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An Approach to the Atomic Bond Connectivity Index for Graphs Under Transformations Fact Over Pendent Paths

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ABSTRACT: Graph theory is a dynamic tool for designing and modeling of an interconnection system by a network/graph. The processor nodes behave as the vertices and the connections between them behave as edges of such graph. The best use of system is decided by its topology. To characterize the topological aspects of underlying interconnection networks or graphs one of the most studied graph invariant is atom bomb connectivity index. To define new networks of our own choice the transformation of graph is an important tool. In this paper we will talk about the transformed family of graphs or networks. Let Ω be the connected graph of n vertices and $\Omega_n^{k,l}$ be made up by attaching the the k number of pendent paths with the fully connected vertices of the graph Ω. By applying the transformations A_α and A_α^β; $0 \leq \alpha \leq l-2$ $0 \leq \beta \leq k-1$ we get the transformed graphs $A_\alpha(\Omega_n^{k,l})$ and $A_\alpha^\beta(\Omega_n^{k,l})$ respectively. In this paper we derive new inequalities for the graph family $\Omega_n^{k,l}$ and transformed graphs $A_\alpha(\Omega_n^{k,l})$ and $A_\alpha^\beta(\Omega_n^{k,l})$ which involves $ABC(\Omega)$. The existence of $ABC(\Omega)$ made the inequalities more general than all formerly defined for ABC index. Additionally, we characterize extremal graphs which make the inequalities compact.

Keywords: Atom bond connectivity index, extremal graph, networks, equivalence classes, graph invariants, transformed graphs, pendent paths.

Contents

1. Introduction

The advancement in technology mainly networking, computer, biological and electrical networks made practicable the accurate data transfer within very small duration. The Internet, social media, biological, ecological and neural networks are few examples of such networks. Telecommunications based on interconnection networks are used to share data files. Similarly, data exchange using computing devices also based on computer network through data linkage; optical fiber cable (OFC) and wireless media such as Wi-Fi. Different algorithms used for directing, arranging/determining numerical calculations and image processing. Multiprocessor interconnection networks (MIN's) are used to design powerful microprocessors and memory chips [1,2].

Graph theory provides a fundamental tool for designing and analyzing such networks. Intuitively the interconnection system modified by graph with processor nodes behave as vertices and connection between them behave as edges of that graph. Graph theory and interconnection networks give a exhaustive understanding of these interrelated topics via their topology. The topology of a graph give information about the fashion by which vertices attached in the graph. The topological indices are graph invariants used to study the graph's topology . Apart from computer networks graph theory regards as a powerful tool in different areas of research, like in database management system, circuit design, secret sharing schemes, coding theory and theoretical chemistry [3]. The topological descriptors of various interconnection networks previously computed in [4,5,6]. Along with inter-connection networks, these invariants are equally important in Chemical graph theory which deals with problems in chemistry using associated graph of chemical compounds [7].

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The study of underlying substance using their graph with the help of graph invariants play an important role in chem-informatics, materials science, pharmaceutical sciences, engineering and so forth [8]. Among theoretical molecular descriptors, topological indices have an affect in chemistry due to the prediction of physio-chemical properties of that substance. Its role in QSPR/QSAR analysis to model physical and chemical properties of molecules is also remarkable [9,10,11]. Actually, topological indices are designed on the ground of transformation which associates a numeric value with the graph which characterizes its topology [12]. In 1947 Harold Wiener proposed first topological index given the name as Wiener index [13].It gives best connection with the boiling points of alkanes. The discovery of Winner index provides emerging research platform to the research community. Interest to maximize the accuracy in prediction of physio-chemical properties with practical results in Quantitative Structure property relationship (QSPR) analysis encouraged to define a large class of topological indices. For the first time, an index defined on the base of the degrees of end vertices edges by Milan Randić named Randić connectivity index [14], as

$$RA(\Omega) = \sum_{uv \in E(\Omega)} \frac{1}{\sqrt{deg_u deg_v}}.$$

Due to this reason, it attains a great attraction of the researchers till now. In 1998 Estrada et al. [15] introduced Atom Bomb Connectivity index

$$ABC(\Omega) = \sum_{st \in E(G)} \sqrt{\frac{d_s + d_t - 2}{d_s d_t}}.$$

The ABC index has excellent correlation with the heat of formation of alkanes [16,17]. Star graph among trees and complete graph in general for fixed number of vertices has maximal value of index. Bounds and extremal characterization of ABC index for underlying families of graphs studied at some extent in [18,19,20].It encouraged us to study ABC index for $\Omega_n^{k,l}$ and transformed graphs $A_\alpha(\Omega_n^{k,l})$ and $A_\alpha^\beta(\Omega_n^{k,l})$ under the fact of transformations A_α and A_α^β; $0 \le \alpha \le l-2$ $0 \le \beta \le k-1$ respectively as recently Muhammad Asif et al. [26] derived the bonds and extermal characterization of graphs and transformed graphs for GA index. We characterize extremal graphs for all of these families of graphs.

2. Title Material

Through out this work, let graph $\Omega_n^{k,l}$ comprises with n-vertex simple connected graph Ω along with k pendent paths of length $l \ge 2$ attached with $v \in \Omega$ having degree $d_v \ge 1$.The order of $\Omega_n^{k,l}$ is $n + kl$, size $m + kl$ and $deg_1 = \delta_\Omega \le deg_2 \le deg_3 \le ... \le \Delta_\Omega + 1$ be its degree sequence.

Let graph $\Omega = \Omega(V, E)$ with degree of vertex $u \in \Omega$, $\delta_\Omega \le d_u \le \Delta_\Omega$ and $\delta_\Omega \le d_v \le \Delta_\Omega + 1$ be the degree of $v \in \Omega_n^{k,l}$. For validity of our proved results we defined following list of useful graphs.

Type-I: Let $\delta_\Omega \le d_u \le \Delta_\Omega$ where $u \in V(\Omega)$. $\Omega_n^{k,l}$ of type-I obtained by attaching pendent paths of length l with vertices of degree $d_u = \Delta_\Omega$ in such a way that the vertices with pendent path are adjacent to the vertices with out pendent paths.

The graph of type-I shown in figure 1.

Type-II: $\Omega_n^{k,l}$ of type-II is the graph of type-I with pendent path attached to all vertices.

The graph of type-II shown in figure 2.

Theorem 2.1 *let graph $\Omega_n^{k,l}$ comprises with n-vertex simple connected graph Ω along with k pendent paths of length $l \ge 2$ attached with $v \in \Omega$ of degree d_v, maximum degree $\Delta_\Omega + 1$ and minimum δ_Ω. Then*

$$ABC(\Omega_n^{k,l}) \ge \frac{k\Delta_\Omega}{2}\left[\frac{\sqrt{2}(\Delta_\Omega + 1)l + \sqrt{2}\Delta_\Omega^{\frac{3}{2}}}{\Delta_\Omega(\Delta_\Omega + 1)} - \sqrt{\frac{\delta_\Omega + \Delta_\Omega - 2}{\delta_\Omega \Delta_\Omega}}\right] + ABC(\Omega).$$

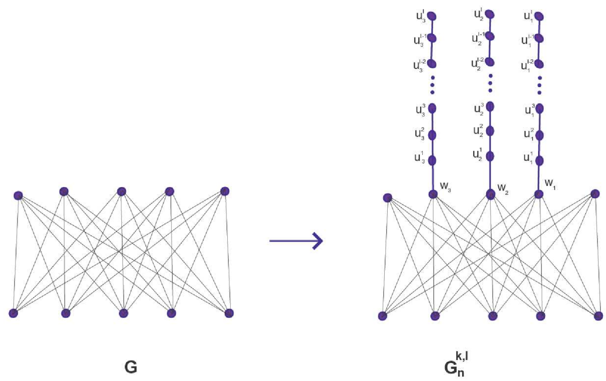

Figure 1: Graph $\Omega_n^{k,l}$ of type-I

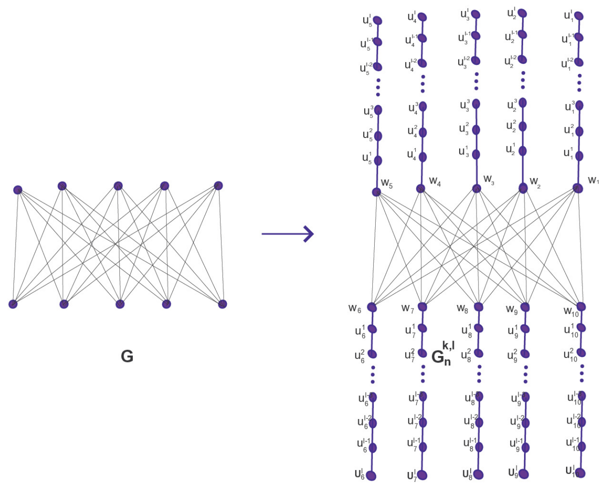

Figure 2: Graph $\Omega_n^{k,l}$ of type-II

Equality holds for graphs of type-II. and

$$ABC(\Omega_n^{k,l}) \leq k\left[\frac{l}{\sqrt{2}} + \Delta_\Omega\sqrt{\frac{2\delta_\Omega - 1}{\delta_{\Omega(\delta_\Omega+1)}}} - \frac{\delta_\Omega}{\Delta_\Omega}\sqrt{2\Delta_\Omega - 2}\right] + ABC(\Omega).$$

Equality holds for graph of type-I.

Proof: let a simple graph Ω of order n, Size m, maximum degree Δ_Ω and minimum δ_Ω . $\Omega_n^{k,l}$ be the graph formed by k number of paths having length l pendent at distinct vertices $u \in \Omega$ such that $1 \leq d_u \leq \Delta_\Omega$. The Atom Bomb Connectivity index of any graph Ω is

$$ABC(\Omega) = \sum_{st\in E(\Omega)} \sqrt{\frac{d_s + d_t - 2}{d_s d_t}}.$$

The construction of $\Omega_n^{k,l}, l \geq 2$ implies $|E(\Omega_n^{k,l})| = m + kl$ and for $st \in E(\Omega_n^{k,l})$ $(d_s + d_t) \in \{3, 4, d_u + 2, d_u + d_v, d_u + d_v + 1\}$. The edge set of $\Omega_n^{k,l}$ partitioned as $A_3 = \{st \in \Omega_n^{k,l} : d_s = 1, d_t = 2\}$, $A_4 = \{st \in \Omega_n^{k,l} : d_s = d_t = 2\}$, $A_{d_u+2} = \{st \in \Omega_n^{k,l} : d_s = d_u + 1, \delta_\Omega \leq d_u \leq \Delta_\Omega, d_t = 2\}$, $A_{d_u+d_v} = \{st \in \Omega_n^{k,l} : \delta_\Omega \leq$

$d_s = d_u, d_t = d_v \leq \Delta_\Omega\}$ and $A_{d_u+d_v+1} = \{st \in \Omega_n^{k,l} : \delta_\Omega \leq d_s = d_u, d_t = d_v + 1, d_v \leq \Delta_\Omega\}$.

$$ABC(\Omega_n^{k,l}) = \sum_{\substack{st\ are\ edges\ of\\ pendent\ paths}} \sqrt{\frac{d_s + d_t - 2}{d_s d_t}} + \sum_{\substack{st\ are\ edges\\ of\ \Omega}} \sqrt{\frac{d_s + d_t - 2}{d_s d_t}}.$$

$$\begin{aligned} ABC(\Omega_n^{k,l}) &= \sum_{st\in A_3} \sqrt{\frac{d_s + d_t - 2}{d_s d_t}} + \sum_{st\in A_4} \sqrt{\frac{d_s + d_t - 2}{d_s d_t}} + \sum_{st\in A_{d_u+2}} \sqrt{\frac{d_s + d_t - 2}{d_s d_t}} \\ &+ \sum_{st\in A_{d_u+d_v+1}} \sqrt{\frac{d_s + d_t - 2}{d_s d_t}} + \sum_{st\in A_{d_u+d_v}} \sqrt{\frac{d_s + d_t - 2}{d_s d_t}} \end{aligned} \tag{2.1}$$

The construction of $(\Omega_n^{k,l})$ implies that the cardinality of A_3 is k $i.e|A_3| = k, |A_4| = k(l-2)$, $|A_{d_u+2}| = k$, $|A_{d_u+d_u+1}| \leq k\Delta_\Omega$ and $|A_{d_u+d_u}| \leq k\Delta_\Omega$. The function $f(x) = \sqrt{\frac{a+x-2}{ax}}$ is deceasing, where $a \leq x$ is a constant . So, for δ_Ω minimum degree of vertices of Ω and maximum degree Δ_Ω , we have, $\sqrt{\frac{2+d_s+1-2}{2(d_s+1)}} = \frac{1}{\sqrt{2}}$, $\sqrt{\frac{(\Delta_\Omega+1)+(\Delta_\Omega+1)-2}{(\Delta_\Omega+1)(\Delta_\Omega+1)}} \leq \sqrt{\frac{d_s+1+d_t-2}{(d_s+1)d_t}}$ and $\sqrt{\frac{d_s+d_t-2}{d_s.d_t}} \geq ABC(\Omega) - \frac{\Delta_\Omega k}{2}\sqrt{\frac{\delta_\Omega+\Delta_\Omega-2}{\delta_\Omega\Delta_\Omega}}$. From equation 2.1 we have

$$ABC(\Omega_n^{k,l}) \geq \frac{k}{\sqrt{2}} + \frac{k(l-2)}{\sqrt{2}} + \frac{k}{\sqrt{2}} + \frac{k\Delta_\Omega\sqrt{2\Delta_\Omega}}{2(\Delta_\Omega+1)} + ABC(\Omega) - \frac{k\Delta_\Omega}{2}\sqrt{\frac{\delta_\Omega + \Delta_\Omega - 2}{\delta_\Omega\Delta_\Omega}}.$$

after simplification we get,

$$ABC(\Omega_n^{k,l}) \geq \frac{k\Delta_\Omega}{2}\left[\frac{\sqrt{2}(\Delta_\Omega+1)l + \sqrt{2}\Delta_\Omega^{\frac{3}{2}}}{\Delta_\Omega(\Delta_\Omega+1)} - \sqrt{\frac{\delta_\Omega + \Delta_\Omega - 2}{\delta_\Omega\Delta_\Omega}}\right] + ABC(\Omega). \tag{2.2}$$

Now, again set $\sqrt{\frac{2+d_s+1-2}{2(d_s+1)}} = \frac{1}{\sqrt{2}}$, $\sqrt{\frac{d_s+1+d_t-2}{(d_s+1)d_t}} \leq \sqrt{\frac{\delta_\Omega+(\delta_\Omega+1)-2}{\delta_\Omega(\delta_\Omega+1)}}$ and $\sum_{st\in A_{d_u+d_v}} \sqrt{\frac{d_s+d_t-2}{d_s d_t}} \leq ABC(\Omega) - k\Delta_\Omega\sqrt{\frac{\Delta_\Omega+\Delta_\Omega-2}{\Delta_\Omega\Delta_\Omega}}$ by the characteristics of $f(x) = \sqrt{\frac{x+y-2}{xy}}$ in equation 2.1 , we get following inequality.

$$ABC(\Omega_n^{k,l}) \leq \frac{k}{\sqrt{2}} + \frac{k(l-2)}{\sqrt{2}} + \frac{k}{\sqrt{2}} + k\Delta_\Omega\sqrt{\frac{2\delta_\Omega - 1}{\delta_\Omega(\delta_\Omega+1)}} + ABC(\Omega) - \frac{k\delta_\Omega}{\Delta_\Omega}\sqrt{2\Delta_\Omega - 2}.$$

after simplification we get,

$$ABC(\Omega_n^{k,l}) \leq k\left[\frac{l}{\sqrt{2}} + \Delta_\Omega\sqrt{\frac{2\delta_\Omega - 1}{\delta_{\Omega(\delta_\Omega+1)}}} - \frac{\delta_\Omega}{\Delta_\Omega}\sqrt{2\Delta_\Omega - 2}\right] + ABC(\Omega). \tag{2.3}$$

The inequalities 2.2 and 2.3 completes the proof. □

The 2.1 shows generalization of the above defined inequalities. One can get more inequalities of their desire by replacing $ABC(\Omega)$ with already defined bonds of ABC index.

Corollary 2.1 *let graph $\Omega_n^{k,l}$ comprises with n-vertex simple connected graph Ω along with k pendent paths of length $l \geq 2$ attached with $v \in \Omega$ of degree d_v, maximum degree $\Delta_\Omega + 1$ and minimum δ_Ω. Then*

$$ABC(\Omega_n^{k,l}) \geq \frac{k\Delta_\Omega}{2}\left[\frac{\sqrt{2}(\Delta_\Omega+1)l + \sqrt{2}\Delta_\Omega^{\frac{3}{2}}}{\Delta_\Omega(\Delta_\Omega+1)} - \sqrt{\frac{\delta_\Omega + \Delta_\Omega - 2}{\delta_\Omega\Delta_\Omega}}\right] + m\frac{\sqrt{2\Delta_\Omega - 2}}{\Delta_\Omega}.$$

Equality holds for regular graph of the type-II

$$ABC(\Omega_n^{k,l}) \leq k\left[\frac{l}{\sqrt{2}} + \Delta_\Omega\sqrt{\frac{2\delta_\Omega - 1}{\delta_{\Omega(\delta_\Omega+1)}}} - \frac{\delta_\Omega}{\Delta_\Omega}\sqrt{2\Delta_\Omega - 2}\right] + m\sqrt{\frac{\delta_\Omega + \Delta_\Omega - 2}{\delta_\Omega \Delta_\Omega}}.$$

Equality holds for regular graph of the type-I.

Proof: Using results of theorem 2.1 and Inequality regarding ABC index proved in as,

$$m\frac{\sqrt{2\Delta_\Omega - 2}}{\Delta_\Omega} \leq ABC(\Omega) \leq m\sqrt{\frac{\delta_\Omega + \Delta_\Omega - 2}{\delta_\Omega \Delta_\Omega}}$$

we get desired results.

□

3. Graph Transformations

Let $H(\Omega) \subset E(\Omega)$, the $\Omega^{'} = \Omega - H$ be the new graph generated by removing set edges of $H(\Omega)$ and $\Omega'' = \Omega - V_1(\Omega)$ be the new graph generated by deleting set of vertices $V_1(\Omega) \subset V(\Omega)$. We use following transformations as used in [25]. These transformations have solid effect over ABC of $\Omega_n^{k,l}$.

Transformation A:

Let $w_j \in V(\Omega), d_{w_j} \geq 1$ for $1 \leq j \leq k \leq n$ and paths pendent at w_j of the form $\{w_j u_j^1, u_j^1 u_j^2, u_j^2 u_j^3,$ $..., u_j^{l-1} u_j^l\}$ comprises $\Omega_n^{k,l}$. Then

$$A(\Omega_n^{k,l}) = \Omega_n^{k,l} - \sum_{j=1}^{k}\{u_j^2 u_j^3, u_j^3 u_j^4 ..., u_j^{l-1} u_j^l\} + \sum_{j=1}^{k}\{w_j u_j^2, u_j^2 u_j^3, ..., u_j^{l-1} u_j^l\}$$

The transformation A shown in figure 3.

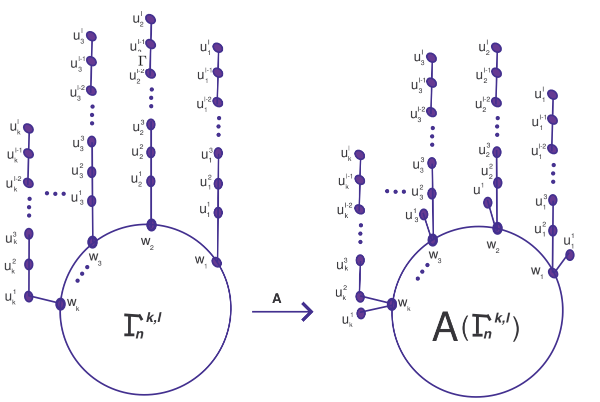

Figure 3: figure
Transformation A

In theorem 3.1 we discuss the effect of transformation A over ABC index.

Theorem 3.1 *let graph $\Omega_n^{k,l}$ comprises with n-vertex simple connected graph Ω along with k pendent paths of length $l \geq 2$ attached with $v \in \Omega$ of degree $d_v \geq 2$, maximum degree of $v \in \Omega_n^{k,l}$ is $\Delta_\Omega + 1$ and minimum δ_Ω. Then*

$$ABC(A_\alpha(\Omega_n^{k,l})) \geq \frac{k\Delta_\Omega}{2}\left[\frac{\sqrt{2}(\Delta_\Omega+\alpha+1)(l-\alpha)+\alpha\sqrt{(\Delta_\Omega+\alpha)(\Delta_\Omega+\alpha+1)}+\Delta_\Omega\sqrt{2(\Delta_\Omega+\alpha)}}{\Delta_\Omega(\Delta_\Omega+\alpha+1)}\right]- \frac{k\Delta_\Omega}{2}\sqrt{\frac{\Delta_\Omega+\delta_\Omega-2}{\Delta_\Omega\delta_\Omega}}+ABC(\Omega).$$

Equality hold for all graphs of type-II.

$$ABC(A_\alpha(\Omega_n^{k,l})) \leq k\left[\frac{l-\alpha}{\sqrt{2}}+\frac{\alpha\sqrt{\delta_\Omega(\delta_\Omega+\alpha)}+\Delta_\Omega\sqrt{(2\delta_\Omega+\alpha-1)}}{\sqrt{\delta_\Omega(\delta_\Omega+\alpha+1)}}-\frac{\delta_\Omega\sqrt{2\Delta_\Omega-2}}{\Delta_\Omega}\right]+ABC(\Omega).$$

Equality holds for all graphs of the type-I.

Proof: let a simple graph Ω of order n, Size m, minimum degree δ_Ω and maximum Δ_Ω . $\Omega_n^{k,l}$ be the graph formed by k number of paths of length l pendent at distinct fully connected vertices of Ω. The Atom bomb connectivity index of any graph Ω is

$$ABC(\Omega)=\sum_{st\in E(G)}\sqrt{\frac{d_s+d_t-2}{d_sd_t}}.$$

The construction of $\Omega_n^{k,l}, l\geq 2$ implies $|E(\Omega_n^{k,l})|=m+kl$. After successive applications of transformation A as A_α, $\alpha\leq l-1$ the edge set of $A_\alpha(\Omega_n^{k,l})$ partitioned as $E_{(d_s+d_t)}(A_\alpha(\Omega_n^{k,l}))$, $(d_s+d_t)\in\{3,4,d_u+\alpha+2,d_u+\alpha+3,d_u+d_v,d_u+\alpha+1+d_v\}$.

$$\begin{aligned}
E_3(A_\alpha(\Omega_n^{k,l})) &= \{st\in\Omega_n^{k,l}: d_s=1, d_t=2\}\\
E_4(A_\alpha(\Omega_n^{k,l})) &= \{st\in\Omega_n^{k,l}: d_s=d_t=2\}\\
E_{d_u+\alpha+2}(A_\alpha(\Omega_n^{k,l})) &= \{st\in\Omega_n^{k,l}: \delta_\Omega\leq d_s=d_u+\alpha+1, d_u\leq\Delta_\Omega, d_t=1\}\\
E_{d_u+\alpha+3}(A_\alpha(\Omega_n^{k,l})) &= \{st\in\Omega_n^{k,l}: \delta_\Omega\leq d_s=d_u+\alpha+1, d_u\leq\Delta_\Omega, d_t=2\}\\
E_{d_u+\alpha+1}(A_\alpha(\Omega_n^{k,l})) &= \{st\in\Omega_n^{k,l}: \delta_\Omega\leq d_s=d_u+\alpha, d_u\leq\Delta_\Omega, d_t=1\}\\
E_{d_u+d_v}(A_\alpha(\Omega_n^{k,l})) &= \{st\in\Omega_n^{k,l}: \delta_\Omega\leq d_s=d_u, d_t=d_v\leq\Delta_\Omega\}\\
E_{d_u+\alpha+1+d_v}(A_\alpha(\Omega_n^{k,l})) &= \{st\in\Omega_n^{k,l}: \delta_\Omega\leq d_v, d_u\leq\Delta_\Omega, d_s=d_u+\alpha+1, d_t=d_v\}.
\end{aligned}$$

$$ABC(A_\alpha(\Gamma_n^{k,l})) = \sum_{\substack{E_{(deg_s+deg_t)}(A_\alpha(\Gamma_n^{k,l}))\\ \subseteq E(A_\alpha(\Gamma_n^{k,l}))}}\ \sum_{\substack{st\in\\ E_{(deg_s+deg_t)}(A_\alpha(\Gamma_n^{k,l}))}}\sqrt{\frac{d_s+d_t-2}{d_sd_t}} \tag{3.1}$$

The cardinality of A_3 is k *i.e*$|E_3(A_\alpha(\Omega_n^{k,l}))|=k, |E_4(A_\alpha(\Omega_n^{k,l}))|=k(l-\alpha-2)$, $|E_{d_u+\alpha+2}(A_\alpha(\Omega_n^{k,l}))|=k\alpha$ and $|E_{d_u+\alpha+3}(A_\alpha(\Omega_n^{k,l}))|=k$. The function $f(x)=\sqrt{\frac{a+x-2}{ax}}$ is deceasing, where $a\leq x$ is a constant . So, for δ_Ω minimum degree of Ω and Δ_Ω maximum, for any graph $\sqrt{\frac{2+d_s+1+\alpha-2}{2(d_s+1+\alpha)}}=\frac{1}{\sqrt{2}}$, $\sqrt{\frac{1+d_s+1+\alpha-2}{1(d_s+1+\alpha)}}\geq\sqrt{\frac{1+\Delta_\Omega+\alpha+1-2}{1(\Delta_\Omega+\alpha+1)}}$, $\sqrt{\frac{(\Delta_\Omega+\alpha+1)+(\Delta_\Omega+\alpha+1)-2}{(\Delta_\Omega+\alpha+1)(\Delta_\Omega+\alpha+1)}}\leq\sqrt{\frac{(d_s+\alpha+1)+d_t-2}{(d_s+\alpha+1)d_t}}$ and $\sum_{st\in A_{d_u+d_v}}\sqrt{\frac{d_s+d_t-2}{d_sd_t}}\geq ABC(\Omega)-\frac{k\Delta_\Omega}{2}\sqrt{\frac{\Delta_\Omega+\delta_\Omega-2}{\Delta_\Omega\delta_\Omega}}$. Substituting these changes in equation 3.1, we have following inequality

$$\begin{aligned}
ABC(A_\alpha(\Omega_n^{k,l})) \geq{}& \frac{k}{\sqrt{2}}+\frac{k(l-\alpha-2)}{\sqrt{2}}+k\alpha\sqrt{\frac{1+\Delta_\Omega+\alpha+1-2}{1(\Delta_\Omega+\alpha+1)}}+\frac{k}{\sqrt{2}}\\
&+\frac{k\Delta_\Omega}{2}\sqrt{\frac{(\Delta_\Omega+\alpha+1)+(\Delta_\Omega+\alpha+1)-2}{(\Delta_\Omega+\alpha+1)(\Delta_\Omega+\alpha+1)}}+ABC(\Omega)-\frac{k\Delta_\Omega}{2}\sqrt{\frac{\Delta_\Omega+\delta_\Omega-2}{\Delta_\Omega\delta_\Omega}}.
\end{aligned}$$

after simplification, we get required result

$$ABC(A_\alpha(\Omega_n^{k,l})) \geq \frac{k\Delta_\Omega}{2}\left[\frac{\sqrt{2}(\Delta_\Omega+\alpha+1)(l-\alpha)+\alpha\sqrt{(\Delta_\Omega+\alpha)(\Delta_\Omega+\alpha+1)}+\Delta_\Omega\sqrt{2(\Delta_\Omega+\alpha)}}{\Delta_\Omega(\Delta_\Omega+\alpha+1)}\right] + \tag{3.2}$$

$$\frac{k\Delta_\Omega}{2}\sqrt{\frac{\Delta_\Omega+\delta_\Omega-2}{\Delta_\Omega\delta_\Omega}}+ABC(\Omega). \tag{3.3}$$

Now, again from equation 3.1 and inequalities $\sqrt{\frac{2+d_s+1+\alpha-2}{2(d_s+1+\alpha)}}=\frac{1}{\sqrt{2}}$, $\sqrt{\frac{1+d_s+1+\alpha-2}{1(d_s+1+\alpha)}} \leq \sqrt{\frac{1+\delta_\Omega+\alpha+1-2}{1(\delta_\Omega+\alpha+1)}}$, $\sqrt{\frac{(\delta_\Omega)+(\delta_\Omega+\alpha+1)-2}{(\delta_\Omega)(\delta_\Omega+\alpha+1)}} \geq \sqrt{\frac{(d_s+\alpha+1)+d_t-2}{(d_s+\alpha+1)d_t}}$ and $\sum_{st\in A_{d_u+d_v}}\sqrt{\frac{d_s+d_t-2}{d_sd_t}} \leq ABC(\Omega)-\frac{k\delta_\Omega}{\Delta_\Omega}\sqrt{2\Delta_\Omega-2}$.

$$\begin{aligned} ABC(A_\alpha(\Omega_n^{k,l})) &\leq \frac{k}{\sqrt{2}}+\frac{k(l-\alpha-2)}{\sqrt{2}}+k\alpha\sqrt{\frac{1+\delta_\Omega+\alpha+1-2}{1(\delta_\Omega+\alpha+1)}}+\frac{k}{\sqrt{2}}+k\Delta\sqrt{\frac{(\delta_\Omega)+(\delta_\Omega+\alpha+1)-2}{(\delta_\Omega)(\delta_\Omega+\alpha+1)}} \\ &+ ABC(\Omega)-\frac{k\delta_\Omega}{\Delta_\Omega}\sqrt{2\Delta_\Omega-2}. \end{aligned}$$

After simplification we get,

$$ABC(A_\alpha(\Omega_n^{k,l})) \leq k\left[\frac{l-\alpha}{\sqrt{2}}+\frac{\alpha\sqrt{\delta_\Omega(\delta_\Omega+\alpha)}+\Delta_\Omega\sqrt{(2\delta_\Omega+\alpha-1)}}{\sqrt{\delta_\Omega(\delta_\Omega+\alpha+1)}}-\frac{\delta_\Omega\sqrt{2\Delta_\Omega-2}}{\Delta_\Omega}\right]+ABC(\Omega) \tag{3.4}$$

The equations 3.2 and 3.4 completes the proof. □

Transformation B:

Let $w_j \in V(\Omega), d_{w_j} \geq 1$ for $1 \leq j \leq k \leq n$ and paths pendent at w_j of the form $\{w_ju_j^1, u_j^1u_j^2, u_j^2u_j^3, ..., u_j^{l-1}u_j^l\}$ comprises $\Omega_n^{k,l}$. Then for fixed vertex w_1

$$B(\Omega_n^{k,l}) = \Omega_n^{k,l}-\{u_j^1u_j^2, u_j^2u_j^3, ..., u_j^{l-1}u_j^l\}+\{w_1u_j^1, u_j^1u_j^2, u_j^2u_j^3, ..., u_j^{l-1}u_j^l\}$$

The transformation B shown in figure 4.

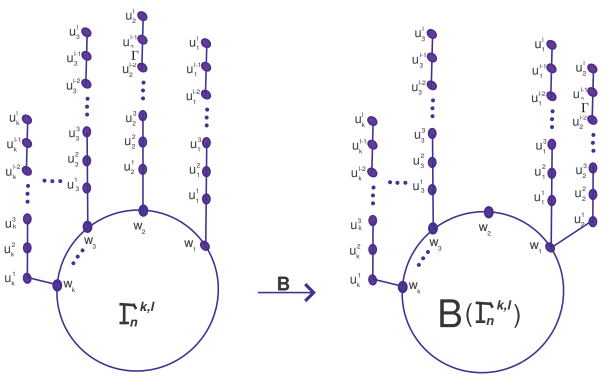

Figure 4: figure
Transformation B for fixed vertex w_1

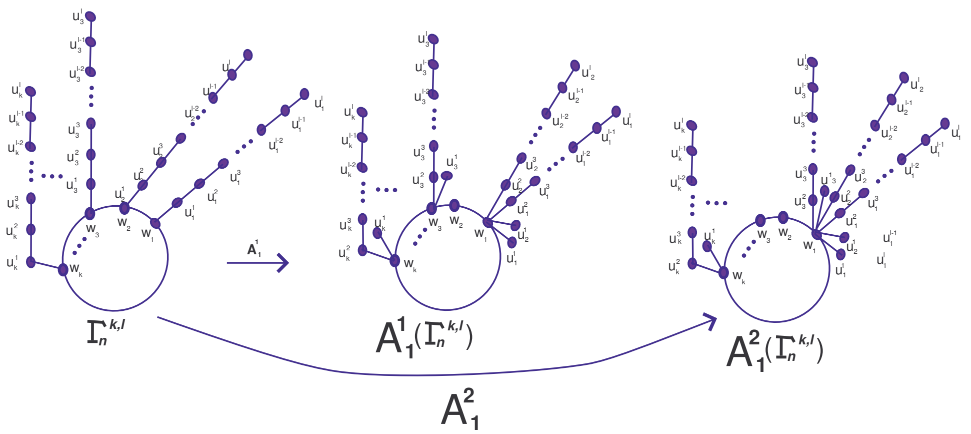

Figure 5: figure
Transformation A_α^β

Transformation A_α^β

Let $0 \leq \alpha \leq l-1$ and $0 \leq \beta \leq k-1$. The transformation A_α^β is the composition of successive applications of transformation A and B as A_α and B_β respectively [25].

In theorem 3.2 we discuss the effect of transformation A_α^β over ABC index.

Theorem 3.2 *let graph $\Omega_n^{k,l}$ comprises with n-vertex simple connected graph Ω along with k pendent paths of length $l \geq 2$ attached with $v \in \Omega$ of degree d_v, maximum degree of $v \in \Omega_n^{k,l}$ is $\Delta_\Omega + 1$ and minimum δ_Ω. Then*

$$\begin{aligned} ABC(A_\alpha^\beta(\Omega_n^{k,l})) \quad & \geq \frac{(k-\beta-1)\sqrt{\Delta_\Omega+\alpha}}{\Delta_\Omega+\alpha+1}\left(\alpha\sqrt{\Delta_\Omega+\alpha+1}+\frac{\Delta_\Omega}{\sqrt{2}}\right)+\frac{1}{2\sqrt{\Delta_\Omega+(\alpha+1)(\beta+1)}} \\ & \left(2\alpha(\beta+1)\sqrt{\Delta_\Omega+(\alpha+1)(\beta+1)-1}+\frac{\Delta_\Omega\sqrt{2\Delta_\Omega+(\alpha+1)(\beta+1)-2}}{\sqrt{\Delta_\Omega+\alpha+1}}\right) \\ + \quad & \frac{k(l-\alpha)}{\sqrt{2}}-\frac{\Delta_\Omega(k-\beta)}{2}\sqrt{\frac{\Delta_\Omega+\delta_\Omega-2}{\Delta_\Omega\delta_\Omega}}+ABC(\Omega) \end{aligned}$$

Equality holds for graph of the type-II with $\alpha = \alpha$ and $\beta = 0$.

$$\begin{aligned} ABC(A_\alpha^\beta(\Omega_n^{k,l})) \quad & \leq \frac{(k-\beta-1)}{\sqrt{\delta_\Omega+\alpha+1}}\left(\alpha\sqrt{\delta_\Omega+\alpha}+\Delta_\Omega\sqrt{\frac{2\delta_\Omega+\alpha-1}{\delta_\Omega}}\right) \\ + \quad & \frac{1}{\sqrt{\delta_\Omega+(\alpha+1)(\beta+1)}}\left(\alpha(\beta+1)\sqrt{\delta_\Omega+(\alpha+1)(\beta+1)-1}+\Delta_\Omega\sqrt{\frac{2\delta_\Omega+(\alpha+1)(\beta+1)-2}{\delta_\Omega}}\right) \\ + \quad & \frac{k(l-\alpha)}{\sqrt{2}}-\frac{\delta_\Omega(k-\beta)}{2}\sqrt{2(\Delta_\Omega-1)}+ABC(\Omega). \end{aligned}$$

Equality holds for graph of the type-I with $\alpha = \alpha$ and $\beta = 0$.

Proof: let a simple graph Ω of order n, Size m, minimum degree δ_Ω and maximum Δ_Ω . $\Omega_n^{k,l}$ be the graph formed by k number of paths of length l pendent at distinct fully connected vertices of Ω. The Atom Bomb Connectivity index of any graph Ω is

$$ABC(\Omega) = \sum_{st\in E(G)} \sqrt{\frac{d_s + d_t - 2}{d_s d_t}}.$$

The construction of $\Omega_n^{k,l}, l \geq 2$ implies $|E(\Omega_n^{k,l})| = m + kl$. Let u^* be the fixed vertex. Applications of transformation A_α^β has an effect over the edge set partition as $E_{(d_s+d_t)}(A_\alpha^\beta(\Omega_n^{k,l}))$, $(d_s+d_t) \in \{3, 4, d_u+\alpha+2, d_u+\alpha+3, d_u+d_v, d_u+\alpha+1+d_v, d_{u^*}+(\beta+1)(\alpha+1)+d_v, d_{u^*}+(\beta+1)(\alpha+1)+1, d_{u^*}+(\beta+1)(\alpha+1)+2\}$.

$$\begin{aligned}
E_3(A_\alpha^\beta(\Omega_n^{k,l})) &= \{st \in \Omega_n^{k,l} : d_s = 1, d_t = 2\}\\
E_4(A_\alpha^\beta(\Omega_n^{k,l})) &= \{st \in \Omega_n^{k,l} : d_s = d_t = 2\}\\
E_{d_u+\alpha+2}(A_\alpha^\beta(\Omega_n^{k,l})) &= \{st \in \Omega_n^{k,l} : \delta_\Omega \leq d_s = d_u + \alpha + 1, d_u \leq \Delta_\Omega, d_t = 1\}\\
E_{d_u+\alpha+3}(A_\alpha^\beta(\Omega_n^{k,l})) &= \{st \in \Omega_n^{k,l} : \delta_\Omega \leq d_s = d_u + \alpha + 1, d_u \leq \Delta_\Omega, d_t = 2\}\\
E_{d_u+d_v}(A_\alpha^\beta(\Omega_n^{k,l})) &= \{st \in \Omega_n^{k,l} : \delta_\Omega \leq d_s = d_u, d_t = d_v \leq \Delta_\Omega\}\\
E_{d_u+\alpha+1+d_v}(A_\alpha^\beta(\Omega_n^{k,l})) &= \{st \in \Omega_n^{k,l} : \delta_\Omega \leq d_v = d_t, d_u \leq \Delta_\Omega, d_s = d_u + \alpha + 1\}\\
E_{d_{u^*}+(\beta+1)(\alpha+1)+d_v}(A_\alpha^\beta(\Omega_n^{k,l})) &= \{st \in \Omega_n^{k,l} : \delta_\Omega \leq d_v = d_t, d_u \leq \Delta_\Omega, d_s = d_u + (\beta + 1)\\
&\quad\ (\alpha + 1)\}\\
E_{d_{u^*}+(\beta+1)(\alpha+1)+1}(A_\alpha^\beta(\Omega_n^{k,l})) &= \{st \in \Omega_n^{k,l} : \delta_\Omega \leq d_v, d_u \leq \Delta_\Omega, d_s = d_u + (\beta + 1)(\alpha + 1)\\
&, \quad d_t = 1\}\\
E_{d_{u^*}+(\beta+1)(\alpha+1)+2}(A_\alpha^\beta(\Omega_n^{k,l})) &= \{st \in \Omega_n^{k,l} : \delta_\Omega \leq d_v, d_u \leq \Delta_\Omega, d_s = d_u + (\beta + 1)(\alpha + 1)\\
&, \quad d_t = 2\}.
\end{aligned}$$

$$ABC(A_\alpha(\Omega_n^{k,l})) = \sum_{\substack{E_{(d_s+d_t)}(A_\alpha^\beta(\Omega_n^{k,l}))\\ \subseteq E(A_\alpha^\beta(\Omega_n^{k,l}))}} \sum_{\substack{st\in\\ E_{(d_s+d_t)}(A_\alpha^\beta(\Omega_n^{k,l}))}} \sqrt{\frac{d_s + d_t - 2}{d_s d_t}} \tag{3.5}$$

The cardinality of E_3 is k $i.e|E_3(A_\alpha^\beta(\Omega_n^{k,l}))| = k, |E_4(A_\alpha^\beta(\Omega_n^{k,l}))| = k(l-\alpha-2)$, $|E_{d_u+\alpha+2}(A_\alpha^\beta(\Omega_n^{k,l}))| = \alpha(k - \beta - 1)$, $|E_{d_u+\alpha+3}(A_\alpha^\beta(\Omega_n^{k,l}))| = k - \beta - 1$, $|E_{d_{u^*}+(\beta+1)(\alpha+1)+1}(A_\alpha^\beta(\Omega_n^{k,l}))| = \alpha(\beta + 1)$ and

$$|E_{d_{u^*}+(\beta+1)(\alpha+1)+2}(A_\alpha^\beta(\Omega_n^{k,l}))| = \beta + 1$$

.

The function $f(x) = \sqrt{\frac{a+x-2}{ax}}$ is deceasing, where $a \leq x$ is a constant . So, for δ_Ω minimum degree of Ω and

Δ_Ω maximum, we have, $\sqrt{\frac{2+(d_s+\alpha+1)-2}{2(d_s+\alpha+1)}} = \frac{1}{\sqrt{2}}$, $\sqrt{\frac{1+(\Delta_\Omega+\alpha+1)-2}{1(\Delta_\Omega+\alpha+1)}} \leq \sqrt{\frac{d_t+(d_s+\alpha+1)}{d_t(d_s+\alpha+1)}}$, $\sqrt{\frac{2+(d_s+(\alpha+1)(\beta+1))-2}{2(d_s+(\alpha+1)(\beta+1))}} = \frac{1}{\sqrt{2}}$, $\sqrt{\frac{1+(d_s+(\alpha+1)(\beta+1))-2}{1(d_s+(\alpha+1)(\beta+1))}} \geq \sqrt{\frac{1+(\Delta_\Omega+(\alpha+1)(\beta+1))-2}{1(\Delta_\Omega+(\alpha+1)(\beta+1))}}$, $\sqrt{\frac{(\Delta_\Omega+\alpha+1)+\Delta_\Omega+(\alpha+1)(\beta+1)-2}{(\Delta_\Omega+\alpha+1)(\Delta_\Omega+(\alpha+1)(\beta+1))}} \leq \sqrt{\frac{(d_t+\alpha+1)(d_s+(\alpha+1)(\beta+1))-2}{(d_t+\alpha+1)(d_s+(\alpha+1)(\beta+1))}}$ and

$$\sum_{st\in A_{d_u+d_v}} \sqrt{\frac{d_s + d_t - 2}{d_s d_t}} \geq ABC(\Omega) + \frac{\Delta_{\Omega(k-\beta-1)}}{2}\sqrt{\frac{(\Delta_\Omega + \alpha + 1) + (\Delta_\Omega + \alpha + 1) - 2}{(\Delta_\Omega + \alpha + 1)(\Delta_\Omega + \alpha + 1)}} - \Delta_\Omega(k - \beta)\sqrt{\frac{\Delta_\Omega + \delta_\Omega - 2}{\Delta_\Omega \delta_\Omega}}.$$

Substituting these changes in equation 3.5, we got the following inequality,

$$\begin{aligned} ABC(A_\alpha^\beta(\Omega_n^{k,l})) &\geq \frac{k}{\sqrt{2}} + \frac{k(l-\alpha-2)}{\sqrt{2}} + \frac{k-\beta-1}{\sqrt{2}} + \alpha(k-\beta-1)\sqrt{\frac{1+(\Delta_\Omega+\alpha+1)-2}{1(\Delta_\Omega+\alpha+1)}} \\ &+ \alpha(\beta+1)\sqrt{\frac{1+(\Delta_\Omega+(\alpha+1)(\beta+1))-2}{1(\Delta_\Omega+(\alpha+1)(\beta+1))}} + \frac{\beta+1}{\sqrt{2}} + \frac{\Delta_\Omega(k-\beta-1)}{2} \\ &\sqrt{\frac{(\Delta_\Omega+\alpha+1)+(\Delta_\Omega+\alpha+1)-2}{(\Delta_\Omega+\alpha+1)(\Delta_\Omega+\alpha+1)}} + \frac{\Delta_\Omega}{2}\sqrt{\frac{(\Delta_\Omega+\alpha+1)+(\Delta_\Omega+(\alpha+1)(\beta+1))-2}{(\Delta_\Omega+\alpha+1)(\Delta_\Omega+(\alpha+1)(\beta+1))}} \\ &+ ABC(\Omega) - \frac{\Delta_\Omega(k-\beta)}{2}\sqrt{\frac{\Delta_\Omega+\delta_\Omega-2}{\Delta_\Omega\delta_\Omega}} \end{aligned}$$

after simplification, we get required result

$$\begin{aligned} ABC(A_\alpha^\beta(\Omega_n^{k,l})) &\geq \frac{(k-\beta-1)\sqrt{\Delta_\Omega+\alpha}}{\Delta_\Omega+\alpha+1}\left(\alpha\sqrt{\Delta_\Omega+\alpha+1} + \frac{\Delta_\Omega}{\sqrt{2}}\right) + \frac{1}{2\sqrt{\Delta_\Omega+(\alpha+1)(\beta+1)}} \\ &\left(2\alpha(\beta+1)\sqrt{\Delta_\Omega+(\alpha+1)(\beta+1)-1} + \frac{\Delta_\Omega\sqrt{2\Delta_\Omega+(\alpha+1)(\beta+1)-2}}{\sqrt{\Delta_\Omega+\alpha+1}}\right) \\ &+ \frac{k(l-\alpha)}{\sqrt{2}} - \frac{\Delta_\Omega(k-\beta)}{2}\sqrt{\frac{\Delta_\Omega+\delta_\Omega-2}{\Delta_\Omega\delta_\Omega}} + ABC(\Omega) \end{aligned} \tag{3.6}$$

Now, again Substituting the following inequalities in equation 3.5,

$$\sqrt{\frac{2+(d_s+\alpha+1)-2}{2(d_s+\alpha+1)}} = \frac{1}{\sqrt{2}}, \sqrt{\frac{1+(\delta_\Omega+\alpha+1)-2}{1(\delta_\Omega+\alpha+1)}} \geq \sqrt{\frac{1+(d_s+\alpha+1)}{1(d_s+\alpha+1)}}$$
$$, \sqrt{\frac{2+(d_s+(\alpha+1)(\beta+1))-2}{2(d_s+(\alpha+1)(\beta+1))}} = \frac{1}{\sqrt{2}}, \sqrt{\frac{1+(d_s+(\alpha+1)(\beta+1))-2}{1(d_s+(\alpha+1)(\beta+1))}} \leq \sqrt{\frac{1+(\delta_\Omega+(\alpha+1)(\beta+1))-2}{1(\delta_\Omega+(\alpha+1)(\beta+1))}}$$
$$, \sqrt{\frac{\delta_\Omega+(\alpha+1)(\beta+1)+\delta_\Omega-2}{\delta_\Omega(\delta_\Omega+(\alpha+1)(\beta+1))}} \geq \sqrt{\frac{d_s+(\alpha+1)(\beta+1)+d_t-2}{(d_s+(\alpha+1)(\beta+1))d_t}}.$$

$$\sum_{st\in A_{d_u+d_v}}\sqrt{\frac{d_s+d_t-2}{d_sd_t}} \leq ABC(\Omega) + \Delta_{\Omega(k-\beta-1)}\sqrt{\frac{\delta_\Omega+(\delta_\Omega+\alpha+1)-2}{\delta_\Omega(\delta_\Omega+\alpha+1)}} - \delta_\Omega(k-\beta)\sqrt{\frac{\Delta_\Omega+\Delta_\Omega-2}{\Delta_\Omega\Delta_\Omega}}.$$

We get,

$$\begin{aligned} ABC(A_\alpha^\beta(\Omega_n^{k,l})) &\leq \frac{k}{\sqrt{2}} + \frac{k(l-\alpha-2)}{\sqrt{2}} + \frac{k-\beta-1}{\sqrt{2}} + \alpha(k-\beta-1)\sqrt{\frac{1+(\delta_\Omega+\alpha+1)-2}{1(\delta_\Omega+\alpha+1)}} \\ &+ \alpha(\beta+1)\sqrt{\frac{1+(\delta_\Omega+(\alpha+1)(\beta+1))-2}{1(\delta_\Omega+(\alpha+1)(\beta+1))}} + \frac{\beta+1}{\sqrt{2}} \\ &+ \Delta_\Omega(k-\beta-1)\sqrt{\frac{\delta_\Omega+(\delta_\Omega+\alpha+1)-2}{\delta_\Omega(\delta_\Omega+\alpha+1)}} + \Delta_\Omega\sqrt{\frac{\delta_\Omega+(\delta_\Omega+(\alpha+1)(\beta+1))-2}{\delta_\Omega[\delta_\Omega+(\alpha+1)(\beta+1)]}} \\ &+ ABC(\Omega) - \delta_\Omega(k-\beta)\sqrt{\frac{\Delta_\Omega+\Delta_\Omega-2}{\Delta_\Omega\Delta_\Omega}}. \end{aligned}$$

after simplification, we get required result

$$
\begin{aligned}
ABC(A_{\alpha}^{\beta}(\Omega_n^{k,l})) \quad & \leq \frac{(k-\beta-1)}{\sqrt{\delta_\Omega+\alpha+1}}\left(\alpha\sqrt{\delta_\Omega+\alpha}+\Delta_\Omega\sqrt{\frac{2\delta_\Omega+\alpha-1}{\delta_\Omega}}\right) \qquad (3.7)\\
&+ \frac{1}{\sqrt{\delta_\Omega+(\alpha+1)(\beta+1)}}\left(\alpha(\beta+1)\sqrt{\delta_\Omega+(\alpha+1)(\beta+1)-1}+\Delta_\Omega\sqrt{\frac{2\delta_\Omega+(\alpha+1)(\beta+1)-2}{\delta_\Omega}}\right)\\
&+ \frac{k(l-\alpha)}{\sqrt{2}}-\frac{\delta_\Omega(k-\beta)}{2}\sqrt{2(\Delta_\Omega-1)}+ABC(\Omega)
\end{aligned}
$$

The inequalities 3.6 and 3.7 completes the proof. □

4. Conclusion

The study of mathematical aspect regarding topological indices is a partially open problem [19,23,24] that for which members family of graphs, certain index has minimal or maximal value? In this work we discussed this fundamental question general graphs with pendent paths for the most studied index named Atom Bomb Connectivity index ABC and develop tight bounds by characterizing graphs. In theorem 3.1 and 3.2, we defined tight bonds for the transformed graphs under the effect of transformations defined in [25].

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