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# On Some Properties of Fibonacci Almost $\mathcal{I}$-Statistical Convergence of Fuzzy Variables in Credibility Space

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ABSTRACT: This paper explores several concepts of ideal and almost convergence for sequences of fuzzy variables within the context of credibility theory. Focusing on Fibonacci almost $\mathcal{I}$-statistical convergence, the study examines it from various perspectives, including almost surely, credibility, mean, distribution, and uniformly almost surely. The paper also investigates the interrelationships between these convergence notions. Additionally, different types of Fibonacci almost $\mathcal{I}$-statistical Cauchy sequences are analyzed in the framework of credibility theory, leading to significant findings.

Key Words: Credibility measure, credibility theory, statistical convergence, ideal convergence.

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## 1. Introduction

Studies on sequences that almost converge have been extensively explored by scholars like Lorentz [19], who made significant contributions to the concept of strong almost convergence. Additional studies on almost convergence can be found in references [3, 7, 11, 20, 22].

Fuzzy theory, introduced by Zadeh [33] provides a flexible mathematical framework for tackling various real-world problems. This framework encompasses possibility theory, which uses fuzzy variables to represent uncertainty and reliability. Kaufmann [10] expanded on fuzzy variables, their distributions, and membership functions. The self-dual measure, essential in credibility theory, combines uncertainty and reliability, offering a powerful tool for comprehensive modeling. Its duality ensures a well-rounded approach to both theoretical analysis and practical applications, influencing fields like finance, engineering, and artificial intelligence.

The self-dual measure enhances credibility theory by refining the analysis of fuzzy events and broadening its practical applicability across multiple fields. Although credibility and uncertainty theories have distinct focuses, they converge in their examination of sequence convergence. Liu and Liu [16] introduced a self-dual measure for credibility theory, which shares similarities with possibility measures in the fuzzy context. Liu [18] also defined various convergence concepts for fuzzy variables, such as almost sure convergence, convergence in credibility, convergence in mean, and convergence in distribution. Convergence concepts in classical measure theory, credibility theory, and probability theory have been thoroughly studied, with valuable contributions from Chen et al. [2] and You [32].

The exploration extends to statistical convergence, initially introduced by Fast [4] as a generalization of ordinary convergence for real sequences. Ideal convergence, which generalizes classical and statistical convergence, was examined by Kostyrko et al. [13] and later further investigated by Savaş and Das [24], and Savaş et al. [26]. For other studies on statistical and ideal convergence, references [5,6,21,25,27,30,31] can be recommended.

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In the field of sequence theory, Kara and Başarir [9] were pioneers in applying the Fibonacci sequence. The Fibonacci numbers exhibit intriguing properties and find applications in various domains such as art, science, and architecture. Many mathematicians, including Başarir et al. [1], have published influential papers on Fibonacci matrices, which have served as inspiration for this study. Additionally, Saha and Tripathy [23] introduced the concept of Fibonacci statistical convergence for sequences of complex uncertain variables.

This work contributes to the field by introducing a novel form of convergence for sequences of fuzzy variables. Section 2 revisits essential definitions and theorems in credibility theory, laying the groundwork for the subsequent exploration of fuzzy variable sequences and credibility space. Section 3 is earmarked for an in-depth investigation into the notion of Fibonacci almost $\mathcal{I}$-statistical convergence of fuzzy variables, with the aim of constructing fundamental properties related to almost convergence in credibility.

## 2. Preliminaries

The credibility measure, denoted as Cr, is deemed a credibility measure if the subsequent axioms are supplied: Let $\Theta$ be a nonempty set, and $\mathcal{P}(\Theta)$ the power set of $\Theta$ (i.e., the larggest algebra over $\Theta$). Each element in $\mathcal{P}$ is referred to as an event. For any $A \in \mathcal{P}(\Theta)$, a credibility measure $\mathrm{Cr}\{A\}$ was presented by Liu and Liu [17] to express the chance that the fuzzy event $A$ occurs. It was proven by Li and Liu [14] that a set function $\mathrm{Cr}\{.\}$ is a credibility measure if the following axioms hold:

Axiom i. $\mathrm{Cr}\{\Theta\} = 1$;

Axiom ii. $\mathrm{Cr}\{A\} \leq \mathrm{Cr}\{B\}$ whenever $A \subset B$;

Axiom iii. Cr is self-dual, i.e., $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1$, for any $A \in \mathcal{P}(\Theta)$;

Axiom iv. $\mathrm{Cr}\{\cup_i A_i\} = \sup_i \mathrm{Cr}\{A_i\}$ for any collection $\{A_i\}$ in $\mathcal{P}(\Theta)$ with

$$\sup_i \mathrm{Cr}\{A_i\} < 0.5.$$

The triple $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ is termed a credibility space. A fuzzy variable is investigated as a function from the credibility space to the set of real numbers by Liu and Liu [17].

**Definition 2.1** *( [17]) The expected value of fuzzy variable $\mu$ expressed as:*

$$E[\mu] = \int_0^{+\infty} \mathrm{Cr}\{\mu \geq r\}\, dr - \int_{-\infty}^{0} \mathrm{Cr}\{\mu \leq r\}\, dr.$$

*This expression holds true provided that at least one of the two integrals is finite.*

Wang and Liu [29] demonstrated that convergence in credibility implies convergence almost surely for a sequence of fuzzy variables. Additionally, Liu [15] investigated that convergence in mean implies convergence in credibility.

**Theorem 2.1** *Let $\mu$ be a fuzzy variable. Then, for any given numbers $t > 0$ and $p > 0$, we have*

$$Cr\{|\mu| \geq t\} \leq \frac{E[|\mu|^p]}{t^p}.$$

The topic of Fibonacci numbers has been extensively discussed in numerous articles and books [8, 12]. The Fibonacci sequence $\{\mathfrak{f}_\mathfrak{m}\}_{\mathfrak{m}=1}^{\infty}$ defined by the linear recurrence equations,

$$\mathfrak{f}_0 = 0 \text{ and } \mathfrak{f}_1 = 1, \mathfrak{f}_\mathfrak{m} = \mathfrak{f}_{\mathfrak{m}-1} + \mathfrak{f}_{\mathfrak{m}-2}.$$

Below are some fundamental properties of the Fibonacci numbers ( [8, 28]).

$$\sum_{\mathfrak{n}=1}^{\mathfrak{m}} \mathfrak{f}_\mathfrak{n} = \mathfrak{f}_{\mathfrak{m}+2} - 1; \mathfrak{m} \geq 1 \sum_{\mathfrak{n}=1}^{\mathfrak{m}} \mathfrak{f}_\mathfrak{n}^2 = \mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}; \mathfrak{m} \geq 1$$

Kara and Basarir [8] put forward the Fibonacci matrix $F = (\mathfrak{f}_{\mathfrak{mn}})_{\mathfrak{n},\mathfrak{m}=1}^{\infty}$ by

$$\mathfrak{f}_{\mathfrak{mn}} = \begin{cases} \frac{\mathfrak{f}_{\mathfrak{n}}^2}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}, & (1 \leq \mathfrak{n} \leq \mathfrak{m}), \\ 0, & (\mathfrak{n} > \mathfrak{m}), \end{cases}$$

that is

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \dots \\ \frac{1}{6} & \frac{1}{6} & \frac{4}{6} & 0 & 0 & \dots \\ \frac{1}{15} & \frac{1}{15} & \frac{4}{15} & \frac{9}{15} & 0 & \dots \\ \frac{1}{40} & \frac{1}{40} & \frac{4}{40} & \frac{9}{40} & \frac{25}{40} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

So, the matrix $F$ is triangular in structure. Additionally, Kara and Başarir [9] specified the Fibonacci sequence space

$$X(F) = \left\{ z = (z_{\mathfrak{n}}) : \frac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}} \sum_{\mathfrak{n}=1}^{\mathfrak{m}} \mathfrak{f}_{\mathfrak{n}}^2 z_{\mathfrak{n}} \in X \right\},$$

for $X$ represents one of the spaces $\ell_\infty, c, c_0$ and $\ell_p$, which correspond to the sets of all bounded, convergent, null sequences, and $p$-absolutely convergent series, respectively.

## 3. Main results

In this section, we expand the idea of Fibonacci almost statistical convergence for a sequence of fuzzy variables by incorporating an ideal. We explore the relationships between these concepts and present several related results.

**Definition 3.1** *The fuzzy variable sequence $\{\varrho_\alpha\}$ is said to be Fibonacci almost $\mathcal{I}$-statistically convergent almost surely to the fuzzy variable $\varrho$ if there exists an event $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that*

$$\left\{ \mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \left\| \frac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left( \varrho_{\alpha+\varsigma}(\gamma) - \varrho(\gamma) \right) \right\| \geq \psi \right\} \right| \geq \varkappa \right\} \in \mathcal{I},$$

*for any $\gamma \in \Theta$, for each $\psi, \varkappa > 0$ and uniformly in $\varsigma$.*

**Definition 3.2** *The fuzzy variable sequence $\{\varrho_\alpha\}$ is said to be Fibonacci almost $\mathcal{I}$-statistically convergent in credibility to $\varrho$ if for any preassigned $\psi, \varkappa, \lambda > 0$*

$$\left\{ \mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \left\| \frac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left( \varrho_{\alpha+\varsigma} - \varrho \right) \right\| \geq \psi \right) \geq \varkappa \right\} \right| \geq \lambda \right\} \in \mathcal{I},$$

*uniformly in $\varsigma$.*

**Definition 3.3** *The fuzzy variable sequence $\{\varrho_\alpha\}$ is said to be Fibonacci almost $\mathcal{I}$-statistically convergent in mean to $\varrho$ if all the fuzzy variables $\varrho$, $\varrho_\alpha$ (for each $\alpha \in \mathbb{N}$) posses finite expected values and for any arbitrarily chosen $\psi, \varkappa > 0$*

$$\left\{ \mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : E\left[ \left\| \frac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left( \varrho_{\alpha+\varsigma} - \varrho \right) \right\| \right] \geq \psi \right\} \right| \geq \varkappa \right\} \in \mathcal{I},$$

*uniformly in $\varsigma$.*

**Definition 3.4** *Assume $\Phi$ and $\Phi_\alpha$ be the credibility distributions of the fuzzy variables $\varrho$, $\varrho_\alpha$ respectively. Then, the fuzzy variable sequence $\{\varrho_\alpha\}$ is said to be Fibonacci almost $\mathcal{I}$-statistically convergent in distribution to $\varrho$ if for any $\psi, \varkappa > 0$, we have*

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \left\| \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left( \Phi_{\alpha+\varsigma}(\mathfrak{y}) - \Phi(\mathfrak{y}) \right) \right\| \geq \psi \right\} \right| \geq \varkappa \right\} \in \mathcal{I},$$

*for all $\mathfrak{y}$ at which $\Phi$ is continuous, uniformly in $\varsigma \in \mathbb{N}$.*

**Definition 3.5** *The fuzzy variable sequence $\{\varrho_\alpha\}$ is said to be Fibonacci almost $\mathcal{I}$-statistically uniformly almost surely to $\varrho$ if there exists some events $\{A_j\}$, $(j \in \mathbb{N})$ each of whose credibility measure approaches zero such that the sequence almost converges uniformly to the same limit in the sense of ideal in $\mathcal{P}(\Theta) - A_j$.*

**Theorem 3.1** *Suppose that the sequence of fuzzy variables $\{\varrho_\alpha\}$ and $\{\upsilon_\alpha\}$ both converge in credibility to $\varrho$ and $\upsilon$, respectively. If there are positive numbers $P_1$, $P$, $T_1$ and $T$ such that $P_1 \leq \|\varrho_\alpha\| \leq P$ and $T_1 \leq \|\upsilon_\alpha\| \leq T$ for any $\alpha$, then*
*(i) $\{\varrho_\alpha + \upsilon_\alpha\}$ is Fibonacci almost $\mathcal{I}$-statistically convergent in credibility to $\varrho + \upsilon$,*
*(ii) $\{\varrho_\alpha - \upsilon_\alpha\}$ is Fibonacci almost $\mathcal{I}$-statistically convergent in credibility to $\varrho - \upsilon$,*
*(iii) $\{\varrho_\alpha \upsilon_\alpha\}$ is Fibonacci almost $\mathcal{I}$-statistically convergent in credibility to $\varrho\upsilon$,*
*(iv) $\left\{\frac{\varrho_\alpha}{\upsilon_\alpha}\right\}$ is Fibonacci almost $\mathcal{I}$-statistically convergent in credibility to $\frac{\varrho}{\upsilon}$.*

**Proof:** (i) Let the sequences $\{\varrho_\alpha\}$ and $\{\upsilon_\alpha\}$ be Fibonacci almost $\mathcal{I}$-statistically convergent in credibility to $\varrho$ and $\upsilon$ respectively. Then, for any preassigned $\psi, \varkappa, \lambda > 0$

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \left\| \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left( \varrho_{\alpha+\varsigma} - \varrho \right) \right\| \geq \frac{\psi}{2} \right) \geq \frac{\varkappa}{2} \right\} \right| \geq \frac{\lambda}{2} \right\} \in \mathcal{I},$$

and

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \left\| \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left( \upsilon_{\alpha+\varsigma} - \upsilon \right) \right\| \geq \frac{\psi}{2} \right) \geq \frac{\varkappa}{2} \right\} \right| \geq \frac{\lambda}{2} \right\} \in \mathcal{I},$$

uniformly in $\varsigma \in \mathbb{N}$.

Since, $P_1 \leq \|\varrho_\alpha\| \leq P$, we have $P_1 \leq \|\varrho\| \leq P$. In a similar way, if $T_1 \leq \|\upsilon_\alpha\| \leq T$, then $T_1 \leq \|\upsilon\| \leq T$. On applying the subadditivity axiom of the credibility measure, we have

$$\begin{aligned}
&\left\{\mathfrak{q} \in \mathbb{N} : \tfrac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \left\| \tfrac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left[ \left( \varrho_{\alpha+\varsigma} + \upsilon_{\alpha+\varsigma} \right) - \left( \varrho + \upsilon \right) \right] \right\| \geq \psi \right) \geq \varkappa \right\} \right| \geq \lambda \right\} \\
&= \left\{\mathfrak{q} \in \mathbb{N} : \tfrac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \left\| \tfrac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left[ \left( \varrho_{\alpha+\varsigma} - \varrho \right) + \left( \upsilon_{\alpha+\varsigma} - \upsilon \right) \right] \right\| \geq \psi \right) \geq \varkappa \right\} \right| \geq \lambda \right\} \\
&\subseteq \left\{\mathfrak{q} \in \mathbb{N} : \tfrac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \left\| \tfrac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left( \varrho_{\alpha+\varsigma} - \varrho \right) \right\| \geq \tfrac{\psi}{2} \right) \geq \tfrac{\varkappa}{2} \right\} \right| \geq \tfrac{\lambda}{2} \right\} \\
&\cup \left\{\mathfrak{q} \in \mathbb{N} : \tfrac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \left\| \tfrac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left( \upsilon_{\alpha+\varsigma} - \upsilon \right) \right\| \geq \tfrac{\psi}{2} \right) \geq \tfrac{\varkappa}{2} \right\} \right| \geq \tfrac{\lambda}{2} \right\} \in \mathcal{I},
\end{aligned}$$

uniformly in $\varsigma \in \mathbb{N}$. Then, we can say $\{\varrho_\alpha + \upsilon_\alpha\}$ is Fibonacci almost $\mathcal{I}$-statistically convergent in credibility to $\mu + \upsilon$.

(ii) It can be proved in a manner similar to (i).

(iii) We see by Axiom iv that

$$\begin{aligned}
&\left\{\mathfrak{q}\in\mathbb{N}:\tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m}\le\mathfrak{q}:\mathrm{Cr}\left(\left\|\tfrac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\varrho_{\alpha+\varsigma}\upsilon_{\alpha+\varsigma}-\varrho\upsilon\right)\right\|\ge\psi\right)\ge\varkappa\right\}\right|\ge\lambda\right\}\\
&\left\{\mathfrak{q}\in\mathbb{N}:\tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m}\le\mathfrak{q}:\mathrm{Cr}\left(\left\|\tfrac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\varrho_{\alpha+\varsigma}\upsilon_{\alpha+\varsigma}-\varrho_{\alpha+\varsigma}\upsilon+\varrho_{\alpha+\varsigma}\upsilon-\varrho\upsilon\right)\right\|\ge\psi\right)\ge\varkappa\right\}\right|\ge\lambda\right\}\\
&\subseteq\left\{\mathfrak{q}\in\mathbb{N}:\tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m}\le\mathfrak{q}:\mathrm{Cr}\left(P\left\|\tfrac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\upsilon_{\alpha+\varsigma}-\upsilon\right)\right\|\ge\tfrac{\psi}{2}\right)\ge\tfrac{\varkappa}{2}\right\}\right|\ge\tfrac{\lambda}{2}\right\}\\
&\cup\left\{\mathfrak{q}\in\mathbb{N}:\tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m}\le\mathfrak{q}:\mathrm{Cr}\left(T\left\|\tfrac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\varrho_{\alpha+\varsigma}-\varrho\right)\right\|\ge\tfrac{\psi}{2}\right)\ge\tfrac{\varkappa}{2}\right\}\right|\ge\tfrac{\lambda}{2}\right\}\\
&\subseteq\left\{\mathfrak{q}\in\mathbb{N}:\tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m}\le\mathfrak{q}:\mathrm{Cr}\left(\left\|\tfrac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\upsilon_{\alpha+\varsigma}-\upsilon\right)\right\|\ge\tfrac{\psi}{2P}\right)\ge\tfrac{\varkappa}{2}\right\}\right|\ge\tfrac{\lambda}{2}\right\}\\
&\cup\left\{\mathfrak{q}\in\mathbb{N}:\tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m}\le\mathfrak{q}:\mathrm{Cr}\left(\left\|\tfrac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\varrho_{\alpha+\varsigma}-\varrho\right)\right\|\ge\tfrac{\psi}{2T}\right)\ge\tfrac{\varkappa}{2}\right\}\right|\ge\tfrac{\lambda}{2}\right\}\in\mathcal{I}_2,
\end{aligned}$$

uniformly in $\varsigma\in\mathbb{N}$. Then, we have the sequence $\{\varrho_\alpha\upsilon_\alpha\}$ is Fibonacci almost $\mathcal{I}$-statistically convergent in credibility to $\varrho\upsilon$.

(iv) We see by Axiom iv that

$$\begin{aligned}
&\left\{\mathfrak{q}\in\mathbb{N}:\tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m}\le\mathfrak{q}:\mathrm{Cr}\left(\left\|\tfrac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\tfrac{\varrho_{\alpha+\varsigma}}{\upsilon_{\alpha+\varsigma}}-\tfrac{\varrho}{\upsilon}\right)\right\|\ge\psi\right)\ge\varkappa\right\}\right|\ge\lambda\right\}\\
&=\left\{\mathfrak{q}\in\mathbb{N}:\tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m}\le\mathfrak{q}:\mathrm{Cr}\left(\left\|\tfrac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\tfrac{\varrho_{\alpha+\varsigma}}{\upsilon_{\alpha+\varsigma}}-\tfrac{\varrho_{\alpha+\varsigma}}{\upsilon}+\tfrac{\varrho_{\alpha+\varsigma}}{\upsilon}-\tfrac{\varrho}{\upsilon}\right)\right\|\ge\psi\right)\ge\varkappa\right\}\right|\ge\lambda\right\}\\
&=\left\{\mathfrak{q}\in\mathbb{N}:\tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m}\le\mathfrak{q}:\mathrm{Cr}\left(\left\|\tfrac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\tfrac{\varrho_{\alpha+\varsigma}(\upsilon-\upsilon_{\alpha+\varsigma})}{\upsilon_{\alpha+\varsigma}\upsilon}+\tfrac{\varrho_{\alpha+\varsigma}-\varrho}{\upsilon}\right)\right\|\ge\psi\right)\ge\varkappa\right\}\right|\ge\lambda\right\}\\
&\subseteq\left\{\mathfrak{q}\in\mathbb{N}:\tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m}\le\mathfrak{q}:\mathrm{Cr}\left(\left\|\tfrac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\tfrac{\varrho_{\alpha+\varsigma}(\upsilon-\upsilon_{\alpha+\varsigma})}{\upsilon_{\alpha+\varsigma}\upsilon}\right)\right\|\ge\tfrac{\psi}{2}\right)\ge\tfrac{\varkappa}{2}\right\}\right|\ge\tfrac{\lambda}{2}\right\}\\
&\cup\left\{\mathfrak{q}\in\mathbb{N}:\tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m}\le\mathfrak{q}:\mathrm{Cr}\left(\left\|\tfrac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\tfrac{\varrho_{\alpha+\varsigma}-\varrho}{\upsilon}\right)\right\|\ge\tfrac{\psi}{2}\right)\ge\tfrac{\varkappa}{2}\right\}\right|\ge\tfrac{\lambda}{2}\right\}\\
&\subseteq\left\{\mathfrak{q}\in\mathbb{N}:\tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m}\le\mathfrak{q}:\mathrm{Cr}\left(\frac{P}{T_1^2}\left\|\tfrac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\upsilon-\upsilon_{\alpha+\varsigma}\right)\right\|\ge\tfrac{\psi}{2}\right)\ge\tfrac{\varkappa}{2}\right\}\right|\ge\tfrac{\lambda}{2}\right\}\\
&\cup\left\{\mathfrak{q}\in\mathbb{N}:\tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m}\le\mathfrak{q}:\mathrm{Cr}\left(\frac{1}{T_1}\left\|\tfrac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\varrho_{\alpha+\varsigma}-\varrho\right)\right\|\ge\tfrac{\psi}{2}\right)\ge\tfrac{\varkappa}{2}\right\}\right|\ge\tfrac{\lambda}{2}\right\}\\
&\subseteq\left\{\mathfrak{q}\in\mathbb{N}:\tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m}\le\mathfrak{q}:\mathrm{Cr}\left(\left\|\tfrac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\upsilon-\upsilon_{\alpha+\varsigma}\right)\right\|\ge\tfrac{\psi T_1^2}{2P}\right)\ge\tfrac{\varkappa}{2}\right\}\right|\ge\tfrac{\lambda}{2}\right\}\\
&\cup\left\{\mathfrak{q}\in\mathbb{N}:\tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m}\le\mathfrak{q}:\mathrm{Cr}\left(\left\|\tfrac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\varrho_{\alpha+\varsigma}-\varrho\right)\right\|\ge\tfrac{\psi T_1}{2}\right)\ge\tfrac{\varkappa}{2}\right\}\right|\ge\tfrac{\lambda}{2}\right\}\in\mathcal{I},
\end{aligned}$$

uniformly in $\varsigma\in\mathbb{N}$. Then, we have $\left\{\frac{\varrho_\alpha}{\upsilon_\alpha}\right\}$ is Fibonacci almost $\mathcal{I}$-statistically convergent in credibility to $\frac{\varrho}{\upsilon}$. □

**Theorem 3.2** *If the sequence $\{\varrho_\alpha\}$ of fuzzy variable is Fibonacci almost $\mathcal{I}$-statistically convergent in mean to $\varrho$, then it is also Fibonacci almost $\mathcal{I}$-statistically convergent in credibility.*

**Proof:** By applying the Markov inequality, we obtain the following for any given $\psi, \varkappa, \lambda > 0$,

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : \operatorname{Cr}\left(\left\|\frac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\varrho_{\alpha+\varsigma}-\varrho\right)\right\| \geq \psi\right) \geq \varkappa\right\}\right| \geq \lambda\right\}$$
$$\subseteq \left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : \left(\frac{E\left[\left\|\frac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}(\varrho_{\alpha+\varsigma}-\varrho)\right\|\right]}{\psi}\right) \geq \varkappa\right\}\right| \geq \lambda\right\} \in \mathcal{I},$$

uniformly in $\varsigma$. Consequently, the sequence $\{\varrho_{\alpha}\}$ is Fibonacci almost $\mathcal{I}$-statistically convergent in credibility. □

**Remark 3.1** *The reverse of the theorem is not true in general. This follows from the following example.*

**Example 3.1** *Consider the credibility space* $(\Theta, \mathcal{P}(\Theta), \operatorname{Cr})$ *to be* $\{\gamma_1, \gamma_2, ...\}$ *with credibility measure* Cr *defined by*

$$\operatorname{Cr}\{A\} = \begin{cases} \sup_{\gamma_{\mathfrak{q}} \in A} \frac{1}{\mathfrak{f}_{\mathfrak{q}+2}}, & \text{if } \sup_{\gamma_{\mathfrak{q}} \in A} \frac{1}{\mathfrak{f}_{\mathfrak{q}+2}} < 0.5, \\ 1 - \sup_{\gamma_{\mathfrak{q}} \in A^c} \frac{1}{\mathfrak{f}_{\mathfrak{q}+2}}, & \text{if } \sup_{\gamma_{\mathfrak{q}} \in A^c} \frac{1}{\mathfrak{f}_{\mathfrak{q}+2}} < 0.5, \\ 0.5, & \text{otherwise.} \end{cases}$$

*We define the fuzzy variable as follows:*

$$\varrho_{\mathfrak{q}}(\gamma) = \begin{cases} \mathfrak{f}_{\mathfrak{q}+2}, & \text{if } \gamma = \gamma_{\mathfrak{q}}, \\ 0, & \text{otherwise,} \end{cases}$$

*for any* $\mathfrak{q} \in \mathbb{N}$, $\varrho \equiv 0$ *and* $\gamma \in A$. *We have for some small number* $\psi, \varkappa, \lambda > 0$,

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : \operatorname{Cr}\left(\left\|\frac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\varrho_{\alpha+\varsigma}-\varrho\right)\right\| \geq \psi\right) \geq \varkappa\right\}\right| \geq \lambda\right\}$$
$$= \left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : \operatorname{Cr}\left(\gamma : \left\|\frac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\varrho_{\alpha+\varsigma}(\gamma)-\varrho(\gamma)\right)\right\| \geq \psi\right) \geq \varkappa\right\}\right| \geq \lambda\right\}$$
$$= \left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : \operatorname{Cr}\{\gamma = \gamma_{\mathfrak{q}}\} \geq \varkappa\right\}\right| \geq \lambda\right\} \in \mathcal{I}.$$

*So, the fuzzy variable sequence* $\{\varrho_{\alpha}\}$ *is Fibonacci almost* $\mathcal{I}$*-statistically convergent in credibility. Again, we have the credibility distribution function*

$$\Phi_{\mathfrak{q}}(\mathfrak{y}) = \begin{cases} 0, & \text{if } \mathfrak{y} < 0, \\ 1 - \frac{1}{\mathfrak{f}_{\mathfrak{q}+2}}, & \text{if } 0 \leq \mathfrak{y} < \mathfrak{f}_{\mathfrak{q}+2}, \\ 1, & \text{if } \mathfrak{y} \geq \mathfrak{f}_{\mathfrak{q}+2}. \end{cases}$$

*Then*

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : E\left[\left\|\frac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\varrho_{\alpha+\varsigma}-\varrho\right)\right\| - 1\right] \geq \psi\right\}\right| \geq \varkappa\right\}$$
$$= \left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : \left[\int_{0}^{\mathfrak{f}_{\mathfrak{q}+2}}\left[1-\left(1-\frac{1}{\mathfrak{f}_{\mathfrak{q}+2}}\right)\right]d\mathfrak{y} - 1\right] \geq \psi\right\}\right| \geq \varkappa\right\}.$$

*Hence, for all* $\psi, \varkappa > 0$, *we have*

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : E\left[\left\|\frac{1}{\mathfrak{f}_{\mathfrak{m}}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_{\alpha}^{2}\left(\varrho_{\alpha+\varsigma}-\varrho\right)\right\|\right] \geq \psi\right\}\right| \geq \varkappa\right\} \in \mathcal{F}(\mathcal{I}),$$

*which is impossible. So, the fuzzy variable sequence* $\{\varrho_{\alpha}\}$ *is not Fibonacci almost* $\mathcal{I}$*-statistically convergent in mean to* $\varrho$.

**Proposition 3.1** *Let $\{\varrho_\alpha\}$ be a sequence of fuzzy variable. Then, it Fibonacci almost $\mathcal{I}$-statisticaly converges in almost surely to $\varrho$ if and only if for any $\psi, \kappa, \lambda > 0$, we have*

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \bigcap_{\mathfrak{n}=1}^{\infty} \bigcup_{\mathfrak{m}=\mathfrak{n}}^{\infty} \left\| \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| \geq \psi \right) \geq \varkappa \right\} \right| \geq \lambda \right\} \in \mathcal{I}.$$

**Proof:** As stated in the definition of Fibonacci almost $\mathcal{I}$-statistically convergence in almost surely of fuzzy variable provides us the existence of a $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \left\| \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma}(\gamma) - \varrho(\gamma)\right) \right\| \geq \psi \right\} \right| \geq \varkappa \right\} \in \mathcal{I},$$

for any $\gamma \in \Theta$, for each $\psi, \varkappa > 0$ and uniformly in $\varsigma$.
Then, for any $\psi > 0$ there exists $\mathfrak{n}$ such that $\|\varrho_{\mathfrak{q}+\varsigma} - \varrho\| < \psi$ where $\mathfrak{q} > \mathfrak{n}$, uniformly in $\varsigma \in \mathbb{N}$ such that

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \bigcup_{\mathfrak{n}=1}^{\infty} \bigcap_{\mathfrak{m}=\mathfrak{n}}^{\infty} \left\| \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| < \psi \right) \geq \varkappa \right\} \right| \geq \lambda \right\} \in \mathcal{F}(\mathcal{I}).$$

From the duality axiom of credibility measure we have

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \bigcap_{\mathfrak{n}=1}^{\infty} \bigcup_{\mathfrak{m}=\mathfrak{n}}^{\infty} \left\| \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| \geq \psi \right) \geq \varkappa \right\} \right| \geq \lambda \right\} \in \mathcal{I},$$

uniformly in $\varsigma$. □

**Proposition 3.2** *Let $\{\varrho_\alpha\}$ be a sequence of fuzzy variable. Then, the sequence $\{\varrho_\alpha\}$ Fibonacci almost $\mathcal{I}$-statistically converges uniformly almost surely to $\varrho$ if and only if for any $\psi, \kappa, \lambda > 0$, such that*

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \bigcup_{\mathfrak{m}=\mathfrak{n}}^{\infty} \left\| \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| \geq \psi \right) \geq \varkappa \right\} \right| \geq \lambda \right\} \in \mathcal{I},$$

*uniformly in $\varsigma$.*

**Proof:** Assume the sequence $\{\varrho_\alpha\}$ of fuzzy variables Fibonacci almost $\mathcal{I}$-statistically converges uniformly almost surely to $\varrho$. Then, for any $\psi > 0$, there exists $\varkappa > 0$, and some events $A_j$ with credibility measure approaching zero such that for any preassigned $\psi > 0$, there exists $\mathfrak{n} \in \mathbb{N}$ with

$$\left\| \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma}(\gamma) - \varrho(\gamma)\right) \right\| < \psi,$$

uniformly in $\varsigma$ for all $\alpha > \mathfrak{n}$ and for any $\gamma \in \mathcal{P}(\Theta) - A_j$. Thus, we have

$$\bigcup_{\mathfrak{m}=\mathfrak{n}}^{\infty} \left\{ \gamma : \left\| \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma}(\gamma) - \varrho(\gamma)\right) \right\| \geq \psi \right\} \subseteq A_j.$$

Taking credibility measure of both the sets in the above, we get

$$\mathrm{Cr}\left( \bigcup_{\mathfrak{m}=\mathfrak{n}}^{\infty} \left\{ \gamma : \left\| \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| \geq \psi \right\} \right) \leq \mathrm{Cr}\left(A_j\right) < \varkappa.$$

Consequently, we can write

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \bigcup_{\mathfrak{m}=\mathfrak{n}}^{\infty} \left\| \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| \geq \psi \right) \geq \varkappa \right\} \right| \geq \lambda \right\} \in \mathcal{I},$$

uniformly in $\varsigma$, for any $\psi, \kappa, \lambda > 0$.
To address the converse aspect, let's posit that the specified conditions are met. Let

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \bigcup_{\mathfrak{m}=\mathfrak{n}}^{\infty} \left\| \frac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| \geq \psi \right) \geq \varkappa \right\} \right| \geq \lambda \right\} \in \mathcal{I},$$

Then, for any $\vartheta > 0$, $s \geq 1$, there exists $s_w > 0$ such that

$$\mathrm{Cr}\left( \bigcup_{\mathfrak{m}=s_w}^{\infty} \left\| \frac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| \geq \frac{1}{s} \right) < \frac{\vartheta}{2^s}.$$

Consider,

$$H = \mathrm{Cr}\left( \bigcup_{s=1}^{\infty} \bigcup_{\mathfrak{m}=s_w}^{\infty} \left\| \frac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| \geq \frac{1}{s} \right).$$

Then,

$$\mathrm{Cr}(H) \leq \mathrm{Cr}\left( \bigcup_{s=1}^{\infty} \bigcup_{\mathfrak{m}=s_w}^{\infty} \left\| \frac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| \geq \frac{1}{s} \right)$$

$$\leq \sum_{s=1}^{\infty} \frac{\vartheta}{2^s} = \vartheta.$$

In addition, we have

$$\sup_{\gamma \in \mathcal{P}(\Theta) - H} \left\| \frac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma}(\gamma) - \varrho(\gamma)\right) \right\| < \frac{1}{q},$$

where $q = 1, 2, 3, ..., \alpha > s_w$ and for any $\gamma \in \mathcal{P}(\Theta) - A_j$. Hence, the proposition is proved. □

**Theorem 3.3** *Let $\{\varrho_\alpha\}$ be an Fibonacci almost $\mathcal{I}$-statistically convergent sequence in uniformly almost surely to $\varrho$. Then, $\{\varrho_\alpha\}$ is an Fibonacci almost $\mathcal{I}$-statistically convergent sequence in almost surely to $\varrho$.*

**Proof:** Assume that $\{\varrho_\alpha\}$ Fibonacci almost $\mathcal{I}$-statistically converges uniformly almost surely to $\varrho$. Then, we get

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \bigcup_{\mathfrak{m}=\mathfrak{n}}^{\infty} \left\| \frac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| \geq \psi \right) \geq \varkappa \right\} \right| \geq \lambda \right\} \in \mathcal{I},$$

Since

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \bigcap_{\mathfrak{n}=1}^{\infty} \bigcup_{\mathfrak{m}=\mathfrak{n}}^{\infty} \left\| \frac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| \geq \psi \right) \geq \varkappa \right\} \right| \geq \lambda \right\}$$
$$\subseteq \left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \bigcup_{\mathfrak{m}=\mathfrak{n}}^{\infty} \left\| \frac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| \geq \psi \right) \geq \varkappa \right\} \right| \geq \lambda \right\} \in \mathcal{I},$$

uniformly in $\varsigma$. Hence, $\{\varrho_\alpha\}$ Fibonacci almost $\mathcal{I}$-statistically converges in almost surely to $\varrho$ by the Proposition 3.2. □

**Theorem 3.4** *The fuzzy variable sequence $\{\varrho_\alpha\}$ which Fibonacci almost $\mathcal{I}$-statistically converges uniformly almost surely to $\varrho$ is also Fibonacci almost $\mathcal{I}$-statistically converges in credibility to $\varrho$.*

**Remark 3.2** *A sequence $\{\varrho_\alpha\}$ of fuzzy variable which is Fibonacci almost $\mathcal{I}$-statistically convergent with respect to almost surely does not necessarily imply that it is Fibonacci almost $\mathcal{I}$-statistically convergent in mean. An illustration of this is provided in the following example.*

**Example 3.2** *Consider the credibility space* $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ *where* $\Theta = \{\gamma_1, \gamma_2, ...\}$ *and* $\mathrm{Cr}\{A\} = \sum_{\gamma_\mathfrak{q} \in A} \frac{1}{2^{\mathfrak{f}_{\mathfrak{q}+2}}}$. *Define the fuzzy variables* $\varrho_\alpha$ *and* $\varrho$ *by*

$$\varrho_\mathfrak{q}(\gamma) = \begin{cases} 2^{\mathfrak{f}_{\mathfrak{q}+2}}, & \text{if } \gamma = \gamma_\mathfrak{q}, \\ 0, & \text{otherwise}, \end{cases}$$

*and* $\varrho \equiv 0$, $\forall \gamma \in \Theta$ *and for* $\mathfrak{q} \in \mathbb{N}$. *Calculating in the same way like earlier example, we get* $\{\varrho_\alpha\}$ *Fibonacci almost* $\mathcal{I}$*-statistical converges to* $\varrho$ *with respect to almost surely. The credibility distribution function of the fuzzy variable sequence* $\{\varrho_\alpha\}$ *is given by*

$$\Phi_\mathfrak{q}(\mathfrak{y}) = \begin{cases} 0, & \text{if } \mathfrak{y} < 0, \\ 1 - \frac{1}{2^{\mathfrak{f}_{\mathfrak{q}+2}}}, & \text{if } 0 \leq \mathfrak{y} < 2^{\mathfrak{f}_{\mathfrak{q}+2}}, \\ 1, & \text{otherwise}. \end{cases}$$

*for* $\mathfrak{q} \in \mathbb{N}$. *So*

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : E\left[\left\| \frac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| - 1\right] \geq \psi \right\} \right| \geq \varkappa \right\}$$

$$= \left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \left[ \int_0^{2^{\mathfrak{f}_{\mathfrak{q}+2}}} \left[1 - \left(1 - \frac{1}{2^{\mathfrak{f}_{\mathfrak{q}+2}}}\right)\right] d\mathfrak{y} - 1 \right] \geq \psi \right\} \right| \geq \varkappa \right\}$$

*uniformly in* $\varsigma \in \mathbb{N}$. *Hence, we have*

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : E\left[\left\| \frac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| - 1\right] \geq \psi \right\} \right| \geq \varkappa \right\} \in \mathcal{I}.$$

*Thus, the fuzzy variable sequence* $\{\varrho_\alpha\}$ *is not Fibonacci almost* $\mathcal{I}$*-statistically convergent in mean to* $\varrho$.

**Remark 3.3** *The notions of Fibonacci almost* $\mathcal{I}$*-statistically convergence in credibility and Fibonacci almost* $\mathcal{I}$*-statistically convergence in almost surely are unrelated to each other. This assertion is exemplified by the subsequent pair of examples.*

**Example 3.3** *A sequence* $\{\varrho_\alpha\}$ *of fuzzy variable which is Fibonacci almost* $\mathcal{I}$*-statistically convergent in almost surely does not necessarily imply that it is Fibonacci almost* $\mathcal{I}$*-statistically convergent in credibility. This distinction becomes apparent on considering the credibility space* $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ *with* $\mathcal{P}(\Theta) = \{\gamma_1, \gamma_2, ...\}$, *and the credibility measurable function as follows:*
*For* $\alpha = 1, 2, ...$ *we define*

$$\varrho_\mathfrak{q}(\gamma) = \begin{cases} \mathfrak{f}_{\mathfrak{q}+2}, & \text{if } \gamma = \gamma_\mathfrak{q}, \\ 0, & \text{otherwise}, \end{cases}$$

*and* $\varrho \equiv 0$.
*Let us the credibility measure by*

$$\mathrm{Cr}\{A\} = \begin{cases} \sup_{\gamma_\mathfrak{q} \in A} \frac{\mathfrak{f}_{\mathfrak{q}+2}}{2\mathfrak{f}_{\mathfrak{q}+2}+1}, & \text{if } \sup_{\gamma_\mathfrak{q} \in A} \frac{\mathfrak{f}_{\mathfrak{q}+2}}{2\mathfrak{f}_{\mathfrak{q}+2}+1} < 0.5, \\ 1 - \sup_{\gamma_\mathfrak{q} \in A^c} \frac{\mathfrak{f}_{\mathfrak{q}+2}}{2\mathfrak{f}_{\mathfrak{q}+2}+1}, & \text{if } \sup_{\gamma_\mathfrak{q} \in A^c} \frac{\mathfrak{f}_{\mathfrak{q}+2}}{2\mathfrak{f}_{\mathfrak{q}+2}+1} < 0.5, \\ 0.5, & \text{if not}. \end{cases}$$

*Then, we obtain*

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \left\| \frac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left(\varrho_{\alpha+\varsigma}(\gamma) - \varrho(\gamma)\right) \right\| \geq \psi \right\} \right| \geq \varkappa \right\} \in \mathcal{I},$$

*for any $\gamma \in \Theta$, for each $\psi, \varkappa > 0$ and uniformly in $\varsigma$. But, for any preassigned $\psi, \varkappa > 0$ and $\lambda \in \left[\frac{1}{2}, 1\right)$,*

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \left\| \frac{1}{\mathfrak{f}_{\mathfrak{m}} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_{\alpha}^{2} \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| \geq \psi \right) \geq \varkappa \right\} \right| \geq \frac{1}{2} \right\} \in \mathcal{I},$$

*uniformly in $\varsigma$. Therefore, $\{\varrho_\alpha\}$ does not Fibonacci almost $\mathcal{I}$-statistical converge in credibility for $\lambda \in \left[0, \frac{1}{2}\right)$.*

**Remark 3.4** *A sequence $\{\varrho_\alpha\}$ of fuzzy variable which is Fibonacci almost $\mathcal{I}$-statisticaly convergent in credibility does not necessarily imply that it is almost $\mathcal{I}_2$-convergent with respect to almost surely. To illustrate the above theorem, we employ the following example:*

**Example 3.4** *With the help of Borel algebra and Lebesgue measure, consider the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to be $[0,1]$. Then, there exists $w \in \mathbb{Z}$ such that $\alpha = 2^w + u$ where $u$ is an integer between 0 and $\min\{2^w\} - 1$. For any $\alpha \in \mathbb{N}$ we establish a fuzzy variable as*

$$\varrho_{\mathfrak{q}}(\gamma) = \begin{cases} 1, & \text{if } \frac{u}{2^w} \leq \gamma \leq \frac{u+1}{2^w}, \quad (u, w \in \mathbb{Z}) \\ 0, & \text{if not,} \end{cases}$$

*and $\varrho \equiv 0$. For given $\psi, \varkappa, \lambda > 0$, one can obtain*

$$\begin{aligned} &\left\{\mathfrak{q} \in \mathbb{N} : \tfrac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \left\| \tfrac{1}{\mathfrak{f}_{\mathfrak{m}} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_{\alpha}^{2} \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| \geq \psi \right) \geq \varkappa \right\} \right| \geq \lambda \right\} \\ &= \left\{\mathfrak{q} \in \mathbb{N} : \tfrac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left( \gamma : \left\| \tfrac{1}{\mathfrak{f}_{\mathfrak{m}} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_{\alpha}^{2} \left(\varrho_{\alpha+\varsigma}(\gamma) - \varrho(\gamma)\right) \right\| \geq \psi \right) \geq \varkappa \right\} \right| \geq \lambda \right\} \\ &= \left\{\mathfrak{q} \in \mathbb{N} : \tfrac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr}\left\{\gamma = \gamma_{\mathfrak{q}}\right\} \geq \varkappa \right\} \right| \geq \lambda \right\} \in \mathcal{I}. \end{aligned}$$

*Then, the sequence $\{\varrho_\alpha\}$ Fibonacci almost $\mathcal{I}$-statisticaly converges in credibility to $\varrho$. Furthermore, for given $\psi, \varkappa > 0$, it can be observed that*

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : E\left[ \left\| \frac{1}{\mathfrak{f}_{\mathfrak{m}} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha=0}^{\mathfrak{m}-1} \mathfrak{f}_{\alpha}^{2} \left(\varrho_{\alpha+\varsigma} - \varrho\right) \right\| \right] \geq \psi \right\} \right| \geq \varkappa \right\} \in \mathcal{I},$$

*uniformly in $\varsigma$. This indicates that the sequence $\{\varrho_\alpha\}$ Fibonacci almost $\mathcal{I}$-statisticaly converges in mean to $\varrho$. Nevertheless, for any $\gamma \in [0,1]$ there exist an infinite number of intervals in the form $\left[\frac{u}{2^w}, \frac{u+1}{2^w}\right]$ that contain $\gamma$. Consequently, $\{\varrho_\alpha\}$ does not exhibit Fibonacci almost $\mathcal{I}$-statisticaly converge in almost surely to $\varrho$.*

Suppose
(1) Fibonacci almost $\mathcal{I}$-statistically convergence almost surely
(2) Fibonacci almost $\mathcal{I}$-statistically convergence in credibility
(3) Fibonacci almost $\mathcal{I}$-statistically convergence in mean
(4) Fibonacci almost $\mathcal{I}$-statistically convergence uniformly almost surely
(5) Fibonacci almost $\mathcal{I}$-statistically convergence in distribution

Then the interrelationships among them are shown in Fig. 1.

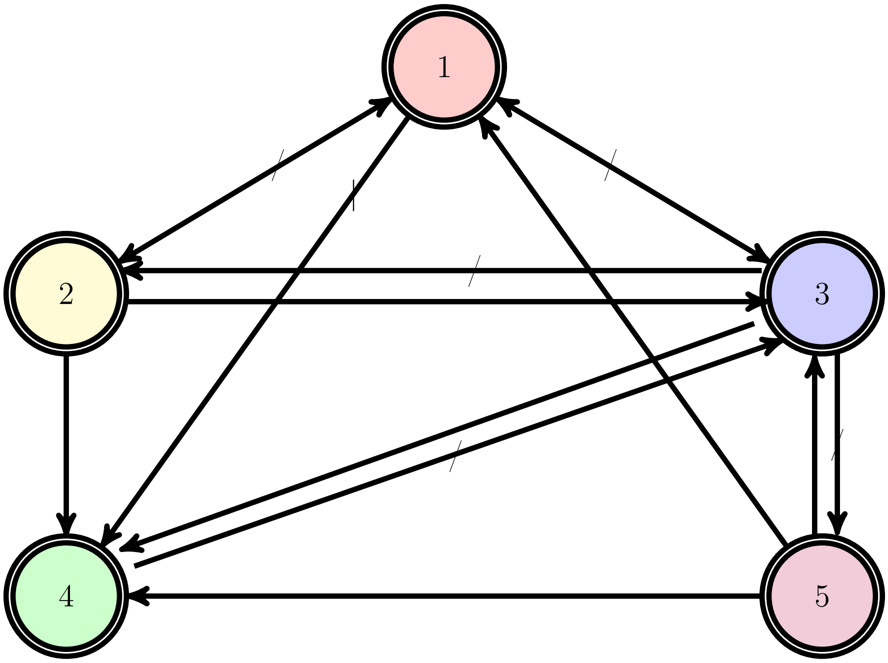


Fig 1 Relationships among Fibonacci $I-$almost statistical concepts

**Definition 3.6** *The sequence of fuzzy variable $\{\varrho_\alpha\}$ is Fibonacci almost $\mathcal{I}$-statistical Cauchy almost surely, provided that for any given $\psi, \varkappa > 0$, there is $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that*

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \left\| \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha_1=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left( \varrho_{\alpha_1+\varsigma}(\gamma) \right) - \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha_2=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left( \varrho_{\alpha_2+\varsigma}(\gamma) \right) \right\| \geq \psi \right\} \right| \geq \varkappa \right\} \in \mathcal{I},$$

*for any $\gamma \in \Theta$, for each $\psi, \varkappa > 0$ and uniformly in $\varsigma$.*

**Definition 3.7** *The sequence of fuzzy variable $\{\varrho_\alpha\}$ is Fibonacci almost $\mathcal{I}$-statistical Cauchy in credibility, if for any preassigned $\psi, \varkappa, \lambda > 0$*

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \mathrm{Cr} \left( \left\| \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha_1=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left( \varrho_{\alpha_1+\varsigma} \right) - \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha_2=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left( \varrho_{\alpha_2+\varsigma} \right) \right\| \geq \psi \right) \geq \varkappa \right\} \right| \geq \lambda \right\} \in \mathcal{I},$$

*uniformly in $\varsigma$.*

**Definition 3.8** *The sequence of fuzzy variable $\{\varrho_\alpha\}$ is Fibonacci almost $\mathcal{I}$-statistical Cauchy in mean, if for any given $\psi, \varkappa > 0$*

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : E \left[ \left\| \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha_1=0}^{\mathfrak{m}-1} \mathfrak{f}_\varsigma^2 \left( \varrho_{\alpha_1+\varsigma} \right) - \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha_2=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left( \varrho_{\alpha_2+\varsigma} \right) \right\| \right] \geq \psi \right\} \right| \geq \varkappa \right\} \in \mathcal{I},$$

*uniformly in $\varsigma$.*

**Definition 3.9** *Let $\Phi_\alpha$ be the fuzzy credibility distributions of fuzzy variables $\varrho_\alpha$. Then, the sequence $\{\varrho_\alpha\}$ is called Fibonacci almost $\mathcal{I}$-statistical Cauchy in distribution if for any $\psi, \varkappa > 0$*

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}} \left| \left\{ \mathfrak{m} \leq \mathfrak{q} : \left\| \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha_1=0}^{\mathfrak{m}-1} \mathfrak{f}_\varsigma^2 \left( \Phi_{\alpha_1+\varsigma} (\mathfrak{y}) \right) - \frac{1}{\mathfrak{f}_\mathfrak{m} \mathfrak{f}_{\mathfrak{m}+1}} \sum_{\alpha_2=0}^{\mathfrak{m}-1} \mathfrak{f}_\alpha^2 \left( \Phi_{\alpha_2+\varsigma} (\mathfrak{y}) \right) \right\| \geq \psi \right\} \right| \geq \varkappa \right\} \in \mathcal{I},$$

*for all $\mathfrak{y}$ at which $\Phi$ is continuous and uniformly in $\varsigma \in \mathbb{N}$.*

**Theorem 3.5** *The fuzzy variable sequence $\{\varrho_\alpha\}$ is Fibonacci almost $\mathcal{I}$-statistically convergent almost surely if and only if it is a Fibonacci almost $\mathcal{I}$-statistical Cauchy sequence almost surely.*

**Proof:** Assume that the sequence $\{\varrho_\alpha\}$ is almost $\mathcal{I}_2$-convergent almost surely to $\varrho$. Then, there exists an event $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : \left\|\frac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_\alpha^2\left(\varrho_{\alpha+\varsigma}(\gamma) - \varrho(\gamma)\right)\right\| \geq \frac{\psi}{2}\right\}\right| \geq \frac{\varkappa}{2}\right\} \in \mathcal{I},$$

for any $\gamma \in \Theta$, for each $\psi, \varkappa > 0$ and uniformly in $\varsigma$.
Therefore, we obtain

$$\begin{aligned}&\left\{\mathfrak{q} \in \mathbb{N} : \tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : \left\|\tfrac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha_1=0}^{\mathfrak{m}-1}\mathfrak{f}_\alpha^2\left(\varrho_{\alpha_1+\varsigma}(\gamma)\right) - \tfrac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha_2=0}^{\mathfrak{m}-1}\mathfrak{f}_\alpha^2\left(\varrho_{\alpha_2+\varsigma}(\gamma)\right)\right\| \geq \psi\right\}\right| \geq \varkappa\right\}\\ &\subseteq \left\{\mathfrak{q} \in \mathbb{N} : \tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : \left\|\tfrac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha_1=0}^{\mathfrak{m}-1}\mathfrak{f}_\alpha^2\left(\varrho_{\alpha_1+\varsigma}(\gamma) - \varrho(\gamma)\right)\right\| \geq \tfrac{\psi}{2}\right\}\right| \geq \tfrac{\varkappa}{2}\right\}\\ &\cup \left\{\mathfrak{q} \in \mathbb{N} : \tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : \left\|\tfrac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha_2=0}^{\mathfrak{m}-1}\mathfrak{f}_\alpha^2\left(\varrho_{\alpha_2+\varsigma}(\gamma) - \varrho(\gamma)\right)\right\| \geq \tfrac{\psi}{2}\right\}\right| \geq \tfrac{\varkappa}{2}\right\} \in \mathcal{I},\end{aligned}$$

uniformly in $\varsigma$. Hence, $\{\varrho_\alpha\}$ is Fibonacci almost $\mathcal{I}$-statistical Cauchy sequence with respect to almost surely.

Conversely, let the $\{\varrho_\alpha\}$ be Fibonacci almost $\mathcal{I}$-statistical Cauchy sequence with respect to almost surely. Then, for any given $\psi, \varkappa > 0$, there is $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : \left\|\frac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha_1=0}^{\mathfrak{m}-1}\mathfrak{f}_\alpha^2\left(\varrho_{\alpha_1+\varsigma}(\gamma)\right) - \frac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha_2=0}^{\mathfrak{m}-1}\mathfrak{f}_\alpha^2\left(\varrho_{\alpha_2+\varsigma}(\gamma)\right)\right\| \geq \psi\right\}\right| \geq \varkappa\right\} \in \mathcal{I},$$

for any $\gamma \in \Theta$, for each $\psi, \varkappa > 0$ and uniformly in $\varsigma$. Taking $\alpha_1 = \alpha_2 = \alpha_0$ in the above equation, we have $\left(\frac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha_0=0}^{\mathfrak{m}-1}\mathfrak{f}_\alpha^2\left(\varrho_{\alpha_0+\varsigma}(\gamma)\right)\right)_{\mathfrak{m}=1}^{\infty}$ becomes a Fibonacci almost $\mathcal{I}$-statistically Cauchy sequence and so Fibonacci almost $\mathcal{I}$-statistically convergent. For any $\psi, \varkappa > 0$, there exists $\gamma \in A$ such that

$$\left\{\mathfrak{q} \in \mathbb{N} : \frac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : \left\|\frac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_\alpha^2\left(\varrho_{\alpha_0+\varsigma}(\gamma) - \varrho(\gamma)\right)\right\| \geq \frac{\psi}{2}\right\}\right| \geq \frac{\varkappa}{2}\right\} \in \mathcal{I},$$

uniformly in $\varsigma$. Then,

$$\begin{aligned}&\left\{\mathfrak{q} \in \mathbb{N} : \tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : \left\|\tfrac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_\alpha^2\left(\varrho_{\alpha+\varsigma}(\gamma) - \varrho(\gamma)\right)\right\| \geq \psi\right\}\right| \geq \varkappa\right\}\\ &\subseteq \left\{\mathfrak{q} \in \mathbb{N} : \tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : \left\|\tfrac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha=0}^{\mathfrak{m}-1}\mathfrak{f}_\alpha^2\left(\varrho_{\alpha+\varsigma}(\gamma)\right) - \tfrac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha_0=0}^{\mathfrak{m}-1}\mathfrak{f}_\alpha^2\left(\varrho_{\alpha_0+\varsigma}(\gamma)\right)\right\| \geq \tfrac{\psi}{2}\right\}\right| \geq \tfrac{\varkappa}{2}\right\}\\ &\cup \left\{\mathfrak{q} \in \mathbb{N} : \tfrac{1}{\mathfrak{q}}\left|\left\{\mathfrak{m} \leq \mathfrak{q} : \left\|\tfrac{1}{\mathfrak{f}_\mathfrak{m}\mathfrak{f}_{\mathfrak{m}+1}}\sum_{\alpha_0=0}^{\mathfrak{m}-1}\mathfrak{f}_\alpha^2\left(\varrho_{\alpha_0+\varsigma}(\gamma) - \varrho(\gamma)\right)\right\| \geq \tfrac{\psi}{2}\right\}\right| \geq \tfrac{\varkappa}{2}\right\} \in \mathcal{I}.\end{aligned}$$

So, $\{\varrho_\alpha\}$ is Fibonacci almost $\mathcal{I}$-statistically convergent with respect to almost surely to $\varrho$. □

The proof of Theorems 3.6, 3.7, 3.8, can be established in manner similar to Theorem 3.5. Therefore, it is omitted.

**Theorem 3.6** *The fuzzy variable sequence $\{\varrho_\alpha\}$ is Fibonacci almost $\mathcal{I}$-statistically convergent in credibility if and only if it is Fibonacci almost $\mathcal{I}$-statistical Cauchy in credibility.*

**Theorem 3.7** *The fuzzy variable sequence $\{\varrho_\alpha\}$ is Fibonacci almost $\mathcal{I}$-statistically convergent in mean if and only if it is Fibonacci almost $\mathcal{I}$-statistical Cauchy in mean.*

**Theorem 3.8** *The fuzzy variable sequence $\{\varrho_\alpha\}$ is Fibonacci almost $\mathcal{I}$-statistically convergent in distribution if and only if it is Fibonacci almost $\mathcal{I}$-statistical Cauchy in distribution.*

## 4. Conclusions

In this study, we have thoroughly examined the notions of ideal convergence and almost convergence for sequences of fuzzy variables within the framework of credibility theory. By investigating Fibonacci almost $\mathcal{I}$-statistical convergence from various perspectives, including almost surely, credibility, mean, distribution, and uniformly almost surely, we have established key relationships between these different forms of convergence.

Additionally, our exploration of Fibonacci almost $\mathcal{I}$-statistical Cauchy sequences within credibility theory has revealed important results that contribute to a deeper understanding of convergence behavior in this context. These findings provide valuable insights for both theoretical development and practical applications in areas involving fuzzy variables and credibility theory.

Future research could extend these results by exploring other types of convergence and their applications in more complex models or specific fields such as decision theory, risk management, and fuzzy data analysis. The work presented here serves as a foundation for further exploration into the intricate behavior of fuzzy sequences under different convergence criteria, opening the door to new directions in both theoretical and applied credibility theory.

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