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Spectral Properties of Identity Graph for Group of Integers Modulo n using Degree-Based Matrices*

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ABSTRACT: This paper investigates the spectral properties of the identity graph associated with the group $\mathbb{Z}_n$, utilizing five degree-based matrices. Specifically, the study employs the maximum and minimum degree, greatest common divisor, and first and second Zagreb matrices. For each case, the characteristic polynomial and the corresponding graph energy are derived. Furthermore, a comparative analysis is conducted between the computed energies and previously established results in the literature.

Key Words: Spectral radius, degree-based matrices, energy of a graph, identity graph, groups $\mathbb{Z}_n$.

Contents

1. Introduction

Most research on graphs defined on groups focuses on computing various parameters related to graph theory. To identify interesting groups, we can use graphs to discover new information about them and to put conditions on the various graphs defined within them. The process of finding properties may also lead us to discover beautiful graphs [6]. Graph-theoretic methods have long been applied to algebraic structures, especially groups, to extract structural and spectral properties through graphical representations. Several studies have recently extended the analysis of graphs defined on groups to explore new classes of graphs, such as co-prime order graphs, identity graphs, and divisor-related graphs, thereby enriching the interplay between algebra and graph theory.

Kandasamy and Smarandache [9] first introduced the concept of identity graphs of groups, establishing an important foundation for representing group elements and their relationships graphically. Subsequent works have expanded this perspective by defining additional group-based graphs and studying their topological and spectral parameters.

In particular, several recent contributions have examined the structure and genus of co-prime order graphs, offering results closely related to identity and commuting graphs. For instance, Hao et al. [8] analyzed the co-prime order graph of a group and established new characterizations of its connectivity and clique number. Later, Li et al. [12] investigated finite groups whose co-prime order graphs have positive genus, thus linking algebraic group properties with graph embeddings. Further algebraic labeling properties were examined by Saini et al. [21], who introduced divisor labeling for co-prime order graphs of finite groups. Related to these, Saini et al. [22] studied co-prime order graphs of finite abelian p-groups, providing insights into their degree distribution and connectivity. A comprehensive overview of graphs defined on groups in terms of element order has also been provided by [1,13], summarizing major directions and applications in this growing field. These developments complement the current investigation of identity graphs and their spectral properties. They collectively highlight the increasing

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importance of degree-based matrices and characteristic polynomials in understanding the underlying algebraic and combinatorial structure of group-based graphs.

These developments complement the current investigation of identity graphs and their spectral properties. They collectively highlight the increasing importance of degree-based matrices and characteristic polynomials in understanding the underlying algebraic and combinatorial structure of group-based graphs. Traditionally, a graph is represented by its adjacency matrix; however, researchers have extended this concept by defining new types of matrices derived from the degree of the graph. Adiga and Smitha [3] discovered the definition of the maximum degree matrix, and one year later, Adiga and Swamy [4] defined the minimum degree matrix of a graph. Then, in 2020, Romdhini and Nawawi [19] discussed these types of graph matrices for commuting graphs that are defined on the dihedral groups. Moreover, researchers also introduced several graph matrices, including the greatest common divisor degree matrix [15] and first and second Zagreb matrices [14] as well as Zagreb index [2]. Romdhini and Nawawi [18] combined these two matrices of commuting graphs for dihedral groups.

As the association of graph and matrix, Gutman [7] pioneered the definition of the energy of a graph based on the eigenvalues of the matrix. Moreover, the exploration of the graph energy results in the energy of a graph is never an odd integer [5]. This result is our baseline for further investigation in this paper. In addition, we can see the justification of the energy value in [17]. The classification of the energy value of the graph is also at our attention, as presented in [11]. The graph can be hyperenergetic if it satisfies the particular requirement that will be explained in the next section.

Therefore, this inspires us to observe multiple graph energies employing distinct matrices, including the maximum and minimum degree, greatest common divisor, and first and second Zagreb matrices for the identity graph. This research focuses on a group of integers modulo n, $\mathbb{Z}_n = \left\{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n-1}\right\}$. We divided the analysis into two cases, odd n and even n.

We manage this paper as follows. In the second section of this paper, we present the basic definition and notation that are useful for the next section. The goals of this research are achieved as presented in the third section. We start with the simplified process of finding the determinant formula. Then, we use this formula for investigating the energy of the identity graph for $\mathbb{Z}_n$ for odd n and for formulating the characteristic polynomial for even n. We eventually derive the conclusion of this research in the last section.

2. Preliminaries

This section serves as a reminder of the fundamental definition and theorems that are valuable for our primary findings. We begin by defining the identity graph.

Definition 2.1 [9] The identity graph of a group G, which is written as Γ_G, is a graph with the elements of the group as its vertices. Two different vertices u and v will be adjacent whenever $uv = e$, and every element of $G\backslash\{e\}$ will be adjacent to the identity element e.

In this study, we refer to the identity graph of $\mathbb{Z}_n$ as $\Gamma_{\mathbb{Z}_n}$. The subsequent two theorems delineate $\Gamma_{\mathbb{Z}_n}$ for both odd and even values of n.

Theorem 2.1 *[9] For $\mathbb{Z}_n = \left\{\overline{0}, \overline{1}, \overline{2}, ..., \overline{n-1}\right\}$ with $n \geq 3$ and n is odd, then $\Gamma_{\mathbb{Z}_n}$ comprises $\frac{n-1}{2}$ of K_3.*

Theorem 2.2 *[9] For $\mathbb{Z}_n = \left\{\overline{0}, \overline{1}, \overline{2}, ..., \overline{n-1}\right\}$ with $n \geq 2$ and n is even, then $\Gamma_{\mathbb{Z}_n}$ comprises $\frac{n-2}{2}$ of K_3 and a K_2.*

Let d_{v_i} be the degree of vertex v_i, which is the number of $\Gamma_{\mathbb{Z}_n}$ vertices that are adjacent to v_i in $\Gamma_{\mathbb{Z}_n}$. We need the degree of every vertex in $\Gamma_{\mathbb{Z}_n}$ to make the matrices of $\Gamma_{\mathbb{Z}_n}$.

Theorem 2.3 *[20] In $\Gamma_{\mathbb{Z}_n}$, if n is an odd integer, then*

1. $d_{\overline{0}} = n - 1$, *and*
2. $d_{\overline{a}} = 2$, *for* $a \neq 0$.

Theorem 2.4 *[20] In $\Gamma_{\mathbb{Z}_n}$, if n is an even integer, then*

1. $d_{\overline{0}} = n-1$,
2. $d_{\overline{\frac{n}{2}}} = 1$, *and*
3. $d_{\overline{a}} = 2$, *for* $a \neq 0, \frac{n}{2}$.

Upon the definition of the maximum degree, minimum degree, greatest common divisor degree, first and second Zagreb matrices, the degree-based matrices of $\Gamma_{\mathbb{Z}_n}$. Here is how the definition goes.

Definition 2.2 [3] The maximum degree matrix of $\Gamma_{\mathbb{Z}_n}$ is $MaxD(\Gamma_{\mathbb{Z}_n}) = [maxd_{ij}]$ in which (i,j)-th entry is

$$maxd_{ij} = \begin{cases} max\{d_{v_i}, d_{v_j}\}, & \text{if } v_i, v_j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.3 [4] The minimum degree matrix of $\Gamma_{\mathbb{Z}_n}$ is $MinD(\Gamma_{\mathbb{Z}_n}) = [mind_{ij}]$ in which (i,j)-th entry is

$$mind_{ij} = \begin{cases} min\{d_{v_i}, d_{v_j}\}, & \text{if } v_i, v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.4 [15] The greatest common divisor degree matrix of $\Gamma_{\mathbb{Z}_n}$ is $GCDD(\Gamma_{\mathbb{Z}_n}) = [gcd_{ij}]$ in which (i,j)-th entry is

$$gcd_{ij} = \begin{cases} g.c.d.\{d_{v_i}, d_{v_j}\}, & \text{if } v_i, v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.5 [14] The first Zagreb matrix of $\Gamma_{\mathbb{Z}_n}$ is $Z_1(\Gamma_{\mathbb{Z}_n}) = [z_{1_{ij}}]$ in which (i,j)-th entry is

$$z_{1_{ij}} = \begin{cases} d_{v_i} + d_{v_j}, & \text{if } v_i, v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.6 [14] The second Zagreb matrix of $\Gamma_{\mathbb{Z}_n}$ is $Z_2(\Gamma_{\mathbb{Z}_n}) = [z_{2_{ij}}]$ in which (i,j)-th entry

$$z_{2_{ij}} = \begin{cases} d_{v_i} \cdot d_{v_j}, & \text{if } v_i, v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic formula of $MaxD(\Gamma_{\mathbb{Z}_n})$ is defined as

$$P_{MaxD(\Gamma_{\mathbb{Z}_n})}(\lambda) = |\lambda I_n - MaxD(\Gamma_{\mathbb{Z}_n})|. \tag{2.1}$$

The solution to the equation $P_{MaxD(\Gamma_{\mathbb{Z}_n})}(\lambda) = 0$ correspond to the eigenvalues of $\Gamma_{\mathbb{Z}_n}$. The definition of graph energy is derived from the eigenvalues of $\Gamma_{\mathbb{Z}_n}$ as seen below.

Definition 2.7 [7] The maximum degree energy of $\Gamma_{\mathbb{Z}_n}$ can be written by

$$E_{MaxD}(\Gamma_{\mathbb{Z}_n}) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ represent the eigenvalues of $MaxD(\Gamma_{\mathbb{Z}_n})$. The energy value of $\Gamma_{\mathbb{Z}_n}$ is deemed hyperenergetic if it exceeds $2(n-1)$ [11].

The spectrum of $\Gamma_{\mathbb{Z}_n}$ corresponding with the maximum degree matrix is

$$Spec_{MaxD}(\Gamma_{\mathbb{Z}_n}) = \{(\lambda_1)^{k_1}, (\lambda_2)^{k_2}, \ldots, (\lambda_n)^{k_n}\},$$

where the multiplicities of the eigenvalues are denoted as $k_1, k_2, \ldots, k_n$. The spectral radius of $\Gamma_{\mathbb{Z}_n}$ that corresponds to the maximum degree matrix is

$$\rho_{MaxD}(\Gamma_{\mathbb{Z}_n}) = max\{|\lambda| : \lambda \in Spec_{MaxD}(\Gamma_{\mathbb{Z}_n})\}.$$

Similarly, the notation for other matrices can be utilized analogously.

3. Main Results

To derive the determinant in Equation 2.1, we require the characteristic formula of a square matrix M to facilitate the computation of the characteristic polynomial of $\Gamma_{\mathbb{Z}_n}$.

Theorem 3.1 *Let a, b are real numbers and M be an $n \times n$ matrix as follows:*

$$M = \begin{pmatrix} 0 & a & a & \dots & a & a \\ a & 0 & 0 & \dots & 0 & b \\ a & 0 & 0 & \dots & b & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a & 0 & b & \dots & 0 & 0 \\ a & b & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then the characteristic polynomial of M can be simplified as follows:

$$P_M(\lambda) = \left(\lambda^2 - b\lambda - a^2(n-1)\right)(\lambda - b)^{\frac{n-3}{2}}(\lambda + b)^{\frac{n-1}{2}}.$$

Proof: Using Equation 2.1 for real numbers a, b, we obtain the characteristic formula of M as seen below.

$$P_M(\lambda) = \begin{vmatrix} \lambda & -a & -a & \dots & -a & -a \\ -a & \lambda & 0 & \dots & 0 & -b \\ -a & 0 & \lambda & \dots & -b & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a & 0 & -b & \dots & \lambda & 0 \\ -a & -b & 0 & \dots & 0 & \lambda \end{vmatrix}.$$

Now let R_i and C_i be the i-th row and column of $P_M(\lambda)$, respectively. By employing subsequent row and column operations, it is necessary to simplify the determinant above with the following steps:

1. Replace the $\left(\frac{n+1}{2}+1\right)$-th row with the result of subtracting the elements of that row from the elements of the $\left(\frac{n+3}{2}-i\right)$-th row; that is, perform the row operation: $R_{\frac{n+1}{2}+1} \longrightarrow R_{\frac{n+1}{2}+i} - R_{\frac{n+3}{2}-i}$, for $i = 1, 2, \dots, \frac{n-1}{2}$.
2. Replace the $\left(\frac{n+3}{2}-i\right)$-th column with the result of the summation of the elements of that column and the elements of the $\left(\frac{n+1}{2}+i\right)$-th column; in other words, perform the column operation: $C_{\frac{n+3}{2}-i} \longrightarrow C_{\frac{n+3}{2}-i} + C_{\frac{n+1}{2}+i}$, for $i = 1, 2, \dots, \frac{n-1}{2}$.
3. Apply the following column operation: $C_1 \longrightarrow C_1 + \frac{a}{\lambda-b}C_2 + \frac{a}{\lambda-b}C_3 + \dots + \frac{a}{\lambda-b}C_{\frac{n+1}{2}}$.

Hence $P_M(\lambda)$ is

$$P_M(\lambda) = \begin{vmatrix} \lambda - \frac{a^2(n-1)}{\lambda-2} & -2a & -2a & \dots & -2a & -2a & -a & -a & \dots & -a & -a \\ 0 & \lambda-b & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -b \\ 0 & 0 & \lambda-b & \dots & 0 & 0 & 0 & 0 & \dots & -b & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda-b & 0 & 0 & -b & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda-b & -b & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda+b & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \lambda+b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \lambda+b & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \lambda+b \end{vmatrix}.$$

We see that, $P_M(\lambda)$ must be

$$P_M(\lambda) = \left(\lambda - \frac{a^2(n-1)}{\lambda - b}\right)(\lambda - b)^{\frac{n-1}{2}}(\lambda + b)^{\frac{n-1}{2}} = \left(\lambda^2 - b\lambda - a^2(n-1)\right)(\lambda - b)^{\frac{n-3}{2}}(\lambda + b)^{\frac{n-1}{2}}.$$

□

Theorem 3.2 *Let*

$$M = \begin{pmatrix} 0 & a & a & \dots & a & c & a & \dots & a & a \\ a & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & b \\ a & 0 & 0 & \dots & 0 & 0 & 0 & \dots & b & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a & 0 & 0 & \dots & 0 & 0 & b & \dots & 0 & 0 \\ c & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ a & 0 & 0 & \dots & b & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a & 0 & b & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ a & b & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

where a, b, c are real numbers. Then the characteristic polynomial of M can be simplified as follows:

$$P_M(\lambda) = \lambda^3 - b\lambda^2 - \left(a^2(n-2) + c^2\right)\lambda + bc^2.$$

Proof: By taking real numbers a, b, c and Equation 2.1, the characteristic formula of M is as follows.

$$P_M(\lambda) = \begin{vmatrix} \lambda & -a & -a & \dots & -a & -c & -a & \dots & -a & -a \\ -a & \lambda & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -b \\ -a & 0 & \lambda & \dots & 0 & 0 & 0 & \dots & -b & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a & 0 & 0 & \dots & \lambda & 0 & -b & \dots & 0 & 0 \\ -c & 0 & 0 & \dots & 0 & \lambda & 0 & \dots & 0 & 0 \\ -a & 0 & 0 & \dots & -b & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a & 0 & -b & \dots & 0 & 0 & 0 & \dots & \lambda & 0 \\ -a & -b & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \lambda \end{vmatrix}.$$

We have to make the above determinant easier through the application of row and column operations.

1. Replace the $\left(\frac{n}{2}+1+i\right)$-th row with the result of subtracting the elements of that row from the elements of the $\left(\frac{n}{2}+1-i\right)$-th row; that is, perform the row operation: $R_{\frac{n}{2}+1+i} \longrightarrow R_{\frac{n}{2}+1+i} - R_{\frac{n}{2}+1-i}$, for $i = 1, 2, \dots, \frac{n}{2} - 1$.

2. Replace the $\left(\frac{n}{2}+1-i\right)$-th column with the result of the summation of the elements of that column and the elements of the $\left(\frac{n}{2}+1+i\right)$-th column; in other words, perform the column operation: $C_{\frac{n}{2}+1-i} \longrightarrow C_{\frac{n}{2}+1-i} + C_{\frac{n}{2}+1+i}$, for $i = 1, 2, \dots, \frac{n}{2} - 1$.

3. Apply the following column operation: $C_1 \longrightarrow C_1 + \frac{a}{\lambda-b}C_2 + \frac{a}{\lambda-b}C_3 + \dots + \frac{a}{\lambda-b}C_{\frac{n}{2}-1} + \frac{c}{\lambda}C_{\frac{n}{2}}$.

consequently, we may express $P_M(\lambda)$ as follows:

$$P_M(\lambda)=\begin{vmatrix} \lambda-\frac{a^2(n-2)}{\lambda-b}-\frac{c^2}{\lambda} & -2a & -2a & \dots & -2a & -a & -a & \dots & -a & -a \\ 0 & \lambda-b & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -b \\ 0 & 0 & \lambda-b & \dots & 0 & 0 & 0 & \dots & -b & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda-b & 0 & -b & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda+b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \lambda+b & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \lambda+b \end{vmatrix}.$$

It implies that

$$\begin{aligned} P_M(\lambda) &= \left(\lambda-\frac{a^2(n-2)}{\lambda-b}-\frac{c^2}{\lambda}\right)(\lambda-b)^{\frac{n}{2}-1}\lambda(\lambda+b)^{\frac{n}{2}-1} \\ &= \left(\lambda^3-b\lambda^2-\left(a^2(n-2)+c^2\right)\lambda+bc^2\right)(\lambda-b)^{\frac{n}{2}-3}(\lambda+b)^{\frac{n}{2}-1}. \end{aligned}$$

□

3.1. The Energy of $\Gamma_{\mathbb{Z}_n}$ for Odd n

In this section, we present the degree-based energy of $\Gamma_{\mathbb{Z}_n}$ for odd n.

Theorem 3.3 *In $\Gamma_{\mathbb{Z}_n}$ for odd n, the maximum degree energy of $\Gamma_{\mathbb{Z}_n}$ is*

$$E_{MaxD}(\Gamma_{\mathbb{Z}_n})=2\left(n-2+\sqrt{1+(n-1)^3}\right).$$

Proof: Using Theorems 2.1 and 2.3 and following Definition 2.2, we may formulate an $n\times n$ maximum degree matrix for $\Gamma_{\mathbb{Z}_n}$ as outlined:

$$MaxD(\Gamma_{\mathbb{Z}_n})=\begin{array}{c} \\ \overline{0} \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{n-2} \\ \overline{n-1} \end{array}\begin{array}{c} \begin{array}{cccccc} \overline{0} & \overline{1} & \overline{2} & \dots & \overline{n-2} & \overline{n-1} \end{array} \\ \begin{pmatrix} 0 & n-1 & n-1 & \dots & n-1 & n-1 \\ n-1 & 0 & 0 & \dots & 0 & 2 \\ n-1 & 0 & 0 & \dots & 2 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & 0 & 2 & \dots & 0 & 0 \\ n-1 & 2 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array}$$

By applying Theorem 3.1 with $a=n-1$ and $b=2$, we obtain the following

$$P_{MaxD(\Gamma_{\mathbb{Z}_n})}(\lambda)=\left(\lambda^2-2\lambda-(n-1)^3\right)(\lambda-2)^{\frac{n-3}{2}}(\lambda+2)^{\frac{n-1}{2}}.$$

The solution of $P_{MaxD(\Gamma_{\mathbb{Z}_n})}(\lambda)=0$ are $\lambda_1=2$ of multiplicity $\frac{n-3}{2}$, $\lambda_2=-2$ of multiplicity $\frac{n-1}{2}$, and $\lambda_{3,4}=1\pm\sqrt{1+(n-1)^3}$ both of multiplicity 1. As a result, the spectrum of $\Gamma_{\mathbb{Z}_n}$ is

$$Spec_{MaxD}(\Gamma_{\mathbb{Z}_n})=\left\{\left(1+\sqrt{1+(n-1)^3}\right)^1,(2)^{\frac{n-3}{2}},(-2)^{\frac{n-1}{2}},\left(1-\sqrt{1+(n-1)^3}\right)^1\right\}.$$

The spectral radius of $\Gamma_{\mathbb{Z}_n}$ is obtained as follows.

$$\rho_{MaxD}(\Gamma_{\mathbb{Z}_n})=1+\sqrt{1+(n-1)^3}.$$

Therefore, the maximum degree energy of $\Gamma_{\mathbb{Z}_n}$ is as follows:

$$\begin{aligned} E_{MaxD}(\Gamma_{\mathbb{Z}_n}) =& \left(\frac{n-3}{2}\right)|2| + \left(\frac{n-1}{2}\right)|-2| + \left|1 \pm \sqrt{1+(n-1)^3}\right| \\ =& 2\left(n-2+\sqrt{1+(n-1)^3}\right). \end{aligned}$$

□

Theorem 3.4 *In $\Gamma_{\mathbb{Z}_n}$ for odd n, the minimum degree energy of $\Gamma_{\mathbb{Z}_n}$ is*

$$E_{MinD}(\Gamma_{\mathbb{Z}_n}) = 2\left(n-2+\sqrt{1+4(n-1)}\right).$$

Proof: From the results of Theorems 2.1 and 2.3 and by Definition 2.3, we may formulate an $n \times n$ minimum degree matrix of $\Gamma_{\mathbb{Z}_n}$ as outlined:

$$MinD(\Gamma_{\mathbb{Z}_n}) = \begin{array}{c} \\ \overline{0} \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{n-2} \\ \overline{n-1} \end{array} \begin{array}{c} \begin{array}{cccccc} \overline{0} & \overline{1} & \overline{2} & \dots & \overline{n-2} & \overline{n-1} \end{array} \\ \begin{pmatrix} 0 & 2 & 2 & \dots & 2 & 2 \\ 2 & 0 & 0 & \dots & 0 & 2 \\ 2 & 0 & 0 & \dots & 2 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 0 & 2 & \dots & 0 & 0 \\ 2 & 2 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array}$$

By Theorem 3.1 with $a = b = 2$, consequently

$$P_{MinD(\Gamma_{\mathbb{Z}_n})}(\lambda) = \left(\lambda^2 - 2\lambda - 4(n-1)\right)(\lambda-2)^{\frac{n-3}{2}}(\lambda+2)^{\frac{n-1}{2}}.$$

The solution of $P_{MinD(\Gamma_{\mathbb{Z}_n})}(\lambda) = 0$ are $\lambda_1 = 2$ of multiplicity $\frac{n-3}{2}$, $\lambda_2 = -2$ of multiplicity $\frac{n-1}{2}$, and $\lambda_{3,4} = 1 \pm \sqrt{1+4(n-1)}$ both of multiplicity 1.As a result, the spectrum of $\Gamma_{\mathbb{Z}_n}$ is

$$Spec_{MinD}(\Gamma_{\mathbb{Z}_n}) = \left\{ \left(1+\sqrt{1+4(n-1)}\right)^1, (2)^{\frac{n-3}{2}}, (-2)^{\frac{n-1}{2}}, \left(1-\sqrt{1+4(n-1)}\right)^1 \right\}.$$

We obtain $\Gamma_{\mathbb{Z}_n}$ is

$$\rho_{MinD}(\Gamma_{\mathbb{Z}_n}) = 1+\sqrt{1+4(n-1)}.$$

Thus, the maximum degree energy of $\Gamma_{\mathbb{Z}_n}$ is obtained below:

$$\begin{aligned} E_{MinD}(\Gamma_{\mathbb{Z}_n}) =& \left(\frac{n-3}{2}\right)|2| + \left(\frac{n-1}{2}\right)|-2| + \left|1 \pm \sqrt{1+4(n-1)}\right| \\ =& 2\left(n-2+\sqrt{1+4(n-1)}\right). \end{aligned}$$

□

Theorem 3.5 *In $\Gamma_{\mathbb{Z}_n}$ for odd n, the greatest common divisor degree energy of $\Gamma_{\mathbb{Z}_n}$ is*

$$E_{GCDD}(\Gamma_{\mathbb{Z}_n}) = 2\left(2(n-2)+\sqrt{4+(n+1)^2(n-1)}\right).$$

Proof: Based on Theorems 2.1 and 2.3, we know that $g.c.d.(n-1,2)=2$, $g.c.d.(2,2)=2$. By using Definition 2.4, setting that an $n \times n$ greatest common divisor degree matrix of $\Gamma_{\mathbb{Z}_n}$ as follows:

$$GCDD(\Gamma_{\mathbb{Z}_n}) = \begin{array}{c} \\ \overline{0} \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{n-2} \\ \overline{n-1} \end{array} \begin{array}{c} \begin{array}{cccccc} \overline{0} & \overline{1} & \overline{2} & \dots & \overline{n-2} & \overline{n-1} \end{array} \\ \begin{pmatrix} 0 & 2 & 2 & \dots & 2 & 2 \\ 2 & 0 & 0 & \dots & 0 & 2 \\ 2 & 0 & 0 & \dots & 2 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 0 & 2 & \dots & 0 & 0 \\ 2 & 2 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array} = MinD(\Gamma_{\mathbb{Z}_n})$$

Therefore, based on Theorem 3.4, the greatest common divisor degree energy of $\Gamma_{\mathbb{Z}_n}$ is as follows:

$$E_{GCDD}(\Gamma_{\mathbb{Z}_n}) = 2\left(n-2+\sqrt{1+4(n-1)}\right).$$

□

Theorem 3.6 *In $\Gamma_{\mathbb{Z}_n}$ for odd n, the first Zagreb energy of $\Gamma_{\mathbb{Z}_n}$ is*

$$E_{Z_1}(\Gamma_{\mathbb{Z}_n}) = 2\left(n-2+\sqrt{1+4(n-1)}\right).$$

Proof: The $n \times n$ first Zagreb matrix of $\Gamma_{\mathbb{Z}_n}$ can be constructed by following Theorems 2.1 and 2.3, and Definition 2.5. It is given below:

$$Z_1(\Gamma_{\mathbb{Z}_n}) = \begin{array}{c} \\ \overline{0} \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{n-2} \\ \overline{n-1} \end{array} \begin{array}{c} \begin{array}{cccccc} \overline{0} & \overline{1} & \overline{2} & \dots & \overline{n-2} & \overline{n-1} \end{array} \\ \begin{pmatrix} 0 & n+1 & n+1 & \dots & n+1 & n+1 \\ n+1 & 0 & 0 & \dots & 0 & 4 \\ n+1 & 0 & 0 & \dots & 4 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n+1 & 0 & 4 & \dots & 0 & 0 \\ n+1 & 4 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array}$$

By using Theorem 3.1 with $a=n+1$ and $b=4$, hence,

$$P_{Z_1(\Gamma_{\mathbb{Z}_n})}(\lambda) = \left(\lambda^2-4\lambda-(n+1)^2(n-1)\right)(\lambda-4)^{\frac{n-3}{2}}(\lambda+4)^{\frac{n-1}{2}}.$$

The solution of $P_{Z_1(\Gamma_{\mathbb{Z}_n})}(\lambda)=0$ are $\lambda_1=4$ of multiplicity $\frac{n-3}{2}$, $\lambda_2=-4$ of multiplicity $\frac{n-1}{2}$, and $\lambda_{3,4}=2\pm\sqrt{4+(n+1)^2(n-1)}$ both of multiplicity 1. As a result, we have

$$Spec_{Z_1}(\Gamma_{\mathbb{Z}_n}) = \left\{\left(2+\sqrt{4+(n+1)^2(n-1)}\right)^1, (4)^{\frac{n-3}{2}}, (-4)^{\frac{n-1}{2}}, \left(2-\sqrt{4+(n+1)^2(n-1)}\right)^1\right\}.$$

We get $\Gamma_{\mathbb{Z}_n}$ is

$$\rho_{Z_1}(\Gamma_{\mathbb{Z}_n}) = 2+\sqrt{4+(n+1)^2(n-1)}.$$

Therefore, the maximum degree energy of $\Gamma_{\mathbb{Z}_n}$ is as follows:

$$\begin{aligned} E_{Z_1}(\Gamma_{\mathbb{Z}_n}) &= \left(\frac{n-3}{2}\right)|4| + \left(\frac{n-1}{2}\right)|-4| + \left|2\pm\sqrt{4+(n+1)^2(n-1)}\right| \\ &= 2\left(2(n-2)+\sqrt{4+(n+1)^2(n-1)}\right). \end{aligned}$$

□

Theorem 3.7 *In $\Gamma_{\mathbb{Z}_n}$ for odd n, the second Zagreb energy of $\Gamma_{\mathbb{Z}_n}$ is*

$$E_{Z_2}(\Gamma_{\mathbb{Z}_n}) = 4\left(n - 2 + \sqrt{1 + (n-1)^3}\right).$$

Proof: Using Definition 2.6 and Theorems 2.1 and 2.3, we have an $n \times n$ second Zagreb matrix of $\Gamma_{\mathbb{Z}_n}$ as follows:

$$Z_2(\Gamma_{\mathbb{Z}_n}) = \begin{array}{c} \\ \overline{0} \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{n-2} \\ \overline{n-1} \end{array} \begin{array}{c} \begin{array}{cccccc} \overline{0} & \overline{1} & \overline{2} & \dots & \overline{n-2} & \overline{n-1} \end{array} \\ \begin{pmatrix} 0 & 2(n-1) & 2(n-1) & \dots & 2(n-1) & 2(n-1) \\ 2(n-1) & 0 & 0 & \dots & 0 & 4 \\ 2(n-1) & 0 & 0 & \dots & 4 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(n-1) & 0 & 4 & \dots & 0 & 0 \\ 2(n-1) & 4 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array}$$

By takin $a = 2(n-1)$ and $b = 4$ in Theorem 3.1, then we get

$$P_{Z_2(\Gamma_{\mathbb{Z}_n})}(\lambda) = \left(\lambda^2 - 4\lambda - 4(n-1)^3\right)(\lambda - 4)^{\frac{n-3}{2}}(\lambda + 4)^{\frac{n-1}{2}}.$$

The solution of $P_{Z_2(\Gamma_{\mathbb{Z}_n})}(\lambda) = 0$ are $\lambda_1 = 4$ of multiplicity $\frac{n-3}{2}$, $\lambda_2 = -4$ of multiplicity $\frac{n-1}{2}$, and $\lambda_{3,4} = 2\left(1 \pm \sqrt{1 + (n-1)^3}\right)$ both of multiplicity 1. Hence,

$$Spec_{Z_2}(\Gamma_{\mathbb{Z}_n}) = \left\{ \left(2\left(1 + \sqrt{1+(n-1)^3}\right)\right)^1, (4)^{\frac{n-3}{2}}, (-4)^{\frac{n-1}{2}}, \left(2\left(1 - \sqrt{1+(n-1)^3}\right)\right)^1 \right\}.$$

We can see that

$$\rho_{Z_2}(\Gamma_{\mathbb{Z}_n}) = 2\left(1 + \sqrt{1 + (n-1)^3}\right).$$

Therefore,

$$\begin{aligned} E_{Z_2}(\Gamma_{\mathbb{Z}_n}) &= \left(\frac{n-3}{2}\right)|4| + \left(\frac{n-1}{2}\right)|-4| + \left|2\left(1 \pm \sqrt{1+(n-1)^3}\right)\right| \\ &= 4\left(n - 2 + \sqrt{1+(n-1)^3}\right). \end{aligned}$$

□

From Theorems 3.3, 3.4, 3.5, 3.6, and 3.7, we can conclude a statement as presented in below:

Corollary 3.1 *For odd n, the energy of $\Gamma_{\mathbb{Z}_n}$ is never an odd integer based on $MaxD(\Gamma_{\mathbb{Z}_n})$, $MinD(\Gamma_{\mathbb{Z}_n})$, $GCDD(\Gamma_{\mathbb{Z}_n})$, $Z_1(\Gamma_{\mathbb{Z}_n})$, and $Z_2(\Gamma_{\mathbb{Z}_n})$.*

In comparing the obtained energies and the requirement of the hyperenergetic graph, an interesting result is obtained.

Corollary 3.2 *For odd n, $\Gamma_{\mathbb{Z}_n}$ is always hyperenergetic based on $MaxD(\Gamma_{\mathbb{Z}_n})$, $MinD(\Gamma_{\mathbb{Z}_n})$, $GCDD(\Gamma_{\mathbb{Z}_n})$, $Z_1(\Gamma_{\mathbb{Z}_n})$, and $Z_2(\Gamma_{\mathbb{Z}_n})$.*

3.2. The Characteristic Polynomial of $\Gamma_{\mathbb{Z}_n}$ for Even n

For even n, we show the characteristic formula of the identity graph for $\mathbb{Z}_n$ based on five matrices.

Theorem 3.8 *In $\Gamma_{\mathbb{Z}_n}$ for even n, the characteristic polynomial of $MaxD(\Gamma_{\mathbb{Z}_n})$ is*

$$P_{MaxD(\Gamma_{\mathbb{Z}_n})}(\lambda) = \left(\lambda^3 - 2\lambda^2 - (n-1)^3\lambda + 2(n-1)^2\right)(\lambda - 2)^{\frac{n}{2}-3}(\lambda + 2)^{\frac{n}{2}-1}.$$

Proof: By taking Definition 2.2 and by Theorem 2.2 and 2.4, we have the maximum degree of $\Gamma_{\mathbb{Z}_n}$ for even n as an $n \times n$ matrix as follows:

$$MaxD(\Gamma_{\mathbb{Z}_n}) = \begin{array}{c} \\ \overline{0} \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{\frac{n}{2}-1} \\ \overline{\frac{n}{2}} \\ \overline{\frac{n}{2}+1} \\ \vdots \\ \overline{n-2} \\ \overline{n-1} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \overline{0} & \overline{1} & \overline{2} & \ldots & \overline{\frac{n}{2}-1} & \overline{\frac{n}{2}} & \overline{\frac{n}{2}+1} & \ldots & \overline{n-2} & \overline{n-1} \end{array} \\ \begin{pmatrix} 0 & n-1 & n-1 & \ldots & n-1 & n-1 & n-1 & \ldots & n-1 & n-1 \\ n-1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 2 \\ n-1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 2 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n-1 & 0 & 0 & \ldots & 0 & 0 & 2 & \ldots & 0 & 0 \\ n-1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ n-1 & 0 & 0 & \ldots & 2 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n-1 & 0 & 2 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ n-1 & 2 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \end{array}.$$

By taking Theorem 3.2 with $a = n-1$ and $b = c = 2$, it is possible to formulate the subsequent expression:

$$P_{MaxD(\Gamma_{\mathbb{Z}_n})}(\lambda) = \left(\lambda^3 - 2\lambda^2 - (n-1)^3\lambda + 2(n-1)^2\right)(\lambda - 2)^{\frac{n}{2}-3}(\lambda + 2)^{\frac{n}{2}-1}.$$

□

Theorem 3.9 *In $\Gamma_{\mathbb{Z}_n}$ for even n, the characteristic formula of $MinD(\Gamma_{\mathbb{Z}_n})$ is*

$$P_{MinD(\Gamma_{\mathbb{Z}_n})}(\lambda) = \left(\lambda^3 - 2\lambda^2 - \left((n-2)(n-1)^2 + 1\right)\lambda + 2\right)(\lambda - 2)^{\frac{n}{2}-3}(\lambda + 2)^{\frac{n}{2}-1}.$$

Proof: By Defintion 2.3 and from the vertex degree of Theorem 2.4 and graph form of Theorem 2.2, we can cosntruct the minimum degree of $\Gamma_{\mathbb{Z}_n}$ for even n as an $n \times n$ matrix as follows:

$$MinD(\Gamma_{\mathbb{Z}_n}) = \begin{array}{c} \\ \overline{0} \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{\frac{n}{2}-1} \\ \overline{\frac{n}{2}} \\ \overline{\frac{n}{2}+1} \\ \vdots \\ \overline{n-2} \\ \overline{n-1} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \overline{0} & \overline{1} & \overline{2} & \ldots & \overline{\frac{n}{2}-1} & \overline{\frac{n}{2}} & \overline{\frac{n}{2}+1} & \ldots & \overline{n-2} & \overline{n-1} \end{array} \\ \begin{pmatrix} 0 & 2 & 2 & \ldots & 2 & 1 & 2 & \ldots & 2 & 2 \\ 2 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 2 \\ 2 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 2 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 0 & 0 & \ldots & 0 & 0 & 2 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ 2 & 0 & 0 & \ldots & 2 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 0 & 2 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ 2 & 2 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \end{array}.$$

Theorem 3.2 with $a = n-1$, $b = 2$, and $c = 1$ implies

$$P_{MinD(\Gamma_{\mathbb{Z}_n})}(\lambda) = \left(\lambda^3 - 2\lambda^2 - \left((n-2)(n-1)^2 + 1\right)\lambda + 2\right)(\lambda - 2)^{\frac{n}{2}-3}(\lambda + 2)^{\frac{n}{2}-1}.$$

□

Theorem 3.10 *In $\Gamma_{\mathbb{Z}_n}$, for even n, the characteristic polynomial of $GCDD(\Gamma_{\mathbb{Z}_n})$ is*

$$P_{GCDD(\Gamma_{\mathbb{Z}_n})}(\lambda) = \left(\lambda^3 - 2\lambda^2 - (n-1)\lambda + 2\right)(\lambda - 2)^{\frac{n}{2}-3}(\lambda + 2)^{\frac{n}{2}-1}.$$

Proof: From Definition 2.4 and according Theorem 2.2 and the degree of every vertex in Theorem 2.4, we have GCDD-matrix of $\Gamma_{\mathbb{Z}_n}$ for even n as an $n \times n$ matrix as follows:

$$GCDD(\Gamma_{\mathbb{Z}_n}) = \begin{array}{c} \\ \overline{0} \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{\frac{n}{2}-1} \\ \overline{\frac{n}{2}} \\ \overline{\frac{n}{2}+1} \\ \vdots \\ \overline{n-2} \\ \overline{n-1} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \overline{0} & \overline{1} & \overline{2} & \dots & \overline{\frac{n}{2}-1} & \overline{\frac{n}{2}} & \overline{\frac{n}{2}+1} & \dots & \overline{n-2} & \overline{n-1} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 2 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 2 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 0 & 2 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 2 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 2 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array}.$$

As a result Theorem 3.2 with $a = c = 1$ and $b = 2$, it is possible to formulate the subsequent expression:

$$P_{GCDD(\Gamma_{\mathbb{Z}_n})}(\lambda) = \left(\lambda^3 - 2\lambda^2 - (n-1)\,\lambda + 2\right)(\lambda - 2)^{\frac{n}{2}-3}(\lambda + 2)^{\frac{n}{2}-1}.$$

□

Theorem 3.11 *In $\Gamma_{\mathbb{Z}_n}$ for even n, the characteristic polynomial of $Z_1(\Gamma_{\mathbb{Z}_n})$ is*

$$P_{Z_1(\Gamma_{\mathbb{Z}_n})}(\lambda) = \left(\lambda^3 - 4\lambda^2 - \left((n-2)(n+1)^2 + n^2\right)\lambda + 4n^2\right)(\lambda - 4)^{\frac{n}{2}-3}(\lambda + 4)^{\frac{n}{2}-1}.$$

Proof: By Definition 2.3, and following Theorems 2.2 and 2.4, we get the first Zagreb matrix of $\Gamma_{\mathbb{Z}_n}$ for even n as an $n \times n$ matrix as follows:

$$Z_1(\Gamma_{\mathbb{Z}_n}) = \begin{array}{c} \\ \overline{0} \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{\frac{n}{2}-1} \\ \overline{\frac{n}{2}} \\ \overline{\frac{n}{2}+1} \\ \vdots \\ \overline{n-2} \\ \overline{n-1} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \overline{0} & \overline{1} & \overline{2} & \dots & \overline{\frac{n}{2}-1} & \overline{\frac{n}{2}} & \overline{\frac{n}{2}+1} & \dots & \overline{n-2} & \overline{n-1} \end{array} \\ \begin{pmatrix} 0 & n+1 & n+1 & \dots & n+1 & n & n+1 & \dots & n+1 & n+1 \\ n+1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 4 \\ n+1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 4 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n+1 & 0 & 0 & \dots & 0 & 0 & 4 & \dots & 0 & 0 \\ n & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ n+1 & 0 & 0 & \dots & 4 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n+1 & 0 & 4 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ n+1 & 4 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array}.$$

Theorem 3.2 with $a = n + 1$, $b = 4$, and $c = n$ implies

$$P_{Z_1(\Gamma_{\mathbb{Z}_n})}(\lambda) = \left(\lambda^3 - 4\lambda^2 - \left((n-2)(n+1)^2 + n^2\right)\lambda + 4n^2\right)(\lambda - 4)^{\frac{n}{2}-3}(\lambda + 4)^{\frac{n}{2}-1}.$$

□

Theorem 3.12 *In $\Gamma_{\mathbb{Z}_n}$, the characteristic formula of $Z_2(\Gamma_{\mathbb{Z}_n})$ for even n is*

$$P_{Z_2(\Gamma_{\mathbb{Z}_n})}(\lambda) = \left(\lambda^3 - 4\lambda^2 - (4n-7)(n-1)^2)\lambda + 4(n-1)^2\right)(\lambda - 4)^{\frac{n}{2}-3}(\lambda + 4)^{\frac{n}{2}-1}.$$

Proof: Based on Theorems 2.2 and 2.4, and following Definition 2.6, we have the second Zagreb matrix of $\Gamma_{\mathbb{Z}_n}$ for even n as an $n \times n$ matrix as follows:

$$Z_2(\Gamma_{\mathbb{Z}_n}) = \begin{array}{c} \\ \overline{0} \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{\frac{n}{2}-1} \\ \overline{\frac{n}{2}} \\ \overline{\frac{n}{2}+1} \\ \vdots \\ \overline{n-2} \\ \overline{n-1} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \overline{0} & \overline{1} & \overline{2} & \dots & \overline{\frac{n}{2}-1} & \overline{\frac{n}{2}} & \overline{\frac{n}{2}+1} & \dots & \overline{n-2} & \overline{n-1} \end{array} \\ \begin{pmatrix} 0 & 2(n-1) & 2(n-1) & \dots & 2(n-1) & n-1 & 2(n-1) & \dots & 2(n-1) & 2(n-1) \\ 2(n-1) & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 4 \\ 2(n-1) & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 4 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2(n-1) & 0 & 0 & \dots & 0 & 0 & 4 & \dots & 0 & 0 \\ n-1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 2(n-1) & 0 & 0 & \dots & 4 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2(n-1) & 0 & 4 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 2(n-1) & 4 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array}.$$

Theorem 3.2 with $a = 2(n-1)$, $b = 4$, and $c = n-1$ implies

$$P_{Z_2(\Gamma_{\mathbb{Z}_n})}(\lambda) = \left(\lambda^3 - 4\lambda^2 - (4n-7)(n-1)^2)\lambda + 4(n-1)^2\right)(\lambda-4)^{\frac{n}{2}-3}(\lambda+4)^{\frac{n}{2}-1}.$$

□

4. Conclusions

This research has presented the energy of the identity graph for $\mathbb{Z}_n$ for odd n and the characteristic polynomial for even n corresponding with five degree-based matrices. We highlight that the obtained energies are hyperenergetic and are never an odd integer. In the future, it is possible to find the degree-based energy of the identity graph for $\mathbb{Z}_n$ for even n.

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