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Efficiency Modeling of Sinusoidal Pythagorean Fuzzy Data Envelopment Analysis

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ABSTRACT: A fuzzy data envelopment analysis (DEA) involves solving an optimization linear programming problem to assess the relative efficiency of a group of homogeneous decision-making units with multiple imprecise inputs and outputs. This paper introduces two efficiency models utilizing the sinusoidal Pythagorean fuzzy DEA methodology. A sinusoidal Pythagorean fuzzy number is proposed to mimic hospital datasets' vague and fluctuating characteristics. The paper presents the following two efficiency models: (i) A Sinusoidal Pythagorean fuzzy CCR model. (ii) A Sinusoidal Pythagorean fuzzy BCC model. The expected value and centroid method for Pythagorean fuzzy sets are employed to solve these models. The performance efficiency of these models is evaluated using data collected from twelve hospitals in India.

Key Words: Data envelopment analysis, Pythagorean fuzzy sets, center of gravity method, imprecise inputs and outputs, health sector.

Contents

1. Introduction

In 1965, Zadeh [39] initiated a generalization form of traditional crisp sets by fuzzy sets concept to model imprecise formation and vagueness. A fuzzy set is characterized by a function known as the membership function, whose range is unit interval $[0, 1]$. Since the fuzzy set describes real-life phenomena more realistically than the crisp set, researchers have studied them in detail. Later, in 1986, Atanassov [5] introduced the concept of intuitionistic fuzzy sets (IFSs) as an extension of Zadeh's fuzzy sets. IFS is an appropriate choice to handle uncertainty because it assigns membership and non-membership degrees to the data. Therefore IFS plays an essential role in several areas such as pattern recognition [24], multi-criteria decision-making (MCDM) [34], clustering [22,19], data envelopment analysis (DEA) [29], [31]

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and so many others. Despite many advantages, IFS's significant limitation is that the sum of membership and non-membership degrees must be less than or equal to one. However, real-life phenomena behave so that the sum of membership and non-membership exceeds one. For example, let a membership and non-membership functions denote by a sinusoidal function $\sin\theta$ and $\cos\theta$, respectively. Clearly, $\sin\theta + \cos\theta \geq 1$. see Fig 1.

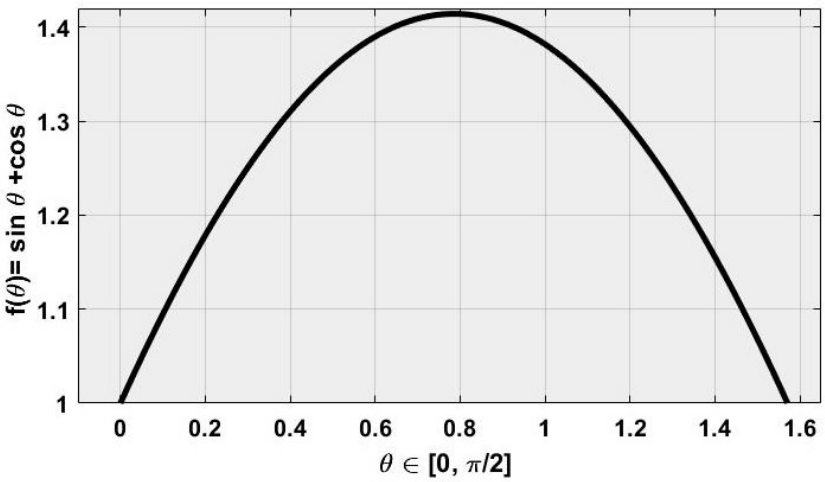

Figure 1: The sum of sine and cosine functions

This limitation motivated Yager [37] to introduce the Pythagorean fuzzy sets (PyFSs) concept. A PyFS is an extension of the IFS that attains the degree of membership and non-membership from unit interval $[0,1]$ provided the square sum of membership and non-membership degree must not exceed one. Mathematically, a PyFS has a quadratic form that expands the membership and non-membership degrees range from unit interval $[0,1]$ into a unit circle. Therefore, a PyFS has wider circumstances to express ambiguity and is more capable than an IFS in depicting vagueness. The theory of PyFS has become a great research domain in the literature. For instance, some of the aggregation operators for the PyFS have been introduced by Yager [38]. Peng et al. [26] represented the definition of distance, similarity, and entropy measures of PyFSs axiomatically. Akram et al. [1] introduced two new techniques to solve linear programming problems with mixed constraints with unrestricted L-R type Pythagorean fuzzy numbers. The related article can be seen in ([12,13], [27], and [35]).

Due to the economic competition scenario between public and private companies, the performance analysis study has been selected as an essential topic for decision-makers and policy-makers. One of the most popular techniques to measure performance analysis is data envelopment analysis (DEA). DEA is a non-parametric technique to measure the relative efficiency of decision-making units (DMUs) that apply multiple inputs to produce multiple outputs. Charnes et al. [7] proposed the traditional DEA model known as the CCR model, a linear programming problem with mixed constraints that measures the relative efficiency of each DMU less than or equal to one. The model was further extended by Banker et al. [6] in the variable return to scale or BCC model. The inputs and outputs are considered crisp numbers in the traditional DEA and most extensions. However, in real-life problems, various factors, such as incomplete and inaccurate information, result in fluctuations that cause uncertainty. Therefore, the theory of fuzzy sets has been selected as a suitable tool to represent incomplete data in DEA (see [28], [8], [20], [25], [14], [15], [30], [18]). In fuzzy set theory, the sum of membership and non-membership degrees equals one. However, in real-life problems, the sum of membership and non-membership degrees is less than one. Therefore, IFS is more appropriate than a fuzzy set for vagueness. The related articles in intuitionistic fuzzy DEA (IFDEA) can be seen in ([9], [3,4,2], [31], [23]). Although IFS is an appropriate technique to deal with imprecise data in which the degree of membership and non-membership is less than or equal to one, there is another possibility that the sum of membership and non-membership degrees may be greater or equal to one. In such cases, the fuzzy set and IFS are useless; instead, the Pythagorean fuzzy set (PyFS) is applicable.

Since Yager introduced the Pythagorean fuzzy set [37], researchers extended and implemented it in several fields, including DEA. For instance, Liang and Xu [21] constructed a new extension of the TOPSIS method under the hesitant Pythagorean fuzzy environment to provide a new interpretation for the elevation of MCDM. Jan et al. [16] introduced a novel concept of complex Pythagorean fuzzy relations (CPyFRs) known as composite, equivalence, and order CPyFRs. A correlation coefficient of PyFSs introduced by Garg [11] that assigns the respective values in the interval range of $[0, 1]$ and further extended into the interval range $[-1, 1]$ by Thao [33]. Wu et al. [36] developed the DEA model in the interval-valued Pythagorean fuzzy environment by converting imprecise and vague information into interval-valued Pythagorean fuzzy numbers through the Likert scale. Fan et al. [10] described the qualitative criteria and developed DEA with undesirable outputs under Pythagorean fuzzy circumstances to deal with the issue of green supplier selection problems. Saini et al. [32] proposed a Pythagorean fuzzy entropy DEA model to represent imprecise data via Pythagorean fuzzy information.

The Pythagorean fuzzy set extends the membership and non-membership degrees range from the unit interval $[0, 1]$ to a unit disc, making it a superior choice over an Intuitionistic Fuzzy Set (IFS) for expressing ambiguity. However, despite this advantage, more research is needed to combine Pythagorean fuzzy sets and DEA models to represent impreciseness with Pythagorean fuzzy numbers. The linear form of an IFS falls short of representing higher-order uncertainties in real-life datasets. This paper introduces the sinusoidal Pythagorean fuzzy number as the first non-linear form to represent dataset ambiguity to address the mentioned gap. In this representation, a sine function defines the membership function, while a cosine function defines the non-membership function. A lemma is provided to prove the proposed sinusoidal as a Pythagorean fuzzy number. Moreover, the conventional Constant Return to Scale (CCR) and Variable Return to Scale (BCC) models have been tailored to suit a Pythagorean fuzzy framework. Ambiguous datasets encompassing inputs and outputs are represented utilizing sinusoidal Pythagorean fuzzy numbers, leading to the development of sinusoidal Pythagorean fuzzy CCR and sinusoidal Pythagorean fuzzy BCC models. We introduce a novel centroid or center of gravity approach for Pythagorean fuzzy sets and establish the expected value for the sinusoidal Pythagorean fuzzy number. As a result, employing these methodologies, we derive Centroid-based sinusoidal Pythagorean fuzzy DEA and Expected-based sinusoidal Pythagorean fuzzy DEA models, serving as extensions of the original CCR and BCC models.

The paper makes the following main contributions:

- We introduce a fusion of DEA and Pythagorean fuzzy sets, leading to the development of sinusoidal Pythagorean fuzzy CCR and sinusoidal Pythagorean fuzzy BCC models.
- We employ a non-linear Pythagorean fuzzy number known as the sinusoidal Pythagorean fuzzy number to handle uncertain inputs and outputs.
- We propose a novel method for determining the centroid of Pythagorean fuzzy sets.
- We employ two solution procedures for solving the sinusoidal Pythagorean fuzzy data envelopment analysis models: the centroid method and the expected value method of the proposed sinusoidal Pythagorean fuzzy numbers.

The paper is structured as follows: Section 2 introduces the fundamental concepts of Pythagorean fuzzy sets and DEA models. In Section 3, the central theoretical part of the paper is discussed. The experimental study is presented in Section 4. Finally, Section 5 concludes the paper.

2. Preliminaries

This section describes the preliminary concepts of Pythagorean fuzzy sets and data envelopment analysis.

2.1. Pythagorean fuzzy set

[37]. Let X be a finite universe discourse set. Then a set P on X is said to be a Pythagorean fuzzy set (PyFS) if it is of the form:
$P = \{\langle x, \mu_P(x), \nu_P(x)\rangle : x \in X\}$ where $\mu_P : X \longrightarrow [0, 1]$ denotes membership function and $\nu_P : X \longrightarrow$

$[0,1]$ denotes non-membership function with the condition: $\mu_P^2(x)+\nu_P^2(x)\leq 1$. The degree of hesitation is $\pi_P(x)=\sqrt{1-\mu_P^2(x)-\nu_P^2(x)}$ (see Fig. 2)

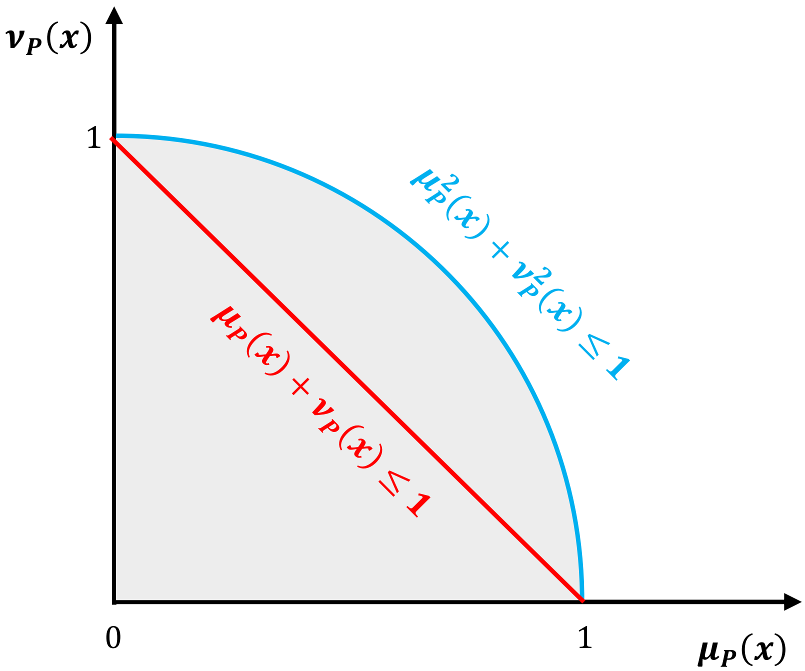

Figure 2: A pictorial representation of Pythagorean fuzzy set

2.2. Pythagorean fuzzy number

[40]. A PyFS in set $\mathbb{R}$ of real numbers is called a Pythagorean fuzzy number (PyFN) if it is of the form:
$\alpha=P(\mu_\alpha,\nu_\alpha)$ here $\mu_\alpha,\nu_\alpha\in[0,1],\pi_\alpha=\sqrt{1-\mu_\alpha^2-\nu_\alpha^2}$, and $\mu_\alpha^2+\nu_\alpha^2\leq 1$.

2.3. Expected value of Pythagorean fuzzy numbers

[40]. Let $\alpha=P(\mu_\alpha,\nu_\alpha)$ be a continuous PyFN such that

$$\mu_\alpha(x)=\begin{cases} f_\alpha(x), & -\infty\leq x\leq x_m \\ g_\alpha(x), & x_m\leq x\leq\infty \end{cases} \tag{2.1}$$

$$\nu_\alpha(x)=\begin{cases} h_\alpha(x), & -\infty\leq x\leq x_m \\ k_\alpha(x), & x_m\leq x\leq\infty \end{cases} \tag{2.2}$$

Here, f_α and k_α are monotonically increasing functions whereas g_α and h_α are monotonically decreasing functions. Then its expected value is defined below [17]:

$$E[P(\mu_\alpha, \nu_\alpha)] = \frac{E[\mu_\alpha] + E[\nu_\alpha]}{2} \tag{2.3}$$

Here

$$E[\mu_\alpha] = x_m + \frac{1}{2}\int_{x_m}^{\infty} g_\alpha(x)dx - \frac{1}{2}\int_{-\infty}^{x_m} f_\alpha(x)dx \tag{2.4}$$

$$E[\nu_\alpha] = x_m + \frac{1}{2}\int_{x_m}^{\infty} k_\alpha(x)dx - \frac{1}{2}\int_{-\infty}^{x_m} h_\alpha(x)dx \tag{2.5}$$

(2.4) and (2.5) are denoted the expected value of a PyFN concerning membership and non-membership grades.

2.4. Data envelopment analysis

Suppose there are n DMUs with m inputs and s outputs. Then, the following matrices are called input matrix and output matrix.

$$X_{m\times n} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}$$

$$Y_{s\times n} = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ y_{s1} & y_{s2} & \cdots & y_{sn} \end{bmatrix}$$

Further, $x_{ik} \in X_{m\times n}, y_{rk} \in Y_{s\times n}$ are the i^{th}, $(i = 1, 2, 3, \cdots m)$ input and r^{th}, $(r = 1, 2, \cdots, s)$ output of k^{th}, $(k = 1, 2, \cdots, n)$ DMU such that $\sum_{i=1}^{m} v_{ik}x_{ik}$ and $\sum_{r=1}^{s} u_{rk}y_{rk}$ are called virtual input and virtual output, respectively. Hence, the mathematical formulation of CCR [7] and BCC [6] models are given below:

2.4.1. The CCR model. Charnes et al. [7]. The CCR (Charnes-Cooper-Rhodes) model, named after its creators, is a constant return-to-scale model that utilizes the abovementioned datasets. It formulates the following linear programming problem model to calculate the efficiency value.

$$\max E_k = \frac{\sum_{r=1}^{s} u_{rk}y_{rk}}{\sum_{i=1}^{m} v_{ik}x_{ik}} \tag{2.6}$$

$$s.t: \frac{\sum_{r=1}^{s} u_{rj}y_{rj}}{\sum_{i=1}^{m} v_{ij}x_{ij}} \leq 1, \quad j = 1, 2, \cdots, n. \tag{2.7}$$

$$u_{rk} \geq 0 \ \forall r, \ 1 \leq r \leq s, \quad v_{ik} \geq 0 \ \forall i, \ 1 \leq i \leq m.$$

Here u_{rk} is the weight of the r^{th} output, and v_{ik} is the weight of the i^{th} input.

2.4.2. The BCC model. Banker et al. [6]. The BCC (Banker-Charnes-Cooper) model, named after its creators, is a variable return-to-scale model that utilizes the aforementioned dataset. It formulates the following linear programming problem model to calculate the efficiency value.

$$\max E_k = \frac{\sum_{r=1}^{s} u_{rk} y_{rk} + u_0}{\sum_{i=1}^{m} v_{ik} x_{ik}} \tag{2.8}$$

$$s.t: \ \frac{\sum_{r=1}^{s} u_{rj} y_{rj} + u_0}{\sum_{i=1}^{m} v_{ij} x_{ij}} \leq 1, \quad j = 1, 2, \cdots, n. \tag{2.9}$$

$$u_{rk} \geq 0 \ \forall r, \ \ 1 \leq r \leq s, \quad v_{ik} \geq 0 \ \forall i, \ \ 1 \leq i \leq m, \quad u_0 \in \mathbb{R}.$$

Here u_{rk} is the weight of the output r^{th} and v_{ik} is the weight of the input i^{th}, and the initial weight u_0 is assigned to k^{th}DMU.

3. Pythagorean fuzzy data envelopment analysis (PyFDEA)

This section is the foundation of the paper and has three subsections. The first subsection defines a sinusoidal Pythagorean fuzzy number and provides a lemma to establish its validity as a Pythagorean fuzzy number (PyFN). The second subsection discusses sinusoidal Pythagorean fuzzy data envelopment analysis. The third subsection explains the proposed models' solution procedure and algorithmic aspects.

3.1. A sinusoidal Pythagorean fuzzy number

Let $P = (x_1, x_2, x_3; x_1', x_2, x_3')$ denotes a Pythagorean fuzzy number. Then it is called a sinusoidal Pythagorean fuzzy number if its membership and non-membership grades are of the form:

$$\mu_P(x) = \begin{cases} \sin\left(\frac{x-x_1}{x_2-x_1} \cdot \frac{\pi}{2}\right), & x_1 \leq x \leq x_2. \\ \sin\left(\frac{x-x_3}{x_2-x_3} \cdot \frac{\pi}{2}\right), & x_2 \leq x \leq x_3. \\ 0, & Otherwise. \end{cases} \tag{3.1}$$

$$\nu_P(x) = \begin{cases} \cos\left(\frac{x-x_1'}{x_2-x_1'} \cdot \frac{\pi}{2}\right), & x_1' \leq x \leq x_2. \\ \cos\left(\frac{x-x_3'}{x_2-x_3'} \cdot \frac{\pi}{2}\right), & x_2 \leq x \leq x_3'. \\ 1, & Otherwise. \end{cases} \tag{3.2}$$

Here, $x_1' \leq x_1 < x_2 < x_3 \leq x_3'$ and $\mu_P^2(x) + \nu_P^2(x) \leq 1 \ \forall x \in [x_1', x_3']$.
See Fig. 3

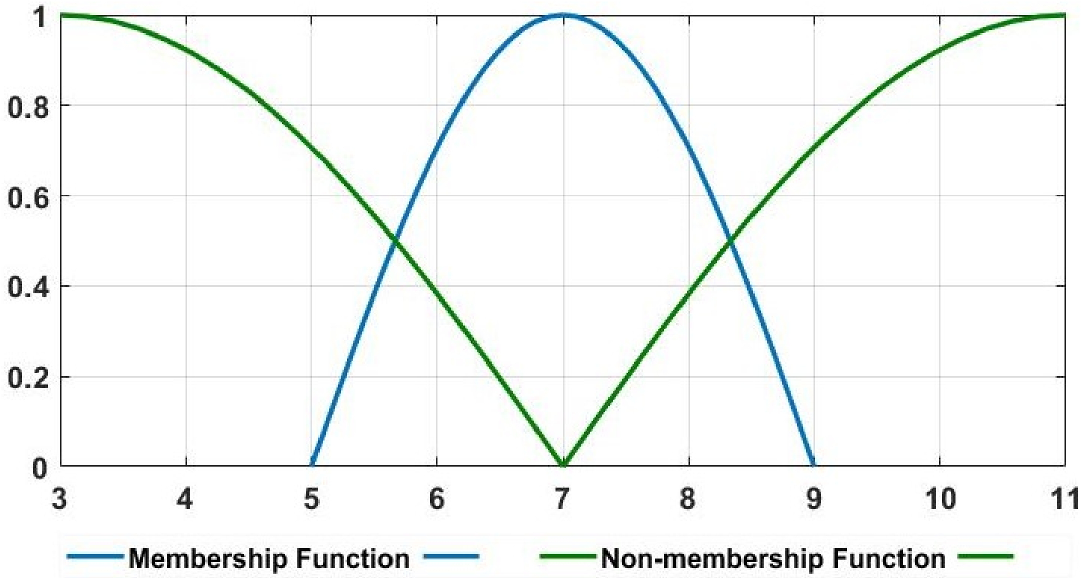

Figure 3: Graphical illustration of the proposed sinusoidal Pythagorean fuzzy number

Lemma 3.1 *The proposed sinusoidal Pythagorean fuzzy number is well-defined.*

Proof: Since $x_1' \leq x_1 \Rightarrow x_1 - x_1' = \gamma, \ \gamma \in \mathbb{R}$.
Trivially, $\frac{x-x_1}{x_2-x_1} \cdot \frac{\pi}{2} \leq \frac{x-x_1+\gamma}{x_2-x_1+\gamma} \cdot \frac{\pi}{2} = \frac{x-x_1'}{x_2-x_1'} \cdot \frac{\pi}{2} \Rightarrow \cos\left(\frac{x-x_1}{x_2-x_1} \cdot \frac{\pi}{2}\right) \geq \cos\left(\frac{x-x_1+\gamma}{x_2-x_1+\gamma} \cdot \frac{\pi}{2}\right)$
Now: $\sin^2\left(\frac{x-x_1}{x_2-x_1} \cdot \frac{\pi}{2}\right) + \cos^2\left(\frac{x-x_1+\gamma}{x_2-x_1+\gamma} \cdot \frac{\pi}{2}\right) \leq \sin^2\left(\frac{x-x_1}{x_2-x_1} \cdot \frac{\pi}{2}\right) + \cos^2\left(\frac{x-x_1}{x_2-x_1} \cdot \frac{\pi}{2}\right) = 1$
A similar process for $x_3 \leq x_3'$.
Hence proved. □

3.2. A sinusoidal Pythagorean fuzzy data envelopment analysis

This subsection outlines a sinusoidal Pythagorean fuzzy data envelopment analysis, wherein inputs and outputs are expressed in the sinusoidal Pythagorean fuzzy form. Denoting the sinusoidal Pythagorean fuzzy inputs and outputs of the k^{th} DMU as $(x_{ik}^L, x_{ik}^M, x_{ik}^U; x_{ik}'^L, x_{ik}^M, x_{ik}'^U)$ and $(y_{rk}^L, y_{rk}^M, y_{rk}^U; y_{rk}'^L, y_{rk}^M, y_{ik}'^U)$, respectively, the following mathematical formulations are provided for the sinusoidal Pythagorean fuzzy CRS and sinusoidal Pythagorean fuzzy VRS models.

3.2.1. A sinusoidal Pythagorean fuzzy CCR model. This model is a modified version of the traditional CCR model. Its purpose is to evaluate the efficiency value of DMUs when there is uncertainty in the inputs and outputs. The proposed approach uses sinusoidal Pythagorean fuzzy numbers to represent these uncertain inputs/outputs. These numbers are defined in equations (3.1) and (3.2), which provide membership and non-membership grades.

$$E_k^{SPyCCR} = \max \sum_{k=1}^{n} \sum_{r=1}^{s} u_{rk}(y_{rk}^L, y_{rk}^M, y_{rk}^U; y_{rk}'^L, y_{rk}^M, y_{ik}'^U) \tag{3.3}$$

$$s.t \ \sum_{k=1}^{n} \sum_{i=1}^{m} v_{ik}(x_{ik}^L, x_{ik}^M, x_{ik}^U; x_{ik}'^L, x_{ik}^M, x_{ik}'^U) = 1. \tag{3.4}$$

$$\sum_{j=1}^{n} \sum_{r=1}^{s} u_{rj}(y_{rj}^L, y_{rj}^M, y_{rj}^U; y_{rj}'^L, y_{rj}^M, y_{ij}'^U) - \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij}(x_{ij}^L, x_{ij}^M, x_{ij}^U; x_{ij}'^L, x_{ij}^M, x_{ij}'^U) \leq 0. \tag{3.5}$$

$$u_{rk} \geq \epsilon \ \ \forall r, v_{ik} \geq \epsilon \ \ \forall i.$$

Here u_{rk} is the r^{th} weight of output, v_{ik} is the i^{th} weight of input, and ϵ is non-Archimedean infinitesimal.

3.2.2. A sinusoidal Pythagorean fuzzy BCC model. This model evaluates the performance efficiency measure of DMUs (Decision-Making Units) based on uncertain inputs and outputs. It is an improved version of the conventional BCC model and uses membership and non-membership grades of the sinusoidal Pythagorean fuzzy number to represent inputs and outputs. The equations (3.1) and (3.2) define the representation of inputs and outputs through the sinusoidal Pythagorean fuzzy number.

$$E_k^{SPyBCC} = \max \sum_{k=1}^{n} \sum_{r=1}^{s} u_{rk}(y_{rk}^L, y_{rk}^M, y_{rk}^U; y_{rk}'^L, y_{rk}^M, y_{ik}'^U) + u_0 \tag{3.6}$$

$$s.t \sum_{k=1}^{n} \sum_{i=1}^{m} v_{ik}(x_{ik}^L, x_{ik}^M, x_{ik}^U; x_{ik}'^L, x_{ik}^M, x_{ik}'^U) = 1. \tag{3.7}$$

$$\sum_{j=1}^{n} \sum_{r=1}^{s} u_{rj}(y_{rj}^L, y_{rj}^M, y_{rj}^U; y_{rj}'^L, y_{rj}^M, y_{ij}'^U) - \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij}(x_{ij}^L, x_{ij}^M, x_{ij}^U; x_{ij}'^L, x_{ij}^M, x_{ij}'^U) + u_0 \leq 0. \tag{3.8}$$

$u_{rk} \geq \epsilon \;\; \forall r, v_{ik} \geq \epsilon \;\; \forall i.$

In the equation, u_{rk} represents the r^{th} weight of the output, v_{ik} represents the i^{th} weight of the input, and ϵ is a non-Archimedean infinitesimal. The variable u_0 remains unconstrained and can assume positive or negative values. When u_0 is positive, the model is classified as exhibiting increasing returns to scale, whereas if u_0 is negative, it indicates decreasing returns to scale.

3.3. Solution Methodology

This section explains the methodology used to solve the proposed models. First, we introduce an extension of the center of gravity method designed explicitly for Pythagorean fuzzy sets. Second, we delve into the expected value concept of the proposed sinusoidal Pythagorean fuzzy number. Lastly, we derive deterministic expressions for the sinusoidal Pythagorean fuzzy CCR and sinusoidal Pythagorean fuzzy BCC models by utilizing both the centroid and expected value of the sinusoidal Pythagorean fuzzy number.

Definition 3.1 *Let $P = \{\langle x, \mu_P(x), \nu_P(x)\rangle : x \in X\}$ denotes a Pythagorean fuzzy set. Then, the extended centroid method is defined as a novel center of gravity method for PyFS.*

$$COG_P(x) = \frac{\int_{-\infty}^{\infty} \mid \mu_P(x) - \pi_P(x) \mid x dx}{\int_{-\infty}^{\infty} \mid \mu_P(x) - \pi_P(x) \mid dx} \tag{3.9}$$

Remark 3.1 *Let us consider the proposed sinusoidal Pythagorean fuzzy number denoted by P, whose membership and non-membership grades are defined by equations (3.1) and (3.2), respectively. Then, we can calculate the $\mid \mu_P(x) - \pi_P(x) \mid$ as follows:*

$$|\mu_P(x) - \pi_P(x)| = \begin{cases} \mid \sqrt{1 - \cos^2(\frac{x - x_1'}{x_2 - x_1'} \cdot \frac{\pi}{2})} \mid, & x_1' \leq x \leq x_1. \\ \mid \sin(\frac{x - x_1}{x_2 - x_1} \cdot \frac{\pi}{2}) - \sqrt{1 - \sin^2(\frac{x - x_1}{x_2 - x_1} \cdot \frac{\pi}{2}) - \cos^2(\frac{x - x_1'}{x_2 - x_1'} \cdot \frac{\pi}{2})} \mid, & x_1 \leq x \leq x_2. \\ \mid \sin(\frac{x - x_3}{x_2 - x_3} \cdot \frac{\pi}{2}) - \sqrt{1 - \sin^2(\frac{x - x_3}{x_2 - x_3} \cdot \frac{\pi}{2}) - \cos^2(\frac{x - x_3'}{x_2 - x_3'} \cdot \frac{\pi}{2})} \mid, & x_2 \leq x \leq x_3. \\ \mid \sqrt{1 - \cos^2(\frac{x - x_3'}{x_2 - x_3'} \cdot \frac{\pi}{2})} \mid, & x_3 \leq x \leq x_3' \end{cases}$$

To simplify things, let's look at the equations we get from the following steps:

$$X_{11} =\left| \sqrt{1-\cos^2(\frac{x-x_1'}{x_2-x_1'}\cdot\frac{\pi}{2})} \right| \tag{3.10}$$

$$X_{12} =\left| \sin(\frac{x-x_1}{x_2-x_1}\cdot\frac{\pi}{2}) - \sqrt{1-\sin^2(\frac{x-x_1}{x_2-x_1}\cdot\frac{\pi}{2})-\cos^2(\frac{x-x_1'}{x_2-x_1'}\cdot\frac{\pi}{2})} \right| \tag{3.11}$$

$$X_{23} =\left| \sin(\frac{x-x_3}{x_2-x_3}\cdot\frac{\pi}{2}) - \sqrt{1-\sin^2(\frac{x-x_3}{x_2-x_3}\cdot\frac{\pi}{2})-\cos^2(\frac{x-x_3'}{x_2-x_3'}\cdot\frac{\pi}{2})} \right| \tag{3.12}$$

$$X_{33} =\left| \sqrt{1-\cos^2(\frac{x-x_3'}{x_2-x_3'}\cdot\frac{\pi}{2})} \right| \tag{3.13}$$

The novel center of gravity, as defined by equation (3.9), has been modified in the following manner:

$$COG_P(x) = \frac{\int_{x_1'}^{x_1} X_{11}\cdot xdx + \int_{x_1}^{x_2} X_{12}\cdot xdx + \int_{x_2}^{x_3} X_{23}\cdot xdx + \int_{x_3}^{x_3'} X_{33}\cdot xdx}{\int_{x_1'}^{x_1} X_{11}\cdot xdx + \int_{x_1}^{x_2} X_{12}dx + \int_{x_2}^{x_3} X_{23}dx + \int_{x_3}^{x_3'} X_{33}dx} \tag{3.14}$$

Assuming $(x_{ik}^L, x_{ik}^M, x_{ik}^U; x_{ik}'^L, x_{ik}^M, x_{ik}'^U)$ and $(y_{rk}^L, y_{rk}^M, y_{rk}^U; y_{rk}'^L, y_{rk}^M, y_{ik}'^U)$ represent imprecise inputs and outputs conveyed by sinusoidal Pythagorean fuzzy numbers, equations (3.10), (3.11), (3.12), and (3.13) undergo modification in the subsequent equations to accommodate imprecise input and imprecise output:

$$X_{11}^k =\left| \sqrt{1-\cos^2(\frac{x-x_{ik}'^L}{x_{ik}^M-x_{ik}'^L}\cdot\frac{\pi}{2})} \right| \tag{3.15}$$

$$X_{12}^k =\left| \sin(\frac{x-x_{ik}^L}{x_{ik}^M-x_{ik}^L}\cdot\frac{\pi}{2}) - \sqrt{1-\sin^2(\frac{x-x_{ik}^L}{x_{ik}^M-x_{ik}^L}\cdot\frac{\pi}{2})-\cos^2(\frac{x-x_{ik}'^L}{x_{ik}^M-x_{ik}'^L}\cdot\frac{\pi}{2})} \right| \tag{3.16}$$

$$X_{23}^k =|\left| \sin(\frac{x-x_{ik}^U}{x_{ik}^M-x_{ik}^U}\cdot\frac{\pi}{2}) - \sqrt{1-\sin^2(\frac{x-x_{ik}^U}{x_{ik}^M-x_{ik}^U}\cdot\frac{\pi}{2})-\cos^2(\frac{x-x_{ik}'^U}{x_{ik}^M-x_{ik}'^U}\cdot\frac{\pi}{2})} \right| \tag{3.17}$$

$$X_{33}^k =\left| \sqrt{1-\cos^2(\frac{x-x_{ik}'^U}{x_{ik}^M-x_{ik}'^U}\cdot\frac{\pi}{2})} \right| \tag{3.18}$$

$$Y_{11}^k =\left| \sqrt{1-\cos^2(\frac{y-y_{rk}'^L}{y_{rk}^M-y_{rk}'^L}\cdot\frac{\pi}{2})} \right| \tag{3.19}$$

$$Y_{12}^k =\left| \sin(\frac{y-y_{rk}^L}{y_{rk}^M-y_{rk}^L}\cdot\frac{\pi}{2}) - \sqrt{1-\sin^2(\frac{y-y_{rk}^L}{y_{rk}^M-y_{rk}^L}\cdot\frac{\pi}{2})-\cos^2(\frac{y-y_{rk}'^L}{y_{rk}^M-y_{rk}'^L}\cdot\frac{\pi}{2})} \right| \tag{3.20}$$

$$Y_{23}^k =\left| \sin(\frac{y-y_{rk}^U}{y_{rk}^M-y_{rk}^U}\cdot\frac{\pi}{2}) - \sqrt{1-\sin^2(\frac{y-y_{rk}^U}{y_{rk}^M-y_{rk}^U}\cdot\frac{\pi}{2})-\cos^2(\frac{y-y_{rk}'^U}{y_{rk}^M-y_{rk}'^U}\cdot\frac{\pi}{2})} \right| \tag{3.21}$$

$$Y_{33}^k =\left| \sqrt{1-\cos^2(\frac{y-y_{rk}'^U}{y_{rk}^M-y_{rk}'^U}\cdot\frac{\pi}{2})} \right| \tag{3.22}$$

The adjusted equation (3.14) now characterizes the precise numerical value associated with the sinusoidal Pythagorean inputs and outputs.

$$\tilde{x}_{ik} = \frac{\int_{x'^L_{ik}}^{x^L_{ik}} X^k_{11} x dx + \int_{x^L_{ik}}^{x^M_{ik}} X^k_{12} \cdot x dx + \int_{x^M_{ik}}^{x^U_{ik}} X^k_{23} \cdot x dx + \int_{x^U_{ik}}^{x'^U_{ik}} X^k_{33} \cdot x dx}{\int_{x'^L_{ik}}^{x^L_{ik}} X^k_{11} dx + \int_{x^L_{ik}}^{x^M_{ik}} X^k_{12} dx + \int_{x^M_{ik}}^{x^U_{ik}} X^k_{23} + \int_{x^U_{ik}}^{x'^U_{ik}} X^k_{33}} \tag{3.23}$$

$$\tilde{y}_{rk} = \frac{\int_{y'^L_{k}}^{y^L_{rk}} Y^k_{11} y dy + \int_{y^L_{rk}}^{y^M_{rk}} Y^k_{12} y dy + \int_{y^M_{rk}}^{y^U_{rk}} Y^k_{23} y dy + \int_{y^U_{rk}}^{y'^U_{rk}} Y^k_{33} y dy}{\int_{y'^L_{k}}^{y^L_{rk}} Y^k_{11} dy + \int_{y^L_{rk}}^{y^M_{rk}} Y^k_{12} dy + \int_{y^M_{rk}}^{y^U_{rk}} Y^k_{23} dy + \int_{y^U_{rk}}^{y'^U_{rk}} Y^k_{33} dy} \tag{3.24}$$

Definition 3.2 *Let $P = (x_L, x_M, x_U; x'_L, x_M, x'_U)$ be the sinusoidal Pythagorean fuzzy number. Its membership and non-membership grades are defined by* (3.1) *and* (3.2)*. The expected value is of the form:*

$$\begin{aligned} E[\mu_P(x)] &= x_M + \frac{1}{2}\int_{x_M}^{x_U} \sin\left(\frac{x - x_U}{x_M - x_U} \cdot \frac{\pi}{2}\right) dx - \frac{1}{2}\int_{x_L}^{x_M} \sin\left(\frac{x - x_L}{x_M - x_L} \cdot \frac{\pi}{2}\right) dx \\ &= \frac{x_L + (\pi - 2)x_M + x_U}{\pi} \end{aligned} \tag{3.25}$$

$$\begin{aligned} E[\nu_P(x)] &= x_M + \frac{1}{2}\int_{x_M}^{x'_U} \cos\left(\frac{x - x'_U}{x_M - x'_U} \cdot \frac{\pi}{2}\right) dx - \frac{1}{2}\int_{x'_L}^{x_M} \cos\left(\frac{x - x'_L}{x_M - x'_L} \cdot \frac{\pi}{2}\right) dx \\ &= \frac{x'_L + (\pi - 2)x_M + x'_U}{\pi} \end{aligned} \tag{3.26}$$

$$EV[(\mu_P(x), \nu_P(x))] = \frac{E[\mu_P(x) + \nu_P(x)]}{2} = \frac{x'_L + x_L + x_U + x'_U + 2(\pi - 2)x_M}{2\pi} \tag{3.27}$$

Assuming $(x^L_{ik}, x^M_{ik}, x^U_{ik}; x'^L_{ik}, x^M_{ik}, x'^U_{ik})$ and $(y^L_{rk}, y^M_{rk}, y^U_{rk}; y'^L_{rk}, y^M_{rk}, y'^U_{ik})$ represent imprecise inputs and outputs conveyed by the sinusoidal Pythagorean fuzzy number, equation (3.27) is provided with their corresponding precise inputs and outputs.

$$\overline{x}_{ik} = \frac{x'^L_{ik} + x^L_{ik} + x^U_{ik} + x'^U_{ik} + 2(\pi - 2)x^M_{ik}}{2\pi} \tag{3.28}$$

$$\overline{y}_{rk} = \frac{y'^L_{rk} + y^L_{rk} + y^U_{rk} + y'^U_{rk} + 2(\pi - 2)y^M_{rk}}{2\pi} \tag{3.29}$$

Property: The expected value of the Pythagorean fuzzy set is a linear operator, i.e., for any two P_1, P_2 Pythagorean fuzzy sets and any two real numbers say λ_1, λ_2:

$$EV[\lambda_1 P_1 + \lambda_2 P_2] = \lambda_1 EV[P_1] + \lambda_2 EV[P_2] \tag{3.30}$$

3.3.1. Centroid-based sinusoidal Pythagorean fuzzy data envelopment analysis. The subsection outlines the centroid-based approach to sinusoidal Pythagorean fuzzy data envelopment analysis. Since an integral is a linear operator, (3.9) ensures that linearity is preserved in the proposed DEA models. As a result, by applying (3.14), the sinusoidal Pythagorean fuzzy CCR model is transformed into the centroid-based sinusoidal Pythagorean fuzzy CCR model. Similarly, the centroid-based sinusoidal Pythagorean fuzzy BCC model is derived from the sinusoidal Pythagorean fuzzy BCC model.

(i) Centroid-based sinusoidal Pythagorean fuzzy CCR model

This model represents a deterministic version of the sinusoidal Pythagorean fuzzy CCR model. Inputs are acquired using equation (3.23), and outputs are obtained using equation (3.24).

$$\tilde{E}_k^{CCR} = \max \sum_{k=1}^{n}\sum_{r=1}^{s} u_{rk}\tilde{y}_{rk} \tag{3.31}$$

$$s.t \sum_{k=1}^{n}\sum_{i=1}^{m} v_{ik}\tilde{x}_{ik} = 1 \tag{3.32}$$

$$\sum_{j=1}^{n}\sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{j=1}^{n}\sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} \leq 0 \tag{3.33}$$

$$u_{rk} \geq \epsilon \forall r,\ \ v_{ik} \geq \epsilon\ \ \forall i.$$

Here u_{rk} and v_{ik} are the r^{th} output and the i^{ih} input weights of k^{th} DMU.

(ii) Centroid-based sinusoidal Pythagorean fuzzy BCC model This is a deterministic form of the sinusoidal Pythagorean BCC model in which inputs and outputs are obtained through (3.23) and (3.24), respectively.

$$\tilde{E}_k^{BCC} = \max \sum_{k=1}^{n}\sum_{r=1}^{s} u_{rk}\tilde{y}_{rk} + \tilde{u}_0 \tag{3.34}$$

$$s.t \sum_{k=1}^{n}\sum_{i=1}^{m} v_{ik}\tilde{x}_{ik} = 1 \tag{3.35}$$

$$\sum_{j=1}^{n}\sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{j=1}^{n}\sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} + \tilde{u}_0 \leq 0 \tag{3.36}$$

$$u_{rk} \geq \epsilon \forall r,\ \ v_{ik} \geq \epsilon\ \ \forall i\ \ \tilde{u}_0 \in \mathbb{R}$$

Here, u_{rk} and v_{ik} represent the weights of the r^{th} output and the i^{th} input, respectively, for the k^{th} DMU. $\tilde{u}_0$ is a free variable that can take on either positive or negative values. Furthermore, if $\tilde{u}_0$ is positive, the model is classified as exhibiting increasing returns to scale. Conversely, if $\tilde{u}_0$ is negative, it indicates decreasing returns to scale.

3.3.2. Expected-based sinusoidal Pythagorean fuzzy data envelopment analysis. The subsection explains the expected-based approach to sinusoidal Pythagorean fuzzy data envelopment analysis. Since an expectation is a linear operator, the equation (3.30) ensures linearity preservation in the proposed DEA models. Therefore, the sinusoidal Pythagorean fuzzy CCR model can be transformed into the Expected-based sinusoidal Pythagorean fuzzy CCR model by applying the equation (3.27). Likewise, the sinusoidal Pythagorean fuzzy BCC model can be converted into the Expected-based sinusoidal Pythagorean fuzzy variable return to scale model by the same equation.

(i) Expected-based sinusoidal Pythagorean fuzzy CCR model
The model is a deterministic form of the sinusoidal Pythagorean fuzzy CCR model in which inputs and outputs are obtained via (3.28) and (3.29), respectively. Hence, its mathematical representation is of the form:

$$\overline{E}_k^{CCR} = \max \sum_{k=1}^{n}\sum_{r=1}^{s} u_{rk}\overline{y}_{rk} \tag{3.37}$$

$$s.t \sum_{k=1}^{n}\sum_{i=1}^{m} v_{ik}\overline{x}_{ik} = 1 \tag{3.38}$$

$$\sum_{j=1}^{n}\sum_{r=1}^{s} u_{rj}\overline{y}_{rj} - \sum_{j=1}^{n}\sum_{i=1}^{m} v_{ij}\overline{x}_{ij} \leq 0 \tag{3.39}$$

$$u_{rk} \geq \epsilon \forall r,\ \ v_{ik} \geq \epsilon\ \ \forall i.$$

Here u_{rk} and v_{ik} are the r^{th} output and the i^{ih} input weights of k^{th} DMU.

(ii) Expected-based sinusoidal Pythagorean fuzzy BCC model

The model is a deterministic form of the sinusoidal Pythagorean fuzzy BCC model in which inputs are obtained via (3.28) and outputs are produced through (3.29). Thus, it has the following mathematical formulation:

$$\overline{E}_k^{BCC} = \max \sum_{k=1}^{n} \sum_{r=1}^{s} u_{rk}\overline{y}_{rk} + u_0 \tag{3.40}$$

$$s.t \ \sum_{k=1}^{n} \sum_{i=1}^{m} v_{ik}\overline{x}_{ik} = 1 \tag{3.41}$$

$$\sum_{j=1}^{n} \sum_{r=1}^{s} u_{rj}\overline{y}_{rj} - \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij}\overline{x}_{ij} + u_0 \leq 0 \tag{3.42}$$

$$u_{rk} \geq \epsilon \forall r, \ \ v_{ik} \geq \epsilon \ \ \forall i \ \ u_0 \in \mathbb{R}$$

Here u_{rk} and v_{ik} represent the r^{th} output and the i^{th} input weights of the k^{th} DMU. u_0 is a free variable with a positive or negative value. In addition, if u_0 is positive, the model has increasing returns to scale. Conversely, if u_0 is negative, the model is said to have decreasing returns to scale.

Algorithm 1:

Algorithm 1 describes the solution procedure for the Centroid-based and Expected-based sinusoidal Pythagorean fuzzy DEA models, which comprises three steps.

1. Illustration of Imprecise Inputs and Outputs: Initially, imprecise inputs and outputs are represented using sinusoidal Pythagorean fuzzy numbers.

2. Transformation of Sinusoidal Pythagorean Fuzzy Inputs/Outputs: Next, the sinusoidal Pythagorean fuzzy inputs and outputs are converted into corresponding crisp inputs and outputs.

3. Efficiency Calculation: Finally, utilizing these crisp inputs and outputs, the efficiency value of each of the four models is computed.

Steps of Algorithm:

1. First step: Collect the inputs and outputs
   - Dataset: Representing by sinusoidal Pythagorean fuzzy numbers.
2. Second step: Transformation process
   - Centroid Method of Pythagorean Fuzzy Sets
     (a) Transform sinusoidal Pythagorean fuzzy inputs via (3.23).
     (b) Transform sinusoidal Pythagorean fuzzy outputs via (3.24).
   - Expected value of Pythagorean Fuzzy sets
     (a) Transform sinusoidal Pythagorean fuzzy inputs through (3.28).
     (b) Transform sinusoidal Pythagorean fuzzy outputs through (3.29).
3. Third step: Efficiency evaluation
   - Calculate the efficiency value of $\tilde{E}_k^{CCR}$ by Centroid-based sinusoidal Pythagorean fuzzy CCR model.
   - Calculate the efficiency value of $\tilde{E}_k^{BCC}$ by Centroid-based sinusoidal Pythagorean fuzzy BCC model.
   - Calculate the efficiency value of $\overline{E}_k^{CCR}$ by Expected-based sinusoidal Pythagorean fuzzy CCR model.
   - Calculate the efficiency value of $\overline{E}_k^{BCC}$ by Expected-based sinusoidal Pythagorean fuzzy BCC model.

4. Experimental analysis

Let's delve into a real-life application problem to experiment with and analyze hospital performance efficiency, utilizing a dataset borrowed from Arya and Yadav [2]. Their study provided two inputs and two outputs of twelve hospitals within an intuitionistic fuzzy environment. We will employ this dataset to measure and analyze the hospitals' performance under Pythagorean fuzzy circumstances.

Efficiency Evaluation of Hospitals: Real-life applications, such as the healthcare sector, often entail uncertainty in their datasets. For instance, there may be a shortage of beds or inadequate healthcare services in a hospital, causing some patients to leave without treatment, which results in uncertainty regarding the number of patients being treated. This fluctuation implies vagueness in a hospital's medical and non-medical staff numbers. Therefore, this paper adopts the proposed sinusoidal Pythagorean fuzzy numbers dataset to address uncertainty in the health sector. In this example, we consider twelve hospitals from the Meerut district of Uttar Pradesh, India, to conduct a performance analysis with two inputs and two outputs. These inputs (number of doctors and number of pharmacists) and outputs (inpatients and outpatients) are represented via sinusoidal Pythagorean fuzzy numbers and illustrated in Table 1.

Table 1: Sinusoidal Pythagorean fuzzy inputs and outputs

DMUs	Input 1	Input 2	Output 1	Output 2
H1	(12,13,16;9,12,17)	(3,5,8;2,5,10)	(3640,3650,3665;3635,3650,3695)	(134130,134137,134140; 134125,134137,134145)
H2	(12,12,12;12,12,12)	(3,5,7;2,5,9)	(4150,4160,4175;4148,4160,4195)	(116060,116062,116070; 116055,116062,116075)
H3	(11,12,15;9,12,18)	(2,4,5;1,4,6)	(4360,4370,4380;4357,4370,4398)	(94060,94066,94070; 94055,94066,94075)
H4	(5,8,11;2,8,14)	(1,1,3;1,1,5)	(485,492,500;483,492,515)	(24325,24329,24334; 24322,24329,24338)
H5	(7,10,13;4,10,16)	(3,4,6;2,4,10)	(2460,2464,2470;2458,2464,2475)	(99745,99748,99750; 99742,99748,99755)
H6	(8,11,14;5,11,17)	(2,3,4;1,3,5)	(1360,1368,1378;1358,1368,1398)	(49398,49401,49405; 49395,49401,49409)
H7	(7,10,11;5,10,15)	(1,2,6;1,2,18)	(1055,1062,1080;1050,1062,1083)	(37769,37772,37776; 37765,37772,37779)
H8	(7,11,14;5,11,17)	(2,4,7;1,4,19)	(1295,1302,1310;1290,1302,1325)	(82838,82841,82845; 82835,82841,82849)
H9	(9,12,15;6,12,18)	(2,5,7;1,5,15)	(1660,1671,1690;1657,1671,16105)	(100590,100596,100600; 100586,100596,100605)
H10	(13,16,19;10,16,22)	(2,4,6;1,4,18)	(1010,1018,1035;1008,1018,1045)	(64349,64351,64358; 64345,64351,64360)
H11	(8,11,14;5,11,17)	(3,5,8;2,5,20)	(1500,1504,1515;1495,1504,1535)	(80050,80056,80060; 80045,80056,80065)
H12	(5,5,11;2,8,15)	(3,4,6;1,4,7)	(1960,1965,1972;1958,1965,1985)	(58160,58167,58170; 58157,58167,58174)

Table 2: Comparison between the proposed models and the models presented by A. Arya [2]

	Proposed Models				A.Arya [2] Models	
DMUs	$\tilde{E}_k^{CCR}$	$\tilde{E}_k^{BCC}$	$\overline{CCR}$	$\overline{BCC}$	CCR	BCC
H1	1.0000	1.0000	1.0000	1.0000	0.7794	0.8833
H2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
H3	1.0000	1.0000	1.0000	1.0000	0.6806	0.7583
H4	0.3881	1.0000	0.4111	1.0000	0.5714	0.5741
H5	0.9348	1.0000	0.9904	1.0000	0.6489	0.6490
H6	0.6371	0.9041	0.6444	0.9361	0.5236	0.5322
H7	0.3435	0.7910	0.3891	0.8328	0.5750	0.5900
H8	0.6883	0.8327	0.7586	0.8533	0.5434	0.7713
H9	1.0000	1.0000	0.9585	0.9702	0.5600	0.7992
H10	0.3769	0.5455	0.4503	0.6011	0.3820	0.4213
H11	0.6820	0.8447	0.7225	0.8298	0.5360	0.7503
H12	0.8195	1.0000	0.7301	1.0000	0.5417	0.6205

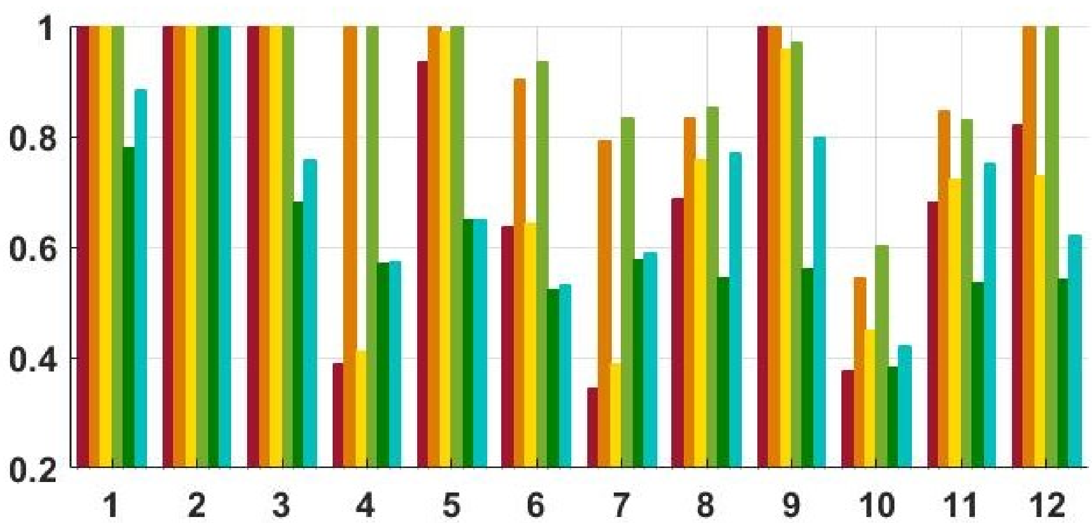

Figure 4: A pictorial representation of Table. 2

In this analysis, we examine hospitals' efficiency based on their output using two models: centroid-based sinusoidal Pythagorean fuzzy DEA and Expected-based sinusoidal Pythagorean fuzzy DEA. We assign efficiency scores using various methods, including $TildeECCR_k$, $TildeEBCC_k$, $overlineECCR_k$, and $overlineEBCC_k$. The efficiency scores measure each hospital's relative performance regarding output1 and output2.

Our findings reveal that hospitals 1, 2, and 9 exhibit the largest output2, while hospitals 4 and 7 have the smallest output1 and output2, respectively. When comparing efficiency scores, we can see that hospitals 7, 4, and 10 yield the most miniature efficiency scores based on CCR models, whereas BCC models assign the lowest efficiency score to hospital 10. The hospitals with the best or optimal efficiency scores under CCR models are 1, 2, and 3, while this extends to hospitals 4, 5, and 12 under BCC models. It is noteworthy that hospital number 9 achieves the optimal efficiency score when the proposed centroid method converts the uncertain inputs and outputs.

Furthermore, we compared the efficiency scores of our proposed models with those from A. Arya [2] models. Our models yield more efficient Decision-Making Units (DMUs) and larger values for inefficient DMUs. Finally, we present a graphical representation of our results in Figure 4, which shows the efficiency scores of Centroid-based sinusoidal Pythagorean fuzzy DEA and Expected-based sinusoidal Pythagorean fuzzy DEA in both the CCR and BCC version. To calculate efficiency scores, we used MATLAB programs to implement the expected value and centroid methods of proposed sinusoidal Pythagorean fuzzy inputs and outputs.

5. Conclusion

Decision Units (DMUs) often encounter challenges from uncertain and imprecise data in real-world decision-making scenarios, impacting their inputs and outputs. This paper introduces a novel approach for assessing DMU efficiency utilizing Pythagorean fuzzy Data Envelopment Analysis (PyFDEA) with sinusoidal Pythagorean fuzzy numbers (SPyFN) to tackle this issue. SPyFN employs non-linear membership and non-membership functions based on sine and cosine functions, enhancing its ability to capture hesitation and ambiguity within the dataset. The paper extends classical CCR and BCC models to include sinusoidal Pythagorean fuzzy CCR and sinusoidal Pythagorean fuzzy BCC models.

This study introduces a new centroid method for Pythagorean fuzzy sets. By leveraging this proposed centroid method alongside the expected value of Pythagorean fuzzy sets, we establish centroid-based sinusoidal Pythagorean fuzzy DEA and Expected-based sinusoidal Pythagorean fuzzy DEA within both the CCR and BCC models. By applying the proposed method to twelve DMUs, we illustrate its applicability and validity. The results demonstrate that our method can assess DMU efficiency scores more accurately and reasonably than existing methods.

However, the paper has some limitations that could be addressed in future research endeavors. For instance, developing a network PyFDEA model based on SPyFN could be explored to capture DMUs' intermediate inputs and outputs. Additionally, extending the proposed method to other DEA models could broaden its applicability and enhance its effectiveness.

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