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The Laplacian Minimum Pendant Dominating Degree Energy of a Graph

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ABSTRACT: In this research paper, we introduce the concept of the laplacian minimum pendant dominating degree energy of a graph, denoted by $LE_P^D(G)$, and compute its value for several well known graph families, including the complete graph, complete bipartite graph, double star graph and dumbbell graph. In addition, we analyze the theoretical upper and lower bounds of $LE_P^D(G)$, shedding light on its behavior and possible range over different categories of graphs.

Key Words: Laplacian minimum pendant dominating degree set, Laplacian minimum pendant dominating degree matrix, Laplacian Mmnimum pendant dominating degree eigenvalues, Laplacian minimum pendant dominating degree energy.

Contents

1. Introduction

The concept of energy of a graph was introduced by I. Gutman [8] in the year 1978. Let G be a graph with n vertices and m edges and let $A = (a_{ij})$ be a adjacency matrix of a graph G. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, arranged in non-increasing order are the eigenvalues of G. As matrix A is real and symmetric, the eigenvalues of G are real with sum equal to zero. The energy $E(G)$ is defined to be the sum of the absolute values of the eigenvalues of G, given as follows:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

I. Gutman and B. Zhou [10] defined the laplacian energy of a graph G in the year 2006. Rajesh Kanna et al. [21] defined minimum dominating energy of a graph. Recently A. R. Nagalakshmi et al. [15] defined degree energy of a graph. Motivated by these papers the present authors defined laplacian minimum pendant dominating degree energy of a graph. Let G be a graph with n vertices and m edges The laplacian matrix of a graph G is denoted by $L = (l_{ij})$, which is a square matrix of order $n \times n$. The elements of the laplacian matrix are defined as follows:

$$l_{ij} = \begin{cases} -1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ d_i, & \text{i = j} \end{cases}$$

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where d_i is the vertex of v_i. The laplacian energy of a graph G is defined using the eigenvalues of its laplacian matrix L_{ij}. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of laplacian matrix and the laplacian energy of G is defined as

$$LE(G) = \sum_{i=1}^{n} \left| \lambda_i - \frac{2m}{n} \right|$$

The concept of minimum pendant dominating energy [17] and laplacian energy [10,12,25] can be analyzed through various bounds and inequalities. The laplacian energy of a graph computed from the eigenvalues [24] of its laplacian matrix, finds wide-ranging and important applications such as in chemistry, high-resolution satellite image classification and segmentation and in uncovering semantic structures within image hierarchies.

2. The Laplacian Minimum Pendant Dominating Degree Energy of a Graph

Let $G = (V, E)$ be a finite, non-empty, simple and undirected graph with p vertices and q edges. The degree of a vertex $p \in V(G)$, denoted by $d(p)$, is the number of edges incident to it. A subset $D \subseteq V(G)$ is called a dominating set, if every vertex in $V(G) - D$ is adjacent to at least one vertex in D. The least number of elements present in a dominating set D is called a domination number, denoted by $\gamma(G)$.

A dominating set S in G is called a pendant dominating set, if the induced sub graph of $< S >$ contains at least one pendant vertex. The minimum cardinality of a pendant dominating set is called pendant domination number, denoted by $\gamma_{pe}(G)$.The laplacian minimum pendant dominating degree matrix of G is the $p \times p$ matrix defined by $L^D_{pe}(G) = L_{ij}$ where

$$L_{ij} = \begin{cases} -2, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } d(v_i) = d(v_j) \\ -1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } d(v_i) \neq d(v_j) \\ 1, & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } d(v_i) = d(v_j) \\ 1, & \text{if } v_i = v_j \text{ for } i = j \text{ and } v_i \in S \\ 1, & \text{if } v_i \sim v_j \text{ and } v_i \in S \\ 0, & \text{otherwise} \end{cases}$$

Here, $d(v_i)$ is the degree of the vertex v_i. The laplacian matrix $L^D_{pe}(G)$ of a graph G is defined as $L^D_{pe}(G) = D^D_{pe}(G) - A^D_{pe}(G)$ where $D^D_{pe}(G)$ and $A^D_{pe}(G)$ are the diagonal and adjacency matrix of G respectively. Let $\beta_1, \beta_2, \ldots, \beta_p$ be the laplacian minimum pendant dominating degree eigenvalues of G. The laplacian minimum pendant dominating degree energy is denoted by $LE^D_{pe}(G)$ and is defined as

$$LE^D_{pe}(G) = \sum_{i=1}^{p} \left| \beta_i - \frac{2q}{p} \right|$$

Here, $\left| \beta_i - \frac{2q}{p} \right|$ represents absolute value of the difference between each eigenvalue β_i and the average degree of the graph G. The average degree of a graph G is computed as $\frac{2q}{p}$ where q is the number of edges and p is the number of vertices in G.

As $L^D_{pe}(G)$ is real and symmetric matrix, the characteristic polynomial of $L^D_{pe}(G)$ is given by $f_p(G, \beta) = (L^D_{pe}(G) - \beta I)$ and the set of eigenvalues $|\beta_1| \geq |\beta_2| \geq \cdots \geq |\beta_p|$ with their algebraic multiplicities $r_1, r_2, \ldots r_p$ of $L^D_{pe}(G)$ is called a laplacian dominating degree spectra of G, denoted by $Spec^D_{pe}(G)$ and is as follows:

$$Spec(L^D_{pe}(G)) = \begin{pmatrix} \beta_1 & \beta_2 & \ldots & \beta_p \\ r_1 & r_2 & \ldots & r_p \end{pmatrix}$$

The laplacian minimum pendant dominating degree energy with algebraic multiplicities r_i is

$$LE^D_{pe}(G) = \sum_{i=1}^{p} \left| \beta_i - \frac{2q}{p} \right| r_i$$

2.1. Example

In Figure 1, Let G be a graph with 7 vertices and 12 edges. The vertex set $\{v_1, v_2, v_3, v_5, v_6\}$ is of degree 4 and the vertex set $\{v_4, v_7\}$ is of degree 2.

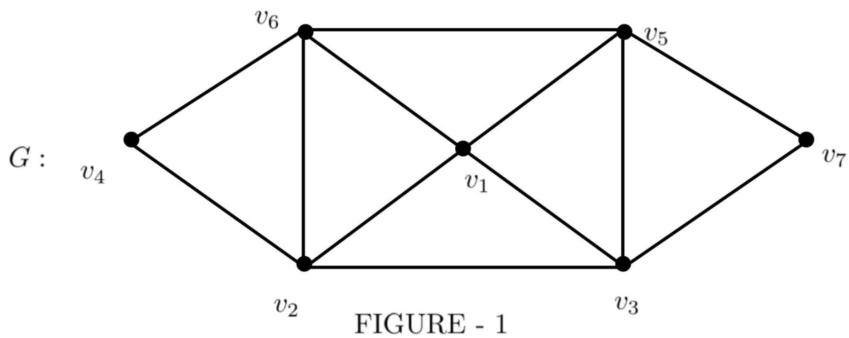

FIGURE - 1

Let $S = \{v_1, v_2, v_7\}$ is the minimum pendant dominating degree set and the associated adjacency matrix, diagonal matrix and laplacian matrix is as follows:

$$A^D_{pe}(G) = \begin{pmatrix} 1 & 2 & 2 & 0 & 2 & 2 & 0 \\ 2 & 1 & 2 & 1 & -1 & 2 & 0 \\ 2 & 2 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 2 & -1 & 2 & 0 & 0 & 2 & 1 \\ 2 & 2 & -1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{pmatrix} \text{ and } D^D_{pe}(G) = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$L^D_{pe}(G) = D^D_{pe}(G) - A^D_{pe}(G) = \begin{pmatrix} 3 & -2 & -2 & 0 & -2 & -2 & 0 \\ -2 & 3 & -2 & -1 & 1 & -2 & 0 \\ -2 & -2 & 4 & 0 & -2 & 1 & -1 \\ 0 & -1 & 0 & 2 & 0 & -1 & 1 \\ -2 & 1 & -2 & 0 & 4 & -2 & -1 \\ -2 & -2 & 1 & -1 & -2 & 4 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 1 \end{pmatrix}$$

The characteristic polynomial of the laplacian matrix is $\beta^7 - 21\beta^6 + 146\beta^5 - 307\beta^4 - 600\beta^3 + 3042\beta^2 - 2602\beta - 885 = 0$. The laplacian minimum pendant dominating degree eigenvalues are $\beta_1 = -0.2586, \beta_2 = -2.5576, \beta_3 = 2.5491, \beta_4 = 2.7083, \beta_5 = 3.5379, \beta_6 = 6.2398, \beta_7 = 8.7811$. The average degree is $\frac{2q}{p} = \frac{2(12)}{7} \approx 3.4286$. Therefore, the laplacian minimum pendant dominating degree energy is $LE^D_{pe}(G) \approx 19.5462$.

Theorem 2.1 *Let G be a bull graph, then* $LE^D_{pe}(G) \approx 11.07541$

Proof: Let G be a simple, undirected bull graph with 5 vertices and 5 edges. The graph is in the form of a triangle with two disjoint pendant edges, which resembles a bull's head with horns. The vertex set $\{v_1, v_2, v_3\}$ is of degree 3 and the vertex set $\{v_4, v_5\}$ is of degree 1.

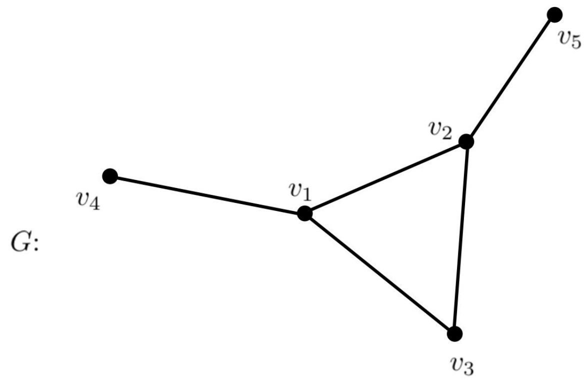

Figure 2: Bull graph with 5 vertices and 5 edges

The laplacian minimum pendant dominating degree set is $S = \{v_1, v_2\}$ and the associated adjacency matrix, diagonal matrix and laplacian matrix is as follows:

$$A^D_{pe}(G) = \begin{pmatrix} 1 & 2 & 1 & 1 & -1 \\ 2 & 1 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & -1 & 0 & -1 \\ -1 & 1 & -1 & -1 & 0 \end{pmatrix} \text{ and } D^D_{pe}(G) = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L^D_{pe}(G) = D^D_{pe}(G) - A^D_{pe}(G) = \begin{pmatrix} 2 & -2 & -1 & -1 & 1 \\ -2 & 2 & -1 & 1 & -1 \\ -1 & -1 & 2 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \end{pmatrix}$$

The characteristic polynomial of the laplacian matrix is $\big(\beta^2 - 4\beta - 4\big)\big(\beta^3 - 4\beta^2 + 4\big) = 0$. The laplacian minimum pendant dominating degree eigenvalues are $\beta_1 = -0.82843, \beta_2 = -0.90321, \beta_3 = 1.19394, \beta_4 = 3.70928, \beta_5 = 4.82843$. The average degree is $\frac{2q}{p} = \frac{2(5)}{5} = 2$. Therefore, the laplacian minimum pendant dominating degree energy is $LE^D_{pe}(G) \approx 11.07541$ $\square$

Theorem 2.2 *Let G be a Peterson graph, then* $LE^D_{pe}(G) \approx 40.73928$

Proof: Let G be a Peterson graph which is non-planar, undirected and regular graph with 10 vertices and 15 edges. The vertex set $\{v_1, v_2, v_3, \ldots, v_{10}\}$ forms a 3-regular graph, where each vertex is of degree 3.

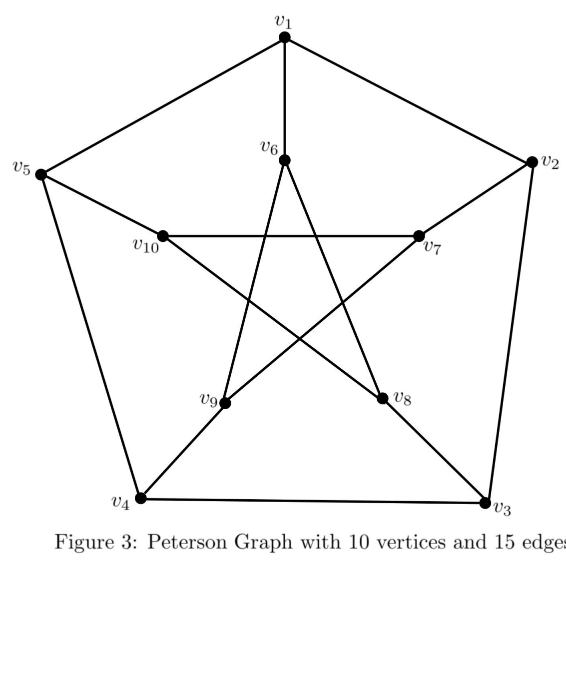

Figure 3: Peterson Graph with 10 vertices and 15 edges

The laplacian minimum pendant dominating degree set is $S = \{v_1, v_6, v_4, v_{10}\}$ and the associated adjacency matrix, diagonal matrix and laplacian matrix is as follows:

$$A_{pe}^D(G) = \begin{pmatrix} 1 & 2 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & -1 \\ 2 & 0 & 2 & -1 & -1 & -1 & 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 2 & -1 & -1 & -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & 1 & 2 & -1 & -1 & -1 & 2 & -1 \\ 2 & -1 & -1 & 2 & 0 & -1 & -1 & -1 & -1 & 2 \\ 2 & -1 & -1 & -1 & -1 & 1 & -1 & 2 & 2 & -1 \\ -1 & 2 & -1 & -1 & -1 & -1 & 0 & -1 & 2 & 2 \\ -1 & -1 & 2 & -1 & -1 & 2 & -1 & 0 & -1 & 2 \\ -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & 1 \end{pmatrix}$$

$$D_{pe}^D(G) = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

$$L_{pe}^D(G) = D_{pe}^D(G) - A_{pe}^D(G) = \begin{pmatrix} 2 & -2 & 1 & 1 & -2 & -2 & 1 & 1 & 1 & 1 \\ -2 & 3 & -2 & 1 & 1 & 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 3 & -2 & 1 & 1 & 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 2 & -2 & 1 & 1 & 1 & -2 & 1 \\ -2 & 1 & 1 & -2 & 3 & 1 & 1 & 1 & 1 & -2 \\ -2 & 1 & 1 & 1 & 1 & 2 & 1 & -2 & -2 & 1 \\ 1 & -2 & 1 & 1 & 1 & 1 & 3 & 1 & -2 & -2 \\ 1 & 1 & -2 & 1 & 1 & -2 & 1 & 3 & 1 & -2 \\ 1 & 1 & 1 & -2 & 1 & -2 & -2 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & -2 & 1 & -2 & -2 & 1 & 2 \end{pmatrix}$$

The characteristic polynomial of the laplacian matrix is $(\beta - 8)(\beta + 1)^2(\beta^2 - 6\beta - 10)(\beta^5 - 15\beta^4 + 46\beta^3 + 120\beta^2 - 266\beta - 414) = 0$. The laplacian minimum pendant dominating degree eigenvalues are $\beta_1 = -1.85085, \beta_2 = -1.37184, \beta_3 = -1.35890, \beta_4 = -1, \beta_5 = -1, \beta_6 = 2.71195, \beta_7 = 7.35890, \beta_8 = 7.60381, \beta_9 = 7.90693, \beta_{10} = 8$. The average degree is $\frac{2q}{p} = \frac{2(15)}{15} = 3$. Therefore, the laplacian minimum pendant dominating degree energy is $LE_{pe}^D(G) \approx 40.73928$ □

3. Fundamental Properties on Eigenvalues of $LE_{pe}^D(G)$

Theorem 3.1 *Let G be a simple graph having the vertex set $V(G) = \{v_1, v_2, v_3, \ldots, v_p\}$, the edge set $E(G)$ and the pendant dominating set $S = \{u_1, u_2, u_3, \ldots, u_p\}$. If $\beta_1, \beta_2, \ldots, \beta_p$ are the eigenvalues of the laplacian minimum pendant dominating degree matrix $L_{pe}^D(G)$, then*

(i) $\sum_{i=1}^{p} \beta_i = 2|E| - |S|$

(ii) $\sum_{i=1}^{p} \beta_i^2 = \sum_{i=1}^{p} (d_i - h_i)^2 + 2\sum_{i<j} (L_{ij})^2$ *where* $h_i = \begin{cases} 1, & \text{if } v_i \in S \\ 0, & \text{otherwise} \end{cases}$

Proof: (i) W. K. T. the sum of the eigenvalues of $L_{pe}^D(G)$ is the trace of $L_{pe}^D(G)$. Therefore, we have

$$\sum_{i=1}^{p}\beta_i = \sum_{i=1}^{p} l_{ii} = \sum_{i=1}^{p} d_i - |S| = 2|E| - |S|$$

(ii) Similarly, the sum of squares of the eigenvalues of $L_{pe}^{D}(G)$ is the trace of $(L_{pe}^{D}(G))^2$. Therefore,

$$\begin{aligned}
\sum_{i=1}^{p}\beta_i^2 &= \sum_{i=1}^{p}\beta_i\sum_{j=1}^{p}\beta_j = \sum_{i=1}^{p}\sum_{j=1}^{p}\beta_i\beta_j = \sum_{i=1}^{p}\sum_{j=1}^{p} l_{ij}l_{ji}\\
&= \sum_{i=1}^{p}(l_{ii})^2 + \sum_{i\neq j} l_{ij}l_{ji} = \sum_{i=1}^{p}(l_{ii})^2 + 2\sum_{i<j}(l_{ij})^2\\
\therefore \sum_{i=1}^{p}\beta_i^2 &= \sum_{i=1}^{p}(d_i - h_i)^2 + 2\sum_{i<j}(L_{ij})^2 \;\; where \;\; h_i = \begin{cases} 1, & \text{if } v_i \in S\\ 0, & \text{otherwise}\end{cases}
\end{aligned}$$

□

Theorem 3.2 *If the sum of the absolute eigenvalues of the laplacian minimum pendant dominating degree matrix $L_{pe}^{D}(G)$ is a rational number, then*

$$\sum_{i=1}^{p}|\beta_i| \equiv |S|(mod\ 2)$$

Proof: Let $\beta_1, \beta_2, \ldots, \beta_p$ be the eigenvalues of the laplacian minimum pendant dominating degree matrix $L_{pe}^{D}(G)$ of a graph G, of which $\beta_1, \beta_2, \ldots, \beta_r$ are positive and the rest of them are non-positive, then

$$\begin{aligned}
\sum_{i=1}^{p}|\beta_i| &= \big(\beta_1 + \beta_2 + \cdots + \beta_r\big) - \big(\beta_{p+1} + \cdots + \beta_p\big)\\
&= 2\big(\beta_1 + \beta_2 + \cdots + \beta_r\big) - \big(\beta_1 + \beta_2 + \cdots + \beta_p\big)\\
&= 2\big(\beta_1 + \beta_2 + \cdots + \beta_r\big) - \sum_{i=1}^{p}\beta_i\\
&= 2\big(\beta_1 + \beta_2 + \cdots + \beta_r\big) - \big(2|E| - |S|\big)\\
&= 2\big(\beta_1 + \beta_2 + \cdots + \beta_r - |E|\big) - |S|\\
\therefore \sum_{i=1}^{p}|\beta_i| &\equiv |S|(mod\ 2)
\end{aligned}$$

□

4. Bounds on $LE_{pe}^{D}(G)$

McLelland's [13] gave upper and lower bounds for ordinary energy of a graph. Similar bounds for $LE_{pe}^{D}(G)$ are given in the following theorem.

Theorem 4.1 *(Upper Bound) Let G be a simple connected graph having p vertices and q edges, then*

$$LE_{pe}^{D}(G) \leq \sqrt{p\left(\sum_{i=1}^{p}(d_i - h_i)^2 + 2\sum_{i<j}(L_{ij})^2\right)} + 2q$$

Proof: The Cauchy's-Schwartz inequality is

$$\left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{p} a_i^2\right)\left(\sum_{i=1}^{p} b_i^2\right)$$

Put $a_i = 1$ and $b_i = |\beta_i|$ in the above inequality

$$\begin{aligned}
\left(\sum_{i=1}^{p}|\beta_i|\right)^2 &\leq \left(\sum_{i=1}^{p}1\right)\left(\sum_{i=1}^{p}|\beta_i|^2\right)\\
&= p\left(\sum_{i=1}^{p}(d_i-h_i)^2 + 2\sum_{i<j}(L_{ij})^2\right)\\
\implies \sum_{i=1}^{p}|\beta_i| &\leq \sqrt{p\left(\sum_{i=1}^{p}(d_i-h_i)^2 + 2\sum_{i<j}(L_{ij})^2\right)}
\end{aligned}$$

By Triangular inequality, we have

$$\begin{aligned}
\left|\beta_i - \frac{2q}{p}\right| &\leq |\beta_i| + \left|\frac{2q}{p}\right| = |\beta_i| + \frac{2q}{p}\\
\sum_{i=1}^{p}\left|\beta_i - \frac{2q}{p}\right| &\leq \sum_{i=1}^{p}|\beta_i| + \sum_{i=1}^{p}\frac{2q}{p}\\
\implies LE_{pe}^{D}(G) &\leq \sum_{i=1}^{p}|\beta_i| + 2q
\end{aligned}$$

From the above result we conclude that

$$LE_{pe}^{D}(G) \leq \sqrt{p\left(\sum_{i=1}^{p}(d_i-h_i)^2 + 2\sum_{i<j}(L_{ij})^2\right)} + 2q$$

□

Theorem 4.2 *(Upper Bound) Let G be a simple connected graph having p vertices and q edges, then*

$$LE_{pe}^{D}(G) \leq \sqrt{p\left(\sum_{i=1}^{p}(d_i-h_i)^2 + 2\sum_{i<j}(L_{ij})^2 - \frac{4q^2}{p} + \frac{4q|S|}{p}\right)}$$

Proof: The Cauchy's-Schwartz inequality is

$$\left(\sum_{i=1}^{p}a_ib_i\right)^2 \leq \left(\sum_{i=1}^{p}a_i^2\right)\left(\sum_{i=1}^{p}b_i^2\right)$$

Put $a_i = 1$ and $b_i = \left|\beta_i - \frac{2q}{p}\right|$ in the above inequality

$$\begin{aligned}
\left(\sum_{i=1}^{p}\left|\beta_i - \frac{2q}{p}\right|\right)^2 &\leq \left(\sum_{i=1}^{p}1\right)\left(\sum_{i=1}^{p}\left|\beta_i - \frac{2q}{p}\right|\right)\\
\left(LE_{pe}^{D}(G)\right)^2 &\leq p\left(\sum_{i=1}^{p}\left(|\beta_i|^2 + \frac{4q^2}{p^2} - 2|\beta_i|\left|\frac{2q}{p}\right|\right)\right)\\
&= p\left(\sum_{i=1}^{p}|\beta_i|^2 + \sum_{i=1}^{p}\frac{4q^2}{p^2} - \frac{4q}{p}\sum_{i=1}^{p}|\beta_i|\right)\\
&= p\left(\sum_{i=1}^{p}|\beta_i|^2 + \sum_{i=1}^{p}\frac{4q^2}{p^2} - \frac{4q}{p}\left(2q-|S|\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= p\left(\sum_{i=1}^{p}(d_i-h_i)^2+2\sum_{i<j}(L_{ij})^2+\frac{4q^2}{p^2}(p)-\frac{8q^2}{p}+\frac{4q|S|}{p}\right)\\
&= p\left(\sum_{i=1}^{p}(d_i-h_i)^2+2\sum_{i<j}(L_{ij})^2+\frac{4q}{p}\left(|S|-q\right)\right)\\
\left(LE^D_{pe}(G)\right)^2 &\le p\left(\sum_{i=1}^{p}(d_i-h_i)^2+2\sum_{i<j}(L_{ij})^2+\frac{4q}{p}\left(|S|-q\right)\right)\\
\Longrightarrow LE^D_{pe}(G) &\le \sqrt{p\left(\sum_{i=1}^{p}(d_i-h_i)^2+2\sum_{i<j}(L_{ij})^2+\frac{4q}{p}\left(|S|-q\right)\right)}
\end{aligned}
$$

$\square$

Theorem 4.3 *(Lower Bound) Let G be a simple connected graph having p vertices and q edges and $\Delta = |det(L^D_{pe}(G))|^{\frac{2}{p}}$, then*

$$
LE^D_{pe}(G) \ge \sqrt{p\left(\sum_{i=1}^{p}(d_i-h_i)^2+2\sum_{i<j}(L_{ij})^2+p(p-1)\Delta^{\frac{2}{p}}\right)-2q}
$$

Proof: Consider

$$
\begin{aligned}
\left(\sum_{i=1}^{p}|\beta_i|\right)^2 &= \left(\sum_{i=1}^{p}|\beta_i|\right)\left(\sum_{i=1}^{p}|\beta_i|\right)\\
\left(\sum_{i=1}^{p}|\beta_i|\right)^2 &= \sum_{i=1}^{p}|\beta_i|^2+\sum_{i\neq j}|\beta_i||\beta_j|\\
\Longrightarrow \sum_{i\neq j}|\beta_i||\beta_j| &= \left(\sum_{i=1}^{p}|\beta_i|\right)^2-\sum_{i=1}^{p}|\beta_i|^2
\end{aligned}
$$

Applying inequality between the Arithmetic and Geometric means for $p(p-1)$ terms

$$
\begin{aligned}
\frac{\sum_{i\neq j}|\beta_i||\beta_j|}{p(p-1)} &\ge \left(\prod_{i\neq j}|\beta_i||\beta_j|\right)^{\frac{1}{p(p-1)}}\\
\sum_{i\neq j}|\beta_i||\beta_j| &\ge p(p-1)\left(\prod_{i\neq j}|\beta_i||\beta_j|\right)^{\frac{1}{p(p-1)}}\\
\left(\sum_{i=1}^{p}|\beta_i|\right)^2-\sum_{i=1}^{p}|\beta_i|^2 &\ge p(p-1)\left(\prod_{i\neq j}|\beta_i||\beta_j|\right)^{\frac{1}{p(p-1)}}\\
\left(\sum_{i=1}^{p}|\beta_i|\right)^2-\sum_{i=1}^{p}(d_i-h_i)^2-2\sum_{i<j}(L_{ij})^2 &\ge p(p-1)\left(\prod_{i\neq j}|\beta_i|^{2(p-1)}\right)^{\frac{1}{p(p-1)}}\\
\left(\sum_{i=1}^{p}|\beta_i|\right)^2-\sum_{i=1}^{p}(d_i-h_i)^2-2\sum_{i<j}(L_{ij})^2 &\ge p(p-1)\left(\prod_{i\neq j}|\beta_i|\right)^{\frac{2}{p}}
\end{aligned}
$$

$$\left(\sum_{i=1}^{p}|\beta_i|\right)^2 \geq \sum_{i=1}^{p}(d_i-h_i)^2+2\sum_{i<j}(L_{ij})^2+p(p-1)|det(L_{pe}^D(G))|^{\frac{2}{p}}$$

$$\implies \left(\sum_{i=1}^{p}|\beta_i|\right) \geq \sqrt{\sum_{i=1}^{p}(d_i-h_i)^2+2\sum_{i<j}(L_{ij})^2+p(p-1)|det(L_{pe}^D(G))|^{\frac{2}{p}}}$$

By Triangular inequality, we have

$$|\beta_i|-\left|\frac{2q}{p}\right| \leq \left|\beta_i-\frac{2q}{p}\right| \quad \forall i$$

$$|\beta_i|-\frac{2q}{p} \leq \left|\beta_i-\frac{2q}{p}\right|$$

$$\sum_{i=1}^{p}|\beta_i|-\sum_{i=1}^{p}\frac{2q}{p} \leq \sum_{i=1}^{p}\left|\beta_i-\frac{2q}{p}\right|$$

$$\sum_{i=1}^{p}|\beta_i|-2q \leq \sum_{i=1}^{p}|\beta_i|-2q$$

$$\implies \sum_{i=1}^{p}|\beta_i|-2q \leq LE_{pe}^D(G) \geq \sum_{i=1}^{p}|\beta_i|-2q$$

Therefore, from the above we conclude that

$$LE_{pe}^D(G) \geq \sqrt{p\left(\sum_{i=1}^{p}(d_i-h_i)^2+2\sum_{i<j}(L_{ij})^2+p(p-1)\Delta^{\frac{2}{p}}\right)}-2q$$

□

Theorem 4.4 *If $\beta_1(G)$ is the largest laplacian minimum pendant dominating degree eigenvalues of $L_{pe}^D(G)$, then*

$$\beta_1(G) \geq \frac{\sum_{i=1}^{p}d_i-|S|-\frac{1}{2}\sum_{i<j}(L_{ij})}{p}$$

Proof: Let P be a non-zero vector and by [4], we have

$$\beta_1(G)=max_{P\neq 0}\left[\frac{P^TAP}{P^TP}\right]$$

Therefore,

$$\beta_1(G) \geq \frac{X^TAX}{X^TX}=\frac{\sum_{i=1}^{p}d_i-|S|-\frac{1}{2}\sum_{i<j}(L_{ij})}{p}$$

$$\implies \beta_1(G) \geq \frac{\sum_{i=1}^{p}d_i-|S|-\frac{1}{2}\sum_{i<j}(L_{ij})}{p}$$

where X is a unit matrix $\left[1,1,\ldots,1\right]^T$ □

Theorem 4.5 *Let G be a simple connected graph having p vertices and q edges. Let $|\beta_1| \geq |\beta_2| \geq \cdots \geq |\beta_p|$ be a non-increasing order of the laplacian minimum pendant dominating degree eigenvalues of $L^D_{pe}(G)$, then*

$$LE^D_{pe}(G) \geq \sqrt{p\left(\sum_{i=1}^{p}(d_i-h_i)^2 + 2\sum_{i<j}(L_{ij})^2 - \alpha(p)\Big(|\beta_1|-|\beta_p|\Big)^2\right)} - 2q$$

where $\alpha(p) = p\lceil \frac{p}{2}\rceil\left(1-\frac{1}{p}\lceil\frac{p}{2}\rceil\right)$ and $[x]$ denotes the integral part of a real number.

Proof: Let $a, a_1, a_2, \ldots, a_p, A$ and $b, b_1, b_2, \ldots, b_p, B$ be real numbers such that $a \leq a_i \leq A$ and $b \leq b_i \leq B$ for all $i = 1, 2, \ldots, p$. Then, the following inequality holds

$$\left|p\sum_{i=1}^{p} a_i b_i - \sum_{i=1}^{p} a_i \sum_{i=1}^{p} b_i\right| \leq \alpha(p)(A-a)(B-b)$$

Put $a_i = b_i = |\beta_i|$, $a = b = |\beta_p|$ and $A = B = |\beta_1|$ in the above inequality

$$\left|p\left(\sum_{i=1}^{p}|\beta_i|^2 - \left(\sum_{i=1}^{p}|\beta_i|\right)^2\right)\right| \leq \alpha(p)\Big(|\beta_1|-|\beta_p|\Big)^2$$

$$\left|p\left(\sum_{i=1}^{p}(d_i-h_i)^2 + 2\sum_{i<j}(L_{ij})^2\right) - \left(\sum_{i=1}^{p}|\beta_i|\right)^2\right| \leq \alpha(p)\Big(|\beta_1|-|\beta_p|\Big)^2$$

$$p\left(\sum_{i=1}^{p}(d_i-h_i)^2 + 2\sum_{i<j}(L_{ij})^2\right) - \left(\sum_{i=1}^{p}|\beta_i|\right)^2 \leq \alpha(p)\Big(|\beta_1|-|\beta_p|\Big)^2$$

$$p\left(\sum_{i=1}^{p}(d_i-h_i)^2 + 2\sum_{i<j}(L_{ij})^2\right) - \alpha(p)\Big(|\beta_1|-|\beta_p|\Big)^2 \leq \left(\sum_{i=1}^{p}|\beta_i|\right)^2$$

$$\left(\sum_{i=1}^{p}|\beta_i|\right)^2 \geq p\left(\sum_{i=1}^{p}(d_i-h_i)^2 + 2\sum_{i<j}(L_{ij})^2\right) - \alpha(p)\Big(|\beta_1|-|\beta_p|\Big)^2$$

$$\implies \left(\sum_{i=1}^{p}|\beta_i|\right) \geq \sqrt{p\left(\sum_{i=1}^{p}(d_i-h_i)^2 + 2\sum_{i<j}(L_{ij})^2\right) - \alpha(p)\Big(|\beta_1|-|\beta_p|\Big)^2}$$

W. K. T. $LE^D_{pe}(G) = \sum_{i=1}^{p}\left|\beta_i - \frac{2q}{p}\right|$. By Triangular inequality, we have

$$LE^D_{pe}(G) \geq \sum_{i=1}^{p}\Big|\beta_i\Big| - \sum_{i=1}^{p}\frac{2q}{p}$$

$$LE^D_{pe}(G) \geq \sum_{i=1}^{p}\Big|\beta_i\Big| - 2q$$

From the above, we obtain result

$$LE^D_{pe}(G) \geq \sqrt{p\left(\sum_{i=1}^{p}(d_i-h_i)^2 + 2\sum_{i<j}(L_{ij})^2 - \alpha(p)\Big(|\beta_1|-|\beta_p|\Big)^2\right)} - 2q$$

$\square$

Theorem 4.6 *Let G be a simple connected graph having p vertices and q edges. Let $|\beta_1| \geq |\beta_2| \geq \cdots \geq |\beta_p| > 0$ be a non-increasing order of the laplacian minimum pendant dominating degree eigenvalues of $L^D_{pe}(G)$, then*

$$LE^D_{pe}(G) \geq \frac{\sum_{i=1}^{p}(d_i - h_i)^2 + 2\sum_{i<j}(L_{ij})^2 + p|\beta_1||\beta_p|}{\big(|\beta_1|+|\beta_p|\big)} - 2q$$

Proof: Let $a_i \neq 0, b_i, r$ and R be real numbers satisfying $ra_i \leq b_i \leq Ra_i$, then the following inequality holds

$$\sum_{i=1}^{p} b_i^2 + rR\sum_{i=1}^{p} a_i \leq \big(r+R\big)\sum_{i=1}^{p} a_i b_i$$

Put $b_i = |\beta_i|$, $a_i = 1$, $r = |\beta_p|$ and $R = |\beta_1|$ in the above inequality, then

$$\sum_{i=1}^{p}|\beta_i|^2 + |\beta_1||\beta_p|\sum_{i=1}^{p} 1 \leq \big(|\beta_1| + |\beta_p|\big)\sum_{i=1}^{p}|\beta_i|$$

$$\sum_{i=1}^{p}(d_i - h_i)^2 + 2\sum_{i<j}(L_{ij})^2 + p|\beta_1||\beta_p| \leq \big(|\beta_1| + |\beta_p|\big)\sum_{i=1}^{p}|\beta_i|$$

$$\frac{\sum_{i=1}^{p}(d_i - h_i)^2 + 2\sum_{i<j}(L_{ij})^2 + p|\beta_1||\beta_p|}{\big(|\beta_1| + |\beta_p|\big)} \leq \sum_{i=1}^{p}|\beta_i|$$

By definition, we have

$$LE^D_{pe}(G) = \sum_{i=1}^{p}\Big|\beta_i - \frac{2q}{p}\Big|$$

$$LE^D_{pe}(G) \geq \sum_{i=1}^{p}\big|\beta_i\big| - \sum_{i=1}^{p}\frac{2q}{p}$$

$$LE^D_{pe}(G) \geq \sum_{i=1}^{p}\big|\beta_i\big| - 2q$$

From the above, we conclude

$$LE^D_{pe}(G) \geq \frac{\sum_{i=1}^{p}(d_i - h_i)^2 + 2\sum_{i<j}(L_{ij})^2 + p|\beta_1||\beta_p|}{\big(|\beta_1|+|\beta_p|\big)} - 2q$$

□

5. $LE^D_{pe}(G)$ for Various Standard Graphs

Theorem 5.1 *For a complete graph K_p, we have $LE^D_{pe}(K_p) = (2p-5) + \sqrt{4p^2 - 4p + 17}$ where $p \geq 3$*

Proof: Let K_p be a complete graph having the vertex set $V(K_p) = \{v_1, v_2, v_3, \ldots, v_p\}$. The minimum pendant dominating degree set is $S = \{v_1, v_2\}$. The associated adjacency and diagonal matrix is

$$A^D_{pe}(K_p) = \begin{pmatrix} 1 & 2 & 2 & \dots & 2 & 2 \\ 2 & 1 & 2 & \dots & 2 & 2 \\ 2 & 2 & 0 & \dots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \dots & 0 & 2 \\ 2 & 2 & 2 & \dots & 2 & 0 \end{pmatrix}_{p\times p} \text{ and } D^D_{pe}(K_p) = \begin{pmatrix} p-1 & 0 & \dots & 0 & 0 \\ 0 & p-1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & p-1 & 0 \\ 0 & 0 & \dots & 0 & p-1 \end{pmatrix}_{p\times p}$$

The laplacian matrix is $L^D_{pe}(K_p) = D^D_{pe}(K_p) - A^D_{pe}(K_p)$

$$L^D_{pe}(K_p) = \begin{pmatrix} p-2 & -2 & -2 & \dots & -2 & -2 \\ -2 & p-2 & -2 & \dots & -2 & -2 \\ -2 & -2 & p-1 & \dots & -2 & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & -2 & \dots & p-1 & -2 \\ -2 & -2 & -2 & \dots & -2 & p-1 \end{pmatrix}_{p\times p}$$

The characteristic polynomial of the laplacian matrix $L^D_{pe}(K_p)$ is

$$f_p(K_p, \beta) = (-1)^p\big(\beta - (p+1)\big)^{p-3}\big(\beta - p\big)\big(\beta^2 - \beta - (p^2 - p + 4)\big) = 0$$

The eigenvalues are $\beta = (p+1)$ with multiplicity $(p-3)$, $\beta = p$ and $\beta = \frac{1\pm\sqrt{4p^2-4p+17}}{2}$ with multiplicity 1 each. The average degree of K_p is

$$\frac{2q}{p} = \frac{2\,\frac{p(p-1)}{2}}{p} = \frac{p(p-1)}{p} = (p-1)$$

Therefore, the laplacian minimum pendant dominating degree energy of K_p is

$$LE^D_{pe}(K_p) = (2p-5) + \sqrt{4p^2 - 4p + 17}$$

□

Theorem 5.2 *For a complete bipartite graph $K_{p,p}$, we have $LE^D_{pe}(K_{p,p}) = (3p-1) + \sqrt{9p^2 + 6p - 11}$ where $p \geq 2$*

Proof: Let $K_{p,p}$ be a complete bipartite graph having the vertex set $V(K_{p,p}) = \{u_1, u_2, \dots, u_p, v_1, v_2, \dots, v_p\}$. The minimum pendant dominating degree set is $S = \{u_1, v_1\}$. The associated adjacency and diagonal matrix is

$$A^D_{pe}(K_{p,p}) = \begin{pmatrix} 1 & -1 & \dots & -1 & 2 & 2 & \dots & 2 \\ -1 & 0 & \dots & -1 & 2 & 2 & \dots & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & 0 & 2 & 2 & \dots & 2 \\ 2 & 2 & \dots & 2 & 1 & -1 & \dots & -1 \\ 2 & 2 & \dots & 2 & -1 & 0 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \dots & 2 & -1 & -1 & \dots & 0 \end{pmatrix}_{2p\times 2p} \text{ and }$$

$$D^D_{pe}(K_{p,p}) = \begin{pmatrix} p & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & p & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & p & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & p & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & p & 0 & \dots & p \end{pmatrix}_{2p\times 2p}$$

The laplacian matrix is $L_{pe}^D(K_{p,p}) = D_{pe}^D(K_{p,p}) - A_{pe}^D(K_{p,p})$

$$L_{pe}^D(K_{p,p}) = \begin{pmatrix} p-1 & 1 & \dots & 1 & -2 & -2 & \dots & -2 \\ 1 & p & \dots & 1 & -2 & -2 & \dots & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & p & -2 & -2 & \dots & -2 \\ -2 & -2 & \dots & -2 & p-1 & 1 & \dots & 1 \\ -2 & -2 & \dots & -2 & 1 & p & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & \dots & -2 & 1 & 1 & \dots & p \end{pmatrix}_{2p\times 2p}$$

The characteristic polynomial of the laplacian matrix $L_{pe}^D(K_{p,p})$ is

$$f_p(K_{p,p}, \beta) = \big(\beta - (p-1)\big)^{2p-4}\big(\beta^2 - (5p-3)\beta + (4p^2 - 9p + 5)\big)\big(\beta^2 - (p-3)\beta - (p-1)\big) = 0$$

The eigenvalues are $\beta = (p-1)$ with multiplicity $(2p-4)$, $\beta = \frac{(5p-3)\pm\sqrt{9p^2+6p-11}}{2}$ and $\beta = \frac{(p-3)\pm\sqrt{p^2-2p+5}}{2}$ with multiplicity 1 each. The average degree of $K_{p,p}$ is

$$\tfrac{2q}{p} = \tfrac{2p^2}{2p} = p$$

Therefore, the laplacian minimum pendant dominating degree energy of $K_{p,p}$ is

$$LE_{pe}^D(K_{p,p}) = (3p-1) + \sqrt{9p^2 + 6p - 11}$$

□

Theorem 5.3 *For a double star graph $S_{p,p}$, we have $LE_{pe}^D(S_{p,p}) = \frac{4p^2-2p-2}{p+1} + 2\sqrt{p^2 + 8p + 4}$ where $p \geq 2$*

Proof: Let $S_{p,p}$ be a double star graph having the vertex set $V(S_{p,p}) = \{v_0, v_1, v_2, \dots, v_{p-1}, u_0, u_1, \dots, u_{p-1}\}$, where v_0 and u_0 are the two central vertices. The minimum pendant dominating degree set is $S = \{v_0, u_0\}$. The associated adjacency and diagonal matrix is

$$A_{pe}^D(S_{p,p}) = \begin{pmatrix} 1 & 1 & \dots & 1 & 2 & 0 & \dots & 0 \\ 1 & 0 & \dots & -1 & 0 & -1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & \dots & 0 & 0 & -1 & \dots & -1 \\ 2 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 0 & -1 & \dots & -1 & 1 & 0 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & \dots & -1 & 1 & -1 & \dots & 0 \end{pmatrix}_{2p\times 2p} \quad \text{and}$$

$$D_{pe}^D(S_{p,p}) = \begin{pmatrix} p+1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & p+1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & p & 0 & \dots & 1 \end{pmatrix}_{2p\times 2p}$$

The laplacian matrix is $L_{pe}^D(S_{p,p}) = D_{pe}^D(S_{p,p}) - A_{pe}^D(S_{p,p})$

$$L^D_{pe}(S_{p,p}) = \begin{pmatrix} p & -1 & \dots & -1 & -2 & 0 & \dots & 0 \\ -1 & 1 & \dots & 1 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 1 & \dots & 1 & 0 & 1 & \dots & 1 \\ -2 & 0 & \dots & 0 & p & -1 & \dots & -1 \\ 0 & 1 & \dots & 1 & -1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \dots & 1 & -1 & 1 & \dots & 1 \end{pmatrix}_{2p\times 2p}$$

The characteristic polynomial of the laplacian matrix $L^D_{pe}(S_{p,p})$ is

$$f_p(S_{p,p},\beta) = \beta^{2p-2}\big(\beta^2 - (3p-2)\beta + (2p^2-5p)\big)\big(\beta^2 - (p+2)\beta - p\big) = 0$$

The eigenvalues are $\beta = 0$ with multiplicity $(2p-2)$, $\beta = \frac{(3p-2)\pm\sqrt{p^2+8p+4}}{2}$ and $\beta = \frac{(p+2)\pm\sqrt{p^2+8p+4}}{2}$ with multiplicity 1 each. The average degree of $S_{p,p}$ is

$$\frac{2q}{p} = \frac{2(2p+1)}{2p+2} = \frac{2(2p+1)}{2(p+1)} = \frac{2p+1}{p+1}$$

Therefore, the laplacian minimum pendant dominating degree energy of $S_{p,p}$ is

$$LE^D_{pe}(S_{p,p}) = \frac{4p^2-2p-2}{p+1} + 2\sqrt{p^2+8p+4}$$

□

Theorem 5.4 *For a dumbbell graph $D_{p,p}$, we have $LE^D_{pe}(D_{p,p}) = \frac{2p^2-2p-4}{p} + \sqrt{4p^2-3} + \sqrt{4p+5}$*

Proof: Let $D_{p,p}$ be a dumbbell graph having the vertex set $V(D_{p,p}) = \{v_0, v_1, v_2, \dots, v_{p-1}, u_0, u_1, \dots, u_{p-1}\}$. The minimum pendant dominating degree set is $S = \{v_0, u_0\}$. The associated adjacency and diagonal matrix is

$$A^D_{pe}(D_{p,p}) = \begin{pmatrix} 1 & 1 & \dots & 1 & 2 & 0 & \dots & 0 \\ 1 & 0 & \dots & -1 & 0 & -1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & \dots & 0 & 0 & -1 & \dots & -1 \\ 2 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 0 & -1 & \dots & -1 & 1 & 0 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & \dots & -1 & 1 & -1 & \dots & 0 \end{pmatrix}_{2p\times 2p} \quad \text{and}$$

$$D^D_{pe}(D_{p,p}) = \begin{pmatrix} p & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & p-1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p-1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & p & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & p-1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & p & 0 & \dots & p-1 \end{pmatrix}_{2p\times 2p}$$

The laplacian matrix is $L^D_{pe}(D_{p,p}) = D^D_{pe}(D_{p,p}) - A^D_{pe}(D_{p,p})$

$$L^D_{pe}(D_{p,p}) = \begin{pmatrix} p-1 & -1 & \dots & -1 & -2 & 0 & \dots & 0 \\ -1 & p-1 & \dots & 1 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 1 & \dots & p-1 & 0 & 1 & \dots & 1 \\ -2 & 0 & \dots & 0 & p-1 & -1 & \dots & -1 \\ 0 & 1 & \dots & 1 & -1 & p-1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \dots & 1 & -1 & 1 & \dots & p-1 \end{pmatrix}_{2p\times 2p}$$

The characteristic polynomial of the laplacian matrix $L^D_{pe}(D_{p,p})$ is

$$f_p(D_{p,p},\beta) = \big(\beta-(p-2)\big)^{2p-4}\big(\beta^2-(4p-7)\beta+(3p^2-14p+13)\big)\big(\beta^2-(2p-1)\beta+(p^2-2p-1)\big) = 0$$

The eigenvalues are $\beta = (p-2)$ with multiplicity $(2p-4)$, $\beta = \frac{(4p-7)\pm\sqrt{4p^2-3}}{2}$ and $\beta = \frac{(2p-1)\pm\sqrt{4p+5}}{2}$ with multiplicity 1 each. The average degree of $D_{p,p}$ is

$$\frac{2q}{p} = \frac{2[p(p-1)+1]}{2p} = \frac{p^2-p+1}{2}$$

Therefore, the laplacian minimum pendant dominating degree energy of $D_{p,p}$ is

$$LE^D_{pe}(D_{p,p}) = \frac{2p^2-2p-4}{p} + \sqrt{4p^2-3} + \sqrt{4p+5}$$

□

6. Conclusion

The laplacian dominating degree distribution plays a crucial role in network analysis. In this study, we investigated its properties using linear algebraic techniques. By introducing the pendant dominating degree matrix, a degree-based representation of a simple graph G, we derived new spectral results. This approach reduced matrix dimension in eigenvalue computations and, in many cases, enabled explicit eigenvalue determination. Additionally, we proposed a novel concept analogous to graph energy, degree energy for which we established theoretical bounds and computed exact values for several graph families.

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