

Key Words: Rough sets, approximation space, hypersemigroup, subhypersemigroup, rough hypersemigroup, rough hyperideal.

Contents

1. Introduction

Rough Sets were proposed in 1982 by Z Pawlak [14] and has served as a powerful mathematical tool for handling uncertain or imprecise data [15]. The study of algebraic properties of rough sets has been investigated by many researchers. For example, Bonikowaski [4] and Iwinski [7] have made important contributions to the fundamental work in this area. In 1994, Biswas and Nanda [1] defined the concepts of rough groups and rough subgroups, focusing only on upper approximations and excluding lower approximations. Miao et al. [13] refined the definitions of rough groups and rough subgroups and demonstrated novel properties related to these concepts. In [11], Bagırmaz and Ozcan have studied on rough semigroup. In [35], Gulay et al. examined Lee rough groups. Later, Bagırmaz et al. [2] introduced the notion of a topological rough group. On the other hand, Kuroki and Wang [8] studied the features of lower and upper approximations using normal subgroups, and many studies have been done on this topic since then [3,5,12,16,17,18]. Kuroki [9] also presented the concept of a rough ideal in semigroups. Davvaz [6] extended this idea to rough subrings according to an ideal in the rings.

Hyperstructures began to be studied after 1934, when the French mathematician Marty [10] introduced the concept of hypergroups. Hyperstructures are a generalization of classical algebraic structures, where the operation between elements is not necessarily defined as mapping a pair of elements to a single element, but rather to a set of elements. This concept extends traditional algebra by allowing more flexibility in the operations. This creates a more flexible and generalized framework for algebraic operations. Since then, many researchers have applied it to many classical topics in mathematics, primarily in algebra [19,21,24,20,22,23,25,26]. One of the important hyperstructures is the hypersemigroup. A hypersemigroup extends the concept of a semigroup by allowing the binary operation to result in sets of elements rather than just single elements, while retaining a form of associativity [27]. Hypersemigroups

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represent a highly active research area within hyperalgebraic structures, with numerous applications spanning automata, probability, geometry, topology, cryptography, coding theory, lattices, binary relations, graphs, hypergraphs, fuzzy and rough set theory, as well as various scientific disciplines including biology, chemistry, and physics [24].

The study of rough approximations in hyperstructures is interesting and open to development [31,32, 33]. In [28], Maryati and Davvaz study rough approximations in hypergraph. Hosseini and Jahanshahi investigated upper and lower approximations concerning a closed subgroup in a join space [29]. Leoreanu-Fotea studied the lower and upper approximations of a subset, with respect to an invertible subhypergroup and defined the notion of a rough subhypergroup [30]. After that, she proposed a methodology for calculating the lower and upper approximations of a subset in a hypergroup, with a particular focus on the role of the completeness operator [34].

The aim of this article is to provide a better understanding of hypersemigroups in the context of rough set theory and to present a framework focusing on the properties of rough approximations within a hypersemigroup. In adittion, some important properties of approximations and these algebraic structures are constructed

2. Preliminaries

This section will present some fundamental definitions and results pertaining to rough sets [14] and hypersemigroups [24].

Let U be a finite nonempty set called universe and R be an equivalence relation on U. The pair (U, R) is called an approximation space. We denote the equivalence class of object a in R by $[a]_R$.

The sets

$\bullet R^*(D) = \{a : [a]_R \cap D \neq \emptyset\}$,
$\bullet R_*(D) = \{a : [a]_R \subseteq D\}$,
$\bullet RBN(D) = R^*(D) - R_*(D)$,

are called upper approximation, lower approximation, and boundary region of D in (U, R), respectively. D is called a rough set in (U, R) iff $RBN(D) \neq \varnothing$. Conversely, if $R^*(D) = R_*(D)$, then D is called a definable set.

A hypersemigroup is an algebraic structure that generalizes the notion of a semigroup. In a semigroup, a binary operation combines any two elements to form a single element. However, in a hypersemigroup, this operation can result in a set of elements.

Let S be a non-empty set and $P^*(S)$ denote the family of nonempty subsets of S. Then, the mapping $\circ : S \times S \longrightarrow P^*(S)$ is said to be a hyperoperation. A hypersemigroup is a set S together with a hyperoperation "$\circ$" that satisfies:

(1) Closure: For all $a, b \in S$, $a \circ b \subseteq S$,
(2) Associativity: $(a \circ b) \circ c = a \circ (b \circ c)$ property holds in $P^*(S)$.

The associativity property can be formally expressed as:

$$\cup_{x \in a \circ b}(x \circ c) = \cup_{y \in b \circ c}(a \circ y).$$

Let S be a hypersemigroup and Ω be an equivalence relation on S. Then, the pair (S, Ω) is called a hyper approximation space. From now on, we shall write ab instead of $a \circ b$, $\forall\, a, b \in S$. If D and F are two subsets of S, we denote by DF the subset composed of all the elements of the form ab, where $a \in D$, $b \in F$.

A nonempty subset A of a hypersemigroup S is said to be a subhypersemigroup of S, if $DD \subseteq D$. A nonempty subset I of a hypersemigroup S is said to be a left (resp. right) hyperideal of S if $SI \subseteq I$ (resp. $IS \subseteq I$). A nonempty subset I of S is called a hyperideal (or a two-sided hyperideal) of S if I is both a left and a right hyperideal of S. A nonempty subset F of S is called a hypper bi-ideal of S if $FSF \subseteq F$.

A congruence Ω on a hypersemigroup S is an equivalence relation that is compatible with the hyperoperation on S. We will transfer the concept of congruence relation to hypersemigroup thanks to the following concepts:

Let S be a hypersemigroup and Ω be an equivalence relation on S. If D and F are non-empty subsets of S, then

- $D\overline{\Omega}F$ means that for all $a \in D$, there exists $b \in F$ such that $a\Omega b$ and for all $b \in F$, there exists $a \in D$ such that $a\Omega b$,
- $D\overline{\overline{\Omega}}F$ means that for all $a \in D$ and $b \in F$, we have $a\Omega b$.

Definition 2.1 *The equivalence relation Ω is called*

- *regular if for all a, a_1, b, b_1 of S, it follows that $a\Omega b$ and $a_1\Omega b_1$ imply $(aa_1)\overline{\Omega}(bb_1)$,*
- *strongly regular if for all a, a_1, b, b_1 of S, it follows that $a\Omega b$ and $a_1\Omega b_1$ imply $(aa_1)\overline{\overline{\Omega}}(bb_1)$.*

Let S be a hypersemigroup and Ω be an equivalence relation on S. We denote $[a]_{\Omega}$ the equivalence class of a. We set $S/\Omega = \{[a]_{\Omega} \mid a \in S\}$.

Definition 2.2 *The equivalence relation Ω is called complete strongly regular if for all $c \in ab \in S$, it follows that $[a]_{\Omega}\ [b]_{\Omega} = [c]_{\Omega}$.*

Proposition 2.1 *Let S be a hypersemigroup and Ω be an equivalent relation on S. Then, the following conditions are satisfied:*

(1) *If Ω is regular, then $(S/\Omega, \otimes)$ is a hypersemigroup with respect to hyperoperation $[a]_{\Omega} \otimes [b]_{\Omega} = \{[c]_{\Omega} | c \in ab\}$ on S/Ω.*

(2) *If Ω is strongly regular, then $(S/\Omega, \otimes)$ is a semigroup with respect to hyperoperation $[a]_{\Omega} \otimes [b]_{\Omega} = [c]_{\Omega}$, $\forall c \in ab$, on S/Ω.*

3. Rough approximations in hypersemigroups

In this section, the upper and lower approximations in hypersemigroups are introduced and some important features are studied.

Definition 3.1 *Given a hypersemigroup S and an equivalent relation Ω on S, the upper and lower approximations of a subset $D \subseteq S$ with respect to Ω are defined as follows:*

Upper Approximation $\Omega^{*}(D) = \{a : [a]_{\Omega} \cap D \neq \emptyset\}$,
Lower Approximation $\Omega_{*}(D) = \{a : [a]_{\Omega} \subseteq D\}$.

Example 3.1 Consider a hypersemigroup $S = \{1, 2, 3\}$ with a hyperoperation defined as follows:

$\circ$	1	2	3
1	$\{1,2\}$	$\{2\}$	$\{1,3\}$
2	$\{2\}$	$\{1\}$	$\{2,3\}$
3	$\{1,3\}$	$\{2,3\}$	$\{3\}$

An equivalence relation on S is defined by
$a\Omega b$ if and only if $a \circ \varrho = b \circ \varrho$ for all $\varrho \in S$.
So that it can be easily calculated that the equivalence classes under this relation are:
Class for 1, 2 and 3: $\{1, 2, 3\}$
Let $D = \{2, 3\}$, then $\Omega^{*}(D) = \{1, 2, 3\} = \Omega_{*}(D)$.

Remark 3.1 Throughout this section, we will assume that Ω is a strongly regular relation in the hyper approximation space (S, Ω).

Example 3.2 Consider a hypersemigroup $S = \{1, 2, 3, 4, 5\}$ with a hyperoperation defined by $a \circ b = \{a, b\}$ for all $a, b \in S$. Let us define the equivalence relation Ω based on the parity (evenness or oddness) of the numbers:
$a\Omega b$ if and only if a and b have the same parity (i.e., both are either even or odd)
Then, the equivalence classes for the elements based on their parity:
Odd numbers: $[1] = [3] = [5] = \{1, 3, 5\}$
Even numbers: $[2] = [4] = \{2, 4\}$
Let $D = \{1, 4, 5\}$, then $\Omega^{*}(D) = \{1, 2, 3, 4, 5\}$ and $\Omega_{*}(D) = \emptyset$.

We can easily derive the following proposition, which includes some basic properties, from Pawlak's [14] Proposition 2.2 and Proposition 2.1 in [9]. For completeness, we will give evidence to a few items.

Proposition 3.1 *Let (S,Ω) be a hyper approximation space. If D and F be nonempty subsets of S, then*

(1) $\Omega_*(D) \subseteq D \subseteq \Omega^*(D)$,
(2) $D \subseteq F$ *implies* $\Omega_*(D) \subseteq \Omega_*(F)$,
(3) $D \subseteq F$ *implies* $\Omega^*(D) \subseteq \Omega^*(F)$,
(4) $\Omega_*(D \cup F) \supseteq \Omega_*(D) \cup \Omega_*(F)$,
(5) $\Omega^*(D \cup F) = \Omega^*(D) \cup \Omega^*(F)$,
(6) $\Omega_*(D \cap F) = \Omega_*(D) \cap \Omega_*(F)$,
(7) $\Omega^*(D \cap F) \subseteq \Omega^*(D) \cap \Omega^*(F)$.

Proof: (1) Let $c \in \Omega_*(D)$. Then $c \in [c]_\Omega \subseteq D$ and so $\Omega_*(D) \subseteq D$. Thus $c \in D$ and $c \in [c]_\Omega \cap D$. Then $c \in \Omega^*(D)$. Hence $\Omega_*(D) \subseteq D \subseteq \Omega^*(D)$. □

(2) Let $D \subseteq F$ and $c \in \Omega_*(D)$. Then $c \in [c]_\Omega \subseteq D$ and so $c \in [c]_\Omega \subseteq B$. Hence $c \in \theta_*(F)$ and so $\Omega_*(D) \subseteq \Omega_*(F)$.

(3) Let $D \subseteq F$ and $c \in \Omega^*(D)$. Then $c \in [c]_\Omega \cap D \neq \emptyset$. Since $D \subseteq F$, we have $c \in [c]_\Omega \cap F \neq \emptyset$. Thus $c \in \theta^*(F)$. Hence $\Omega^*(D) \subseteq \Omega^*(F)$.

(4) By (2) we have $\Omega_*(D) \subseteq \Omega_*(D \cup F)$ and $\Omega_*(F) \subseteq \Omega_*(D \cup F)$.Thus $\Omega_*(D) \cup \Omega_*(F) \subseteq \Omega_*(D \cup F)$.

(5) Let $c \in \Omega^*(D \cup F)$. Then $c \in [c]_\Omega \cap (D \cup F)$ and so $c \in [c]_\Omega \cap D$ or $c \in [c]_\Omega \cap F$. Thus $c \in \Omega^*(D)$ or $c \in \Omega^*(F)$. Therefore $c \in \Omega^*(D) \cup \Omega^*(F)$.

On the other hand, if $c \in \Omega^*(D) \cup \Omega^*(F)$ then $c \in \Omega^*(D)$ or $c \in \Omega^*(F)$. Thus $c \in [c]_\Omega \cap D$ or $c \in [c]_\Omega \cap B$ and so $c \in [c]_\Omega \cap (D \cup F)$. Therefore $c \in \Omega^*(D \cup F)$. Hence $\Omega^*(D \cup F) = \Omega^*(D) \cup \Omega^*(F)$.

Proposition 3.2 *Let (S,Ω) be a hyper approximation space. If D and F be nonempty subsets of S, then*

$$\Omega^*(D)\,\Omega^*(F) \subseteq \Omega^*(DF).$$

Proof: *Let $c \in \Omega^*(D)\,\Omega^*(F)$. Then, there exist $c_1 \in \Omega^*(D)$ and $c_2 \in \Omega^*(F)$ such that $c \in c_1c_2$. Hence, there exist elements $a \in [c_1]_\Omega \cap D$ and $b \in [c_2]_\Omega \cap F$. Thus $a \in [c_1]_\Omega$, $a \in D$, $b \in [c_2]_\Omega$ and $b \in F$. Since Ω strongly regular relation, we have $a\Omega c_1$, $b\Omega c_2$ and $ab\overline{\overline{\Omega}}c_1c_2$. Thus, there exists $t \in ab$ such that $t \in [c]_\Omega$. So $t \in [c]_\Omega \cap DF$ and we have $c \in \Omega^*(DF)$. Hence $\Omega_*(D)\,\Omega_*(F) \subseteq \Omega_*(DF)$.* □

Proposition 3.3 *Let (S,Ω) be a hyper approximation space and Ω be a complete strongly regular relation. If D and F be nonempty subsets of S, then*

$$\Omega_*(D)\,\Omega_*(F) \subseteq \Omega_*(DF).$$

Proof: *Let $c \in \Omega_*(D)\,\Omega_*(F)$. Then, there exist $c_1 \in \Omega_*(D)$ and $c_2 \in \Omega_*(F)$ such that $c \in c_1c_2$. Thus, $[c_1]_\Omega \subseteq D$ and $[c_2]_\Omega \subseteq F$. Since Ω is a complete strongly regular relation, for all $c \in c_1c_2$ we have $[c_1]_\Omega\,[c_2]_\Omega = [c]_\Omega \subseteq DF$ and so $c \in \Omega_*(D)\,\Omega_*(F)$. Hence $\Omega_*(D)\,\Omega_*(F) \subseteq \Omega_*(DF)$* □

3.1. Rough hyperideals

In this section, the concept of rough hyperideals is presented and some important characterisations are obtained.

Definition 3.2 *Let (S,Ω) be a hyper approximation space. Then a nonempty subset D of S is called an upper (a lower) rough subhypersemigroup of S if $\Omega^*(D)$ (respectively, $\Omega_*(D)$) is a subhypersemigroup of S. And I is called an upper rough (a lower) left (right, two-sided) hyperideal of S if $\Omega^*(D)$ (respectively, $\Omega_*(D)$) is a left (right, two-sided) hyperideal of S.*

Proposition 3.4 *Let (S,Ω) be a hyper approximation space. Then,*

(1) *If D is a subhypersemigroup of S, then $\Omega^{*}(D)$ is a subhypersemigroup of S,*

(2) *If I is a left (right, two-sided) hyperideal of S, then $\Omega^{*}(I)$ is a left (right, two-sided) hyperideal of S.*

Proof: (1) *Let D be a subhypersemigroup of S. By Proposition 3.1 (1), we have that $A\subseteq\Omega^{*}(D)$. By Proposition 3.2 , we have that $\Omega^{*}(D)\Omega^{*}(D)\subseteq\Omega^{*}(DD)$. Since D is a subhypersemigroup, we have that $DD\subseteq D$ and from Proposition 3.1 (5), we have that $\Omega^{*}(DD)\subseteq\Omega^{*}(D)$. Hence, $\Omega^{*}(D)\Omega^{*}(D)\subseteq\Omega^{*}(D)$. This means that $\Omega^{*}(D)$ is a subhypersemigroup of S.*

(2) *Let I be a left hyperideal of S. We will show that $S\Omega^{*}(I)\subseteq\Omega^{*}(I)$. Since $\Omega^{*}(S)=S$ and by Proposition 3.2 we have that $S\Omega^{*}(I)=\Omega^{*}(S)\Omega^{*}(I)\subseteq\Omega^{*}(SI)$. On the other hand, since $SI\subseteq I$ and $\Omega^{*}(SI)\subseteq\Omega^{*}(I)$ from Proposition 3.1 (5), we obtain $S\Omega^{*}(I)\subseteq\Omega^{*}(I)$. Other cases are easily done in a similar manner.* □

Proposition 3.5 *Let (S,Ω) be a hyper approximation space and Ω be a complete strongly regular relation. Then,*

(1) *If D is a subhypersemigroup of S, then $\varnothing\neq\Omega_{*}(D)$ is a subhypersemigroup of S,*

(2) *If I is a left (right, two-sided) hyperideal of S, then $\varnothing\neq\Omega_{*}(I)$ is a left (right, two-sided) hyperideal of S.*

Proof: (1) *Let D be a subhypersemigroup of S and $\varnothing\neq\Omega_{*}(D)$. It is obvious that $DD\subseteq D$. Then, we have that $\Omega_{*}(D)\Omega_{*}(D)\subseteq\Omega_{*}(DD)\subseteq\Omega_{*}(D)$ from Proposition 3.3 and Proposition 3.1 (4).*

(2) *Let I be a left hyperideal of S and $\varnothing\neq\Omega_{*}(I)$. We will show that $S\Omega_{*}(I)\subseteq\Omega_{*}(I)$. It is obvious that $\Omega_{*}(S)=S$. Then, we have that $S\Omega_{*}(I)\subseteq\Omega_{*}(S)\Omega_{*}(I)\subseteq\Omega_{*}(SI)\subseteq\Omega_{*}(I)$ from Proposition 3.3 and Proposition 3.1 (4). Hence $\Omega_{*}(I)$ is a left hyperideal of S. Other cases are easily done in a similar manner.* □

Proposition 3.6 *Let (S,Ω) be a hyper approximation space. If F is a hyper bi-ideal of S, then $\Omega^{*}(F)$ is a hyper bi-ideal of S.*

Proof: Let F be a hyper bi-ideal of S, i.e., $FSF\subseteq F$. We will show that $\Omega^{*}(F)S\Omega^{*}(F)\subseteq\Omega^{*}(F)$. By Proposition 3.1 (1) and by Proposition 3.2, we have that $F\subseteq\Omega^{*}(F)$ and $\Omega^{*}(F)S\Omega^{*}(F)\subseteq\Omega^{*}(FS)\Omega^{*}(F)\subseteq\Omega^{*}(FSF)$. So, we obtain $\Omega^{*}(FSF)\subseteq\Omega^{*}(F)$ from Proposition 3.1 (5). Hence $\Omega^{*}(F)$ is a hyper bi-ideal of S. □

Proposition 3.7 *Let (S,Ω) be a hyper approximation space and Ω be a complete strongly regular relation. If F is a hyper bi-ideal of S, then $\varnothing\neq\Omega_{*}(F)$ is a hyper bi-ideal of S.*

Proof: *Let F be a hyper bi-ideal of S. We will show that $\Omega_{*}(F)S\Omega_{*}(F)\subseteq\Omega_{*}(F)$. Thus, it follows from Proposition 3.1. (1),(4) and Proposition 3.3 that $\Omega_{*}(F)S\Omega_{*}(F)\subseteq\Omega_{*}(F)\Omega_{*}(S)\Omega_{*}(F)\subseteq\Omega_{*}(FSF)\subseteq\Omega_{*}(F)$. Hence, $\varnothing\neq\Omega_{*}(F)$ is a hyper bi-ideal of S.* □

Proposition 3.8 *Let (S,Ω) be a hyper approximation space. Let R be a right hyperideal of S and L be a left hyperideal of S. Then*

$$\Omega^{*}(RL)\subseteq\Omega^{*}(R)\cap\Omega^{*}(L).$$

Proof: *Let R be a right hyperideal of S and L be a left hyperideal of S, then $RL\subseteq RS\subseteq R$ and $RL\subseteq SL\subseteq B$. Thus $RL\subseteq R\cap L$. Thus, it follows from Proposition 3.1. (5) and (7) that*

$$\Omega^{*}(RL)\subseteq\Omega^{*}(R\cap L)\subseteq\Omega^{*}(R)\cap\Omega^{*}(L).$$

□

Proposition 3.9 *Let (S,Ω) be a hyper approximation space. Let R be a right hyperideal of S and L be a left hyperideal of S.Then*

$$\Omega_*(RL) \subseteq \Omega_*(R) \cap \Omega_*(L).$$

Proof: *By following Propositions 3.1 (4) and (6), it is similar to the above proposition.* □

4. Rough sets in a quotient hypersemigroup

In this last section, rough sets in a quotient hypersemigroup are studied in detail together with their structural properties.

Definition 4.1 *Let S be a hypersemigroup and Ω be an equivalent relation on S. Then, we can present the upper and lower approximations of the subset $A \subseteq S$ with respect to Ω on S/Ω as follows:*

$$\Omega^*(A) = \{[a]_\Omega \in S/\Omega : [a]_\Omega \cap A \neq \emptyset\}, \quad \Omega_*(A) = \{[a]_\Omega \in S/\Omega : [a]_\Omega \subseteq A\}.$$

Example 4.1 Consider a hypersemigroup $Z_3 = \{[\overline{0}], [\overline{1}], [\overline{2}]\}$, which is the set of integers modulo 3, with a hyperoperation defined by $a \circ b = \{\overline{a+b}\}$ for all $a, b \in Z_3$. Let us define the equivalence relation Ω: $a\Omega b$ if and only if $a \circ \omega = b \circ \omega$ for all $\omega \in Z_3$
. Then, the equivalence classes for the elements :
$[\overline{0}] = [\overline{1}] = [\overline{2}] = \{[\overline{0}], [\overline{1}], [\overline{2}]\}$
Let $D = \left\{[\overline{0}], [\overline{2}]\right\}$, then $\Omega^*(D) = \{[\overline{0}], [\overline{1}], [\overline{2}]\}$ and $\Omega_*(D) = \emptyset$.

Proposition 4.1 *Let (S,Ω) be a hyper approximation space. If D is a subhypersemigroup of S, then $\Omega^*(D)$ is a subhypersemigroup of S/Ω.*

Proof: *Let $[c_1]_\Omega, [c_2]_\Omega \in \Omega^*(D)$. It is obvious that $[c_1]_\Omega, [c_2]_\Omega \in S/\Omega$ and by Proposition 2.1, we have that $[c_1]_\Omega \otimes [c_2]_\Omega = [c]_\Omega$, $\forall c \in c_1c_2$. Then $[c_1]_\Omega \cap D \neq \emptyset$ and $[c_2]_\Omega \cap D \neq \emptyset$. Thus there exist elements $a, b \in S$ such that $a \in [c_1]_\Omega \cap D$ and $b \in [c_2]_\Omega \cap D$. Then $a \in [c_1]_\Omega, a \in D, b \in [c_2]_\Omega$, and $b \in D$. Because D is a subhypersemigroup of S, $ab \subseteq DD \subseteq D$. Since Ω is a strongly regular relation, we have $a\Omega c_1$, $b\Omega c_2$ and $ab\overline{\overline{\Omega}}c_1c_2$. Thus, there exist $t \in ab \subseteq D$ such that $t \in [c]_\Omega$. So $t \in [c]_\Omega \cap D$ and we have $[c]_\Omega \in \Omega^*(D)$. Hence $\Omega^*(D)$ is a subhypersemigroup of S/Ω.* □

Proposition 4.2 *Let (S,Ω) be a hyper approximation space and Ω be a complete strongly regular relation. If D is a subhypersemigroup of S, then $\varnothing \neq \Omega_*(D)$ is a subhypersemigroup of S/Ω.*

Proof: *Let $[c_1]_\Omega, [c_2]_\Omega \in \Omega_*(D)$. Thus $[c_1]_\Omega \subseteq D$ and $[c_2]_\Omega \subseteq D$. By Proposition 2.1, we have that $[c_1]_\Omega \otimes [c_2]_\Omega = [c]_\Omega$, $\forall c \in c_1c_2$. On the other hand, since D is a subhypersemigroup of S, for all $c \in c_1c_2, c \in c_1c_2 \subseteq [c_1]_\Omega [c_2]_\Omega = [c]_\Omega \subseteq DD \subseteq D$. Therefore $[c]_\Omega \in \Omega_*(D)$. Hence $\Omega_*(D)$ is a subhypersemigroup of S/Ω.* □

Proposition 4.3 *Let (S,Ω) be a hyper approximation space. If I is a left (right, two-sided) hyperideal of S, then $\Omega^*(I)$ is a left (right, two-sided) hyperideal of S/Ω.*

Proof: *Let $[c_1]_\Omega$ and $[c_2]_\Omega$ be any elements of $\Omega^*(I)$ and S/Ω, respectively. By Proposition 2.1, we have that $[c_2]_\Omega \otimes [c_1]_\Omega = [c]_\Omega, \forall c \in c_2c_1$. We will show that $[c]_\Omega \in \Omega^*(I)$. Since,$[c_1]_\Omega \cap I \neq \varnothing$ there exists an element $a \in [c_1]_\Omega \cap I$ such that $a \in [c_1]_\Omega$ and $a \in I$. Then, since I is a left ideal of S, $ba \subseteq [c_2]_\Omega I \subseteq SI \subseteq I$, where $b \in [c_2]_\Omega$. On the other hand, since Ω is a strongly regular relation, we have $a\Omega c_1$, $b\Omega c_2$ and $ba\overline{\overline{\Omega}}c_2c_1$. Thus there exist $t \in ba \subseteq I$ such that $t \in [c]_\Omega$. So $t \in [c]_\Omega \cap I$ and we have $[c]_\Omega \in \Omega^*(I)$. Hence $\Omega^*(I)$ is a left hyperideal of S/Ω. Other cases are easily done in a similar manner.* □

Proposition 4.4 *Let (S,Ω) be a hyper approximation space and Ω be a complete strongly regular relation.If I is a left (right, two-sided) hyperideal of S, then $\varnothing \neq \Omega_*(I)$ is a left (right, two-sided) hyperideal of S/Ω.*

Proof: *Let $[c_1]_\Omega$ and $[c_2]_\Omega$ be any elements of $\Omega_*(I)$ and S/Ω, respectively. By Proposition 2.1, we have that $[c_2]_\Omega \otimes [c_1]_\Omega = [c]_\Omega, \forall c \in c_2c_1$. We will show that $[c]_\theta \in \Omega_*(I)$. Since, $[c_1]_\Omega \subseteq I$ there exists an element $a \in [c_1]_\Omega \subseteq I$. Then, since I is a left ideal of S, $ba \subseteq [c_2]_\Omega I \subseteq SI \subseteq I$, where $b \in [c_2]_\Omega$. On the other hand, since Ω is a complete strongly regular relation, $[c_2]_\Omega [c_1]_\Omega = [c]_\Omega, \forall c \in c_2c_1$. So $[c]_\Omega \subseteq I$ and we have $[c]_\Omega \in \Omega_*(I)$. Hence $\Omega_*(I)$ is a left hyperideal of S/Ω. Other cases are easily done in a similar manner.* □

Proposition 4.5 *Let (S,Ω) be a hyper approximation space. If F is a hyper bi-ideal of S, then $\Omega^*(F)$ is a hyper bi-ideal of S/Ω.*

Proof: *Let $[c_1]_\Omega$ and $[c_2]_\Omega$ be any elements of $\Omega^*(F)$, and $[c_3]_\Omega$ be any elements of S/Ω. By Proposition 2.1, we have that*

$$([c_1]_\Omega \otimes [c_3]_\Omega) \otimes [c_2]_\Omega = [c]_\Omega \otimes [c_2]_\Omega = [b]_\Omega, \forall c \in c_1c_3, \forall b \in cc_2 \subseteq c_1c_3c_2$$

. We will show that $[b]_\Omega \in \Omega^*(F)$. Since, $[c_1]_\Omega \cap F \neq \varnothing$ and $[c_2]_\Omega \cap F \neq \varnothing$ there exist elements $x \in [c_1]_\Omega \cap F, y \in [c_2]_\Omega \cap F$ such that $x \in [c_1]_\Omega$,$y \in [c_2]_\Omega$ and $x, y \in F$. Let $z \in [c_3]_\Omega$. Then F is a hyper bi-ideal of S,

$$xzy \subseteq [c_1]_\Omega [c_3]_\Omega [c_2]_\Omega \subseteq FSF \subseteq F.$$

On the other hand, since Ω is a strongly regular relation, we have $x\Omega c_1$, $z\Omega c_3$ and $xz\overline{\overline{\Omega}}c_1c_3$. For all $c \in c_1c_3$, $t \in xz$ we have $t\Omega c$, and so $ty\overline{\overline{\Omega}}cc_2$. Thus there exist $k \in tz \subseteq xzy \subseteq I$ such that $k \in [b]_\Omega$, where $b \in cc_2 \subseteq c_1c_3c_2$. So $k \in [b]_\Omega \cap F$ and we have $[b]_\Omega \in \Omega^*(F)$. Hence $\Omega^*(F)$ is a hyper bi-ideal of S/Ω □

Proposition 4.6 *Let (S,Ω) be a hyper approximation space and Ω be a complete strongly regular relation.If F is a hyper bi-ideal of S, then $\varnothing \neq \Omega_*(F)$ is a hyper bi-ideal of S/Ω.*

Proof: *Let $[c_1]_\Omega$ and $[c_2]_\Omega$ be any elements of $\Omega_*(F)$, and $[c_3]_\Omega$ be any elements of S/θ. By Proposition 2.1, we have that*

$$([c_1]_\Omega \otimes [c_3]_\Omega) \otimes [c_2]_\Omega = [c]_\Omega \otimes [c_2]_\Omega = [b]_\Omega, \forall c \in c_1c_3, \forall b \in cc_2 \subseteq c_1c_3c_2$$

. We will show that $[b]_\Omega \in \Omega_*(F)$. Then F is a hyper bi-ideal of S,

$$[c_1]_\Omega [c_3]_\Omega [c_2]_\Omega \subseteq FSF \subseteq F.$$

On the other hand, since Ω is a complete strongly regular relation, $[c_1]_\Omega [c_3]_\Omega [c_2]_\Omega = [b]_\Omega, \forall c \in c_1c_3, \forall b \in cc_2 \subseteq c_1c_3c_2$. So $[b]_\Omega \subseteq F$ and we have $[b]_\Omega \in \Omega_*(F)$. Hence $\Omega_*(F)$ is a hyper bi-ideal of S/Ω. □

5. Conclusions

Considering rough approximations together with hypersemigroups allows hypersemigroups to be studied in situations where exact values and operations may be uncertain or ambiguous. This approach will be useful in applications where only approximate information is available, such as information systems, data analysis, and decision making. In this context, rough approximations are studied in a hypersemigroup in this study. Then, their important properties are proven and examples are given. In addition, the concepts of rough subhypersemigroup and ideal are given. Finally, rough approximations are discussed in a a quotient hypersemigroup.

Declarations

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