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On Rough Perfect Mappings

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ABSTRACT: In this paper we depend on the work of Z. Pawalak [5] who provided us with the definition and the first basic information about what so called later, the rough set theory. Here we introduce the notion of rough perfect mappings and show that such mappings can transfer some rough properties between rough topological spaces. Also, it is proved that a rough perfect map can transfer a rough compactly generated property between rough spaces instead of a rough homeomorphism, which was proved in [4]. A concept of a rough locally compact space is defined, and shown that a rough surjective continuous function between such space and a rough Hausdorff space is a rough perfect map. In addition, it is shown that a rough perfect map can preserve the rough Hausdorff property. Finally, some new rough topological properties are defined and studied with appropriate detailed proofs, which are needed throughout this work.

Keywords: Rough perfect maps, rough compactness, rough compactly generated, rough spaces.

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1. Introduction

For a long time, issues of imperfect data have been handled by some logicians and mathematicians who are interested in the enrichment of knowledge. Recently, these issues have become an essential subject for researchers like Polish computer scientist Z. Pawalak, who is concerned with artificial intelligence [5]. Pawalak has dealt with rough set theory as a new method to solve imperfect data problems using the boundary region of a set [10]. It is assumed that a set is rough if its boundary is not null; otherwise, it is exact. So, if the set is rough, then it is not precisely defined because of a lack of knowledge. Two systematic works are used in rough topological spaces. Firstly, Mathew and John [2] defined rough topology and some basic properties of rough topological spaces, and some related notions like rough closed sets, rough closure, . . . , etc., are studied. Secondly, Dhivya and Divya [5] defined connectedness and compactness in rough topological spaces and discussed some properties of separation axioms. In this paper, rough perfect mapping is defined between two topological spaces as a rough surjective, rough closed and rough continuous mapping satisfying that is rough compact for each in the codomain space. Rough perfect mappings transfer rough compact property between rough spaces. In [4], a rough compactly generated space has been defined, and it has been shown that we need a rough homeomorphism to preserve this property; we proved that a rough perfect map can do the same work. Also, we defined some needed rough topological properties, like rough injective, rough surjective, rough bijective mappings, and rough locally compact spaces, and proved some related results such as, every rough locally compact rough Hausdorff space is a rough compactly generated and, the intersection of a rough compact set and a rough closed set is rough compact and more.
Throughout this work, we will write top, Haus., sp., map, cont., loc., comp. , approx., and equiv., instead of topological, Hausdorff, space, mapping, continuous, locally, compact, approximation, and equivalence, respectively.

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2. Basics

Definition 2.1 *[1] Assume that (X, τ) is a Haus. top sp., then it is called compactly generated (or k-space) if each subset F of X satisfying the property that $F \cap C$ is closed in C (or consequently, in X) for all comp. set C in X then F is closed.*

Definition 2.2 *[3] A closed surjective cont. map $f : (X, \tau) \to (Y, \delta)$ is named perfect (or proper) if, to each $y \in Y$, the set $f^{-1}(y)$ is comp. in X.*

Consider U to be a set of elements (non-empty) named the universe with an equiv. relation R named the indiscernibility relation on the set U, so the ordered pair (U, R) is called an approximation space (approx. sp.) [9]. Suppose X is a set in U, to characterise the set X considering R, we determine the sets called lower approximation (l. approx.) and upper approximation (u. approx.).

Definition 2.3 *[6] The equiv. class with respect the relation R, which contains the element a denoted by $[a]$, is called a granule obtained by R, includes the basic informational region that we can realise based on R.*

Definition 2.4 *[6] The lower (l.) approx. of $X \subseteq U$ means all elements x such that its equiv. class is in X. That is, $\underline{RX} = \{x \in U : [x] \subseteq X\}$.*

Definition 2.5 *[6] The upper(u.) approx. of $X \subseteq U$ is all elements x such that its equiv. class intersects X. That is, $\overline{RX} = \{x \in U : [x] \cap X \neq \emptyset\}$.*

Definition 2.6 *[6] The difference set $\overline{RX} - \underline{RX}$, denoted by $BD(X)$, is named the boundary region of X.*

Definition 2.7 *[6] A set $X \subseteq U$ is named a rough set (r- set) when its boundary region is non-empty. Otherwise, it's called an exact set.*

The notation $RX = (\underline{RX}, \overline{RX})$ will be used to denote a r-subset X of an approx. space (U, R).

Definition 2.8 *[2] Let $RX = (\underline{RX}, \overline{RX})$ be a r- subset of an approx. space (U, R). Let $(\underline{\tau}, \overline{\tau})$ be topologies that include just exact sets in $\underline{RX}$ and $\overline{RX}$ respectively. Then $\tau = (\underline{\tau}, \overline{\tau})$, as an ordered pair, is named rough topology (r- topology) on the r-set $RX = (\underline{RX}, \overline{RX})$. Furthermore, (RX, τ) called rough topological space (r- top sp.). In a r-topology $\tau = (\underline{\tau}, \overline{\tau})$, the topology $\underline{\tau}$ is the lower rough topology (l. r-topology) and $\overline{\tau}$ is the upper rough topology (u. r- topology) of RX. A set $A = (\underline{A}, \overline{A}) \subseteq RX$ is said to be rough open (r-open) if $\underline{A} \in \underline{\tau}$ and $\overline{A} \in \overline{\tau}$ respectively.*

Example 2.1 *[2] Suppose $RX = (\underline{RX}, \overline{RX})$ is a r-set. Define $\underline{\tau} = \{A \subseteq \underline{RX} : A \text{ is exact}\}$ and $\overline{\tau} = \{B \subseteq \overline{RX} : B \text{ is exact}\}$. Therefore, each of $\underline{\tau}$ and $\overline{\tau}$ represent topologies on $\underline{RX}$ and $\overline{RX}$ respectively. In this case, the r- topology $\tau = (\underline{\tau}, \overline{\tau})$ on RX is called discrete r- topology on RX.*

Example 2.2 *[2] Consider $RX = (\underline{RX}, \overline{RX})$ as a r-set and suppose $\underline{\tau} = \{\emptyset, \underline{RX}\}$ while $\overline{\tau} = \{\emptyset, \overline{RX}\}$. It is straightforward that $\underline{\tau}$ and $\overline{\tau}$ represent topologies on $\underline{RX}$ and $\overline{RX}$ respectively. The r-topology $\tau = (\underline{\tau}, \overline{\tau})$ called indiscrete r- topology on RX.*

Definition 2.9 *[5] A subset $A = (\underline{A}, \overline{A})$ of a r-top space (RX, τ) where $RX = (\underline{RX}, \overline{RX})$ and $\tau = (\underline{\tau}, \overline{\tau})$ is named lower r- closed (l. r- closed) when $\underline{A^C} = \underline{RX} - \underline{A} \in \underline{\tau}$. Also A is named upper r- closed (u. r-closed) when $\overline{A^C} = \overline{RX} - \overline{A} \in \overline{\tau}$. The set A is called r- closed when it is both above l. r- closed and u. r- closed.*

Definition 2.10 *[5] Let $RX = (\underline{RX}, \overline{RX})$ be a r-top space with topology $\tau = (\underline{\tau}, \overline{\tau})$. A set $\underline{N_1} \subseteq \underline{RX}$ is named a $\underline{\tau}$- neighborhood of a point $x \in X$ if and only if there is an open set $\underline{G_1}$ of $\underline{RX}$ such that $x \in \underline{G_1} \subset \underline{N_1}$. Similarly, a set $\overline{N_2} \subseteq \overline{RX}$ is named a $\overline{\tau}$- neighborhood of a point $x \in X$ if and only if there is an open set $\overline{G_2}$ of $\overline{RX}$ such that $x \in \overline{G_2} \subset \overline{N_2}$. At the same time if $\underline{N_1} \subseteq \underline{RX} \subseteq \overline{N_2} \subseteq \overline{RX}$, then $N = (\underline{N_1}, \overline{N_2})$ is named a rough (r-) τ-neighborhood of $x \in X$.*

Definition 2.11 *[5] Let $RX = (\underline{RX}, \overline{RX})$ be a r-top sp. with topology $\tau = (\underline{\tau}, \overline{\tau})$. Then $\underline{RX}$ is named lower Hausdorff space (l. Haus. sp.) if for any different points $x, y \in RX$, there is open sets $\underline{U}$ and $\underline{V}$ such that $x \in \underline{U}$ and $y \in \underline{V}$ and $\underline{U} \cap \underline{V} = \emptyset$. Also $\overline{RX}$ is named upper Hausdorff space (u. Haus. sp.) if for any different points $x, y \in \overline{RX}$, there is open sets $\overline{U}$ and $\overline{V}$ such that $x \in \overline{U}$ and $y \in \overline{V}$ and $\overline{U} \cap \overline{V} = \emptyset$. Then RX is rough Hausdorff space (r- Haus. sp.).*

Definition 2.12 *[5] Suppose (RX, τ) and (RY, δ) are r- spaces with topologies $\tau = (\underline{\tau}, \overline{\tau})$ and $\delta = (\underline{\delta}, \overline{\delta})$ respectively. A map $f_* : (\underline{RX}, \underline{\tau}) \to (\underline{RY}, \underline{\delta})$ is lower continuous(l. cont.) at $p \in RX$ if and only if , every $\underline{\delta}$ -neighborhood $\underline{H}$ of $f_*(p)$ having $\underline{\tau}$ -neighborhood $\underline{G}$ of p from $\underline{RX}$ such that $f_*(p) \in f_*(\underline{G}) \subseteq \underline{H}$. Also $f^* : (\overline{RX}, \overline{\tau}) \to (\overline{RY}, \overline{\delta})$ is upper continuous (u. cont.) at $q \in RX$ if and only if, every $\overline{\delta}$-neighborhood $\overline{H}$ of $f^*(q)$ having $\overline{\tau}$-neighborhood $\overline{G}$ of q in $\overline{RX}$ such that, $f^*(q) \in f^*(\overline{G}) \subseteq \overline{H}$. So, $f = (f_*, f^*) : (RX, \tau) \to (RY, \delta)$ is called rough continuous (r- cont.) at p when both f_* and f^* are l. cont. and u. cont. at p respectively. We call $f = (f_*, f^*) : (RX, \tau) \to (RY, \delta)$ r- cont., when f is r-cont. at each $p \in RX$.*

Definition 2.13 *[5] Let $RX = (\underline{RX}, \overline{RX})$ be a r-set. If each open cover of $\underline{RX}$ has a finite subcover, then $\underline{RX}$ is named a lower compact set (l. comp. set). Similarly, if each open cover of $\overline{RX}$ has a finite subcover, then $\overline{RX}$ is called an upper compact set (u. comp. set). Then $RX = (\underline{RX}, \overline{RX})$ is called a r-compact space (r-comp. sp.) if both $\underline{RX}$ and $\overline{RX}$ are comp. sets.*

Definition 2.14 *[8] Let (RX, τ) and (RY, δ) be two r- top spaces with topologies $\tau = (\underline{\tau}, \overline{\tau})$ and $\delta = (\underline{\delta}, \overline{\delta})$ respectively. Let $f = (f_*, f^*) : (RX, \tau) \to (RY, \delta)$ be a map. Then, f is rough closed (r- closed) map, when each of f_* and f^* is a closed map. Similarly, f is rough open (r- open) map, when each of f_* and f^* is an open map.*

Definition 2.15 *Let $RX = (\underline{RX}, \overline{RX})$ and $RY = (\underline{RY}, \overline{RY})$ be r- top spaces with topologies $\tau = (\underline{\tau}, \overline{\tau})$ and $\delta = (\underline{\delta}, \overline{\delta})$ respectively. A map $f_* : \underline{RX} \to \underline{RY}$ is named lower injective (l. inj.) if for each $x_1, x_2 \in \underline{RX}$ such that $x_1 = x_2$, then $f_*(x_1) = f_*(x_2)$. Also $f^* : \overline{RX} \to \overline{RY}$ is named upper injective (u. inj.) if for each $x_1, x_2 \in \overline{RX}$ such that $x_1 = x_2$, then $f^*(x_1) = f^*(x_2)$. Consequently, the map $f = (f_*, f^*) : RX \to RY$ is named a rough injective (r- inj.) map.*

Definition 2.16 *Suppose,$RX = (\underline{RX}, \overline{RX})$ and $RY = (\underline{RY}, \overline{RY})$ are r-top spaces with topologies $\tau = (\underline{\tau}, \overline{\tau})$ and $\delta = (\underline{\delta}, \overline{\delta})$ respectively. A map $f_* : \underline{RX} \to \underline{RY}$ is named lower surjective (l. surj.) if for any $y \in \underline{RY}$ there is $x \in \underline{RX}$ such that $f_*(x) = y$. Also $f^* : \overline{RX} \to \overline{RY}$ is named upper surjective (u. surj.) for each $y \in \overline{RY}$ there is $x \in \overline{RX}$ such that $f^*(x) = y$. Then, the map $f = (f_*, f^*) : RX \to RY$ is said to be a rough surjective(r-surj.) map.*

Definition 2.17 *Let $RX = (\underline{RX}, \overline{RX})$ and $RY = (\underline{RY}, \overline{RY})$ be r-top spaces with topologies $\tau = (\underline{\tau}, \overline{\tau})$ and $\delta = (\underline{\delta}, \overline{\delta})$ respectively. A map $f_* : \underline{RX} \to \underline{RY}$ is named lower bijective (l. bij.) if it is l. inj. and l. surj. Also, the map $f^* : \overline{RX} \to \overline{RY}$ is named upper bijective (u. bij.) if it is u. inj. and u. surj. Then the map $f = (f_*, f^*) : RX \to RY$ is named a rough bijective (r-bij.) map.*

Definition 2.18 *[8] Let (RX, τ) and (RY, δ) be two r- top spaces with topologies $\tau = (\underline{\tau}, \overline{\tau})$, $\delta = (\underline{\delta}, \overline{\delta})$ and let $f = (f_*, f^*) : (RX, \tau) \to (RY, \delta)$ be a map. Then $f = (f_*, f^*)$ is named a rough homeomorphism (r-homeo.), if both f_* and f^* are homeomorphisms. If such map exists then the two spaces are said to be rough (r-) equivalent.*

Remark 2.1 *[8] Let (RX, τ) and (RY, δ) be two r- top spaces with topologies $\tau = (\underline{\tau}, \overline{\tau})$, $\delta = (\underline{\delta}, \overline{\delta})$ and $f = (f_*, f^*) : (RX, \tau) \to (RY, \delta)$ is a map. When $f = (f_*, f^*)$ is a r-homeo. then:*

- *f is r- bij.,*
- *f is r-cont.,*
- *f is r- open,*
- *f is r- closed.*

3. Results

Definition 3.1 *Assume that,* (RX,τ) *and* (RY,δ) *are r- top spaces with topologies* $\tau=(\underline{\tau},\overline{\tau})$, $\delta=(\underline{\delta},\overline{\delta})$ *and* $f=(f_*,f^*):(RX,\tau)\to(RY,\delta)$ *be r-surjective r- cont. map. Then,* f *is a rough perfect (r- perfect) map if* f *is r- closed and* $f^{-1}(q)=(f_*^{-1}(q),f^{*-1}(q))$ *is r- comp. for each* q *in* RY*.*

Theorem 3.1 *Let* $RX=(\underline{RX},\overline{RX})$ *be r-comp. space. If* $RY=(\underline{RY},\overline{RY})$ *is a r-Haus. space, then any r- surjective r-cont. map* $f=(f_*,f^*):RX\to RY$ *is r-perfect.*

Proof: Let y_0 be a point of $\underline{RY}$. Since $\underline{RY}$ is Haus., so $\{y_0\}$ is a closed set. This means $f_*^{-1}(\{y_0\})$ exists and is a closed set in $\underline{RX}$. But $\underline{RX}$ is comp. therefore, $f_*^{-1}(\{y_0\})$ is a comp. set in $\underline{RX}$. Now we try to show that f_* is closed. Let $K=(\underline{K},\overline{K})$ be a r-closed set in RX. But, $\underline{K}$ is closed in $\underline{RX}$, so it is comp. Then, $f_*(\underline{K})$ is comp. in $\underline{RY}$. But $\underline{RY}$ is Hausdorff, so $f_*(\underline{K})$ is closed. □

By the same way, we can prove that $f^{*-1}(\{y_0\})$ is a comp. set in $\overline{RX}$ for each point $y_0\in\overline{RY}$ and f^* is closed. Thus, f is r-perfect.

Theorem 3.2 *Suppose* (RX,τ) *and* (RY,δ) *are r- top spaces with topologies* $\tau=(\underline{\tau},\overline{\tau})$, $\delta=(\underline{\delta},\overline{\delta})$ *and let* $f=(f_*,f^*):(RX,\tau)\to(RY,\delta)$ *is a r- perfect map. If* $RY=(\underline{RY},\overline{RY})$ *is a r- comp., so* $RX=(\underline{RX},\overline{RX})$ *is r- comp.*

Proof: Consider the r- comp. sp. (RY,δ). This means each of $\underline{RY}$ and $\overline{RY}$ are comp. sets. Suppose there is an open cover $\Omega=\{\underline{U_\alpha}:\ \alpha\in I\}$ of $\underline{RX}$. Then by the preceding definition, $f_*^{-1}(y)$ is comp. for each $y\in\underline{RY}$, Ω covers $f_*^{-1}(y)$ and hence, there exists a finite set $B_y\subseteq I$ such that $f_*^{-1}(y)\subseteq\bigcup_\alpha\{\underline{U_\alpha}:\alpha\in B_y\}=\underline{V_y}$. Since f_* is closed then $f_*(\underline{RX}-\underline{V_y})$ is closed in $\underline{RY}$. Let $\underline{W_y}=\underline{RY}-f_*(\underline{RX}-\underline{V_y})$ which is an open neighborhood of y in $\underline{RY}$ and $f_*^{-1}(\underline{W_y})\subseteq\underline{V_y}$ and the collection $\{\underline{W_y}\}$ covers $\underline{RY}$. But $\underline{RY}$ is comp., then we can choose a finite subcover $\underline{W_{y_1}},\dots,\underline{W_{y_n}}$ covering $\underline{RY}$. Then $\underline{V_{y_1}},\dots,\underline{V_{y_n}}$ covers $\underline{RX}$. But, each $\underline{V_y}$ is a finite union of sets $\underline{U_\alpha}$, we also have a finite subcover by sets in $\{\underline{U_\alpha}\}$. Thus $\underline{RX}$ is comp. By the same way, we can prove that $\overline{RX}$ is also comp., and this shows, RX is r- comp. □

Now according to the above theorem, we can show the following:

Remark 3.1 *For any r- perfect map* $f:RX\to RY, f^{-1}(C)$ *is r- comp. in* RX *for every r- comp.* C *of* RY*.*

The converse of the above remark will be discussed later on Theorem 3.6

Definition 3.2 *[4] Let* (RX,τ) *be a r-Haus. sp. where* $RX=(\underline{RX},\overline{RX})$ *and* $\tau=(\underline{\tau},\overline{\tau})$*. Then* $(\overline{RX},\overline{\tau})$ *is called an upper (u.) compactly generated if for each subset* $\overline{W}$ *of* $\overline{RX}$ *satisfying* $\overline{W}\cap\overline{K}$ *is closed in* $\overline{K}$ *(or consequently, in* $\overline{RX}$ *) for every comp. set* $\overline{K}$ *of* $\overline{RX}$ *then* $\overline{W}$ *is closed. Also,* $(\underline{RX},\underline{\tau})$ *named lower (l.) compactly generated if for each subset* $\underline{W}$ *of* $\underline{RX}$ *satisfying* $\underline{W}\cap\underline{K}$ *is closed in* $\underline{K}$ *(or consequently, in* $\underline{RX}$*) for every comp. set* $\underline{K}$ *of* $\underline{RX}$ *then* $\underline{W}$ *is closed. Then* (RX,τ) *is called r- compactly generated space if* $\overline{RX}$ *is u. compactly generated and* $\underline{RX}$ *is l. compactly generated.*

Remark 3.2 *The theorem below has been proved in [4] provided that the function* f *is r-homeo., we will prove that r-compactly generated property will transfer by a r-perfect map.*

Theorem 3.3 *Let* (RX,τ) *and* (RY,δ) *be r- Haus. top spaces with topologies* $\tau=(\underline{\tau},\overline{\tau})$, $\delta=(\underline{\delta},\overline{\delta})$ *and let* $f=(f_*,f^*):(RX,\tau)\to(RY,\delta)$ *be a r- perfect map. If* RX *is a r- compactly generated then so is* RY*.*

Proof: Assume that, RX is a r- compactly generated space and that, M is a r-subset of RY. Suppose C to be a r- comp. set in RY such that $M\cap C$ is r- closed set. Since $f:(RX,\tau)\to(RY,\delta)$ is a r- perfect, so f is r- surj. map, that is, $f^{-1}(M)$ is a r- set in RX. Also, $f^{-1}(C)$ is a r- comp. set of RX (Remark 3.1) and, $f^{-1}(M\cap C)=f^{-1}(M)\cap f^{-1}(C)$ is closed in RX. Then, $f^{-1}(M)$ is r- closed set in RX because of RX is r- compactly generated. Hence, $f(f^{-1}(M))=M$ is r-closed subset of RY . Which means that RY is a r- compactly generated. □

The converse direction is true when f is r- homeo. [4].

Theorem 3.4 *Every finite r- set in a r- Haus. sp.* $RX = (\underline{RX}, \overline{RX})$ *is r- closed.*

Proof: We only need to show that any r-singleton set $\{x_0\}$ is r-closed. Let us discuss the proof in $\underline{RX}$. Consider x is another point in $\underline{RX}$ such that $x \neq x_0$. But, RX is a Haus. sp. so, there exist disjoint open sets $\underline{U}$ and $\underline{V}$ where, $x_0 \in \underline{U}$ and $x \in \underline{V}$. Because $\underline{V} \cap \{x_0\} = \emptyset$, then $x \notin cl\{x_0\}$ which is satisfies for each x in $\underline{RX} - \{x_0\}$. This means $cl\{x_0\} = \{x_0\}$. Thus, $\{x_0\}$ is closed in $\underline{RX}$. □

In same way, we can see $\{x_0\}$ is closed in $\overline{RX}$. This explains that $\{x_0\}$ is r- closed.

Theorem 3.5 *A r- closed set in a r- comp. sp. is r- comp.*

Proof: Suppose, $B = (\underline{B}, \overline{B})$ is a r-closed set in a r- comp. sp. $RX = (\underline{RX}, \overline{RX})$. Suppose that $\mathcal{A} = \{(\underline{\mathcal{A}}, \overline{\mathcal{A}})\}$ a family of r- open sets in RX such that $\underline{B} \subset \cup\{\underline{A} : \underline{A} \in \underline{\mathcal{A}}\}$ and $\overline{B} \subset \cup\{\overline{A} : \overline{A} \in \overline{\mathcal{A}}\}$. The open collection $\underline{\mathcal{A}} \cup \{\underline{RX} - \underline{B}\}$ covers the comp. space $\underline{RX}$, so we can choose a finite set $\underline{A_1}, \ldots, \underline{A_n} \in \underline{\mathcal{A}}$ such that $\underline{RX} = (\underline{RX} - \underline{B}) \cup \underline{A_1} \cup \ldots \cup \underline{A_n}$. This shows that $\underline{B} \subset \underline{A_1} \cup \ldots \cup \underline{A_n}$ and this means that $\underline{B}$ is comp.. Similarly, we can see that, $\overline{B}$ is comp., so $B = (\underline{B}, \overline{B})$ is r- comp.. □

Definition 3.3 *Let* $RX = (\underline{RX}, \overline{RX})$ *be r- top sp. with topology* $\tau = (\underline{\tau}, \overline{\tau})$. *Then the sp.* RX *is called rough locally compact (r- loc. comp.) if both* $\underline{RX}$ *and* $\overline{RX}$ *are loc. comp. spaces. That is, each* $x \in \underline{RX}$ *has a comp.* $\underline{\tau}$*- neighborhood, and each* $y \in \overline{RX}$ *has a comp.* $\overline{\tau}$*- neighborhood.*

Theorem 3.6 *If* (RX, τ) *is a r- Haus. sp. and* (RY, δ) *r- loc. comp. r-Haus. sp., then a r- surj. r- cont. map* $f = (f_*, f^*) : (RX, \tau) \rightarrow (RY, \delta)$ *is r- perfect if and only if,* $f^{-1}(G) = (f_*^{-1}(\underline{G}), f^{*-1}(\overline{G}))$ *is r- comp. in* $RX = (\underline{RX}, \overline{RX})$ *for any r-comp. set* $G = (\underline{G}, \overline{G}) \subseteq RY$.

Proof: Suppose, f is a r- perfect map, so according to (Remark 3.1), we obtain $f^{-1}(G)$ is r- comp. for every r-comp. $G \subseteq RY$. Suppose, $f^{-1}(G)$ is r-comp. in RX for every r-comp. $G \subseteq RY$. Hence, $f^{-1}(g) = (f_*^{-1}(g), f^{*-1}(g))$ is r- comp. for each $g \in RY$. So, $f_*^{-1}(g)$ is comp. in $\underline{RX}$. Now, by Definition 3.1, it is enough to show that, the map $f = (f_*, f^*)$ is r-closed. Let $x \in \underline{RX}$, so there is a comp. neighborhood $\underline{M}$ of $f_*(x)$ in $\underline{RY}$. Now $f_*^{-1}(\underline{M})$ is a comp. neighborhood of x in $\underline{RX}$, and this leads to that, $\underline{RX}$ is loc. comp.. Let us denote to the one-point p compactifications of $\underline{RX}$ and $\underline{RY}$ by Com $\underline{RX}$ and Com $\underline{RY}$ respectively, and define a function $\tilde{f}_* : \mathrm{Com}\underline{RX} \rightarrow \mathrm{Com}\underline{RY}$ by $\tilde{f}_*(x) = f_*(x)$ for every $x \in \underline{RX}$ and $\tilde{f}_*(p_{\underline{RX}}) = p_{\underline{RY}}$. It is obvious that the function $\tilde{f}_*$ is cont.. Suppose, $\underline{U}$ is an open set in Com $(\underline{RY})$. Now when $\underline{U} \subseteq \underline{RY}$, we have, $\underline{U}$ is open in $\underline{RY}$. Hence, $\tilde{f}_*^{-1}(\underline{U}) = f_*^{-1}(\underline{U})$ is open subset of $\underline{RX}$ and also open subset of Com$(\underline{RX})$. When $p_{\underline{RY}} \in \underline{U}$, then $\underline{C} = \mathrm{Com}(\underline{RY}) - \underline{U}$ is comp. and hence, $f_*^{-1}(\underline{C})$ is comp. in $\underline{RX}$. Since $\underline{RX}$ is Haus., that means $f_*^{-1}(\underline{C})$ is closed in $\underline{RX}$. Thus $\tilde{f}_*^{-1}(\underline{U}) = \mathrm{Com}(\underline{RX}) - f_*^{-1}(\underline{C})$ is open subset of Com $(\underline{RX})$. Now when $\underline{F}$ is closed set in $\underline{RX}$, then $\underline{F} \cup \{p_{\underline{RX}}\}$ is closed subset of Com ($\underline{RX}$) and hence it is comp.. Consequently, $\tilde{f}_*(\underline{F} \cup \{p_{\underline{RX}}\})$ is comp. and therefore, closed subset of Com ($\underline{RY}$). Hence, $f_*(\underline{F}) = \underline{RY} \cap \tilde{f}_*(\underline{F} \cup \{p_{\underline{RX}}\})$ is closed subset of $\underline{RY}$. By a same manner, we can find $f^*(\overline{F})$ is closed subset of $\overline{RY}$. Thus, $f = (f_*, f^*)$ is r- proper, and the proof is complete.

□

Proposition 3.1 *Every r-loc. comp. r-Haus. space is r- compactly generated.*

Proof: Let $RX = (\underline{RX}, \overline{RX})$ be r- loc. comp. r-Haus. sp. and $F = (\underline{F}, \overline{F}) \subset RX$. For each r- comp. set $C = (\underline{C}, \overline{C})$ of RX, we assume $\underline{F} \cap \underline{C}$ and $\overline{F} \cap \overline{C}$ are closed sets in $\underline{RX}$ and $\overline{RX}$, respectively. We will show that $\underline{RX}$ is compactly generated. Let $x \in \underline{RX} - \underline{F}$ be any point and let $\underline{M}$ be a comp. neighborhood of the point x. By assumption, $(\underline{RX} - \underline{F}) \cap \underline{M}$ is an open set in $\underline{M}$, then $(\underline{RX} - \underline{F}) \cap \mathrm{Int}\underline{M} = \underline{G}$ is open subset of Int $\underline{M}$. Since Int $\underline{M}$ is open subset of $\underline{RX}$ then $\underline{G}$ is an open neighborhood of x which contained in $\underline{RX} - \underline{F}$. This means $\underline{RX} - \underline{F}$ is open set, hence $\underline{F}$ is closed and $\underline{RX}$ is compactly generated. Similarly, we can prove $\overline{RX}$ is also compactly generated and hence RX is r-compactly generated. □

Theorem 3.7 *Let $f=(f_*,f^*): RX=(\underline{RX},\overline{RX})\to RY=(\underline{RY},\overline{RY})$ be a r- surj. r- cont. map. If RY is r- compactly generated and $f^{-1}(C=(\underline{C},\overline{C}))$ is r-comp. in RX for every r- comp. $C\subseteq RY$, then f is r-perfect map.*

Proof: To prove this proposition, we need to show that f is r- closed. Since RY is r- compactly generated, this means that RY is r-Haus. sp. [4]. Let $K=(\underline{K},\overline{K})$ be r-closed in RX and $C=(\underline{C},\overline{C})$ r-comp. in RY. Then $f^{-1}(C)=(f_*^{-1}(\underline{C}),f^{*-1}(\overline{C}))$ is r- comp. in RX and then so is $f^{-1}(C)\cap K=B=(\underline{B},\overline{B})$. Now $f(B)=C\cap f(K)$ is r- comp. in RY. But RY is r- Haus., so $f(B)$ is r- closed subset of RY. Because, RY is r- compactly generated , so $f(K)$ is r-closed in RY. This means f is r- closed and then f is r- perfect. ☐

Corollary 3.1 *If (RX,τ) is a r-space and (RY,δ) r-compactly generated space, then a r- surj. r- cont. map $f:(RX,\tau)\to(RY,\delta)$ is r- perfect if and only if, $f^{-1}(G)$ is r- comp. in RX for every r- comp. set $G\subseteq RY$.*

Proof: Let f be a r- perfect map, hence by Remark (3.1), we see $f^{-1}(G)$ is r- comp. in RX for each r- comp. set $G\subseteq RY$. Conversely, when $f^{-1}(G)$ is r-comp. in RX for each r- comp. set $G\subseteq RY$ and RY is r-compactly generated then, by theorem (3.7), we see that f is r- perfect. ☐

Theorem 3.8 *Let $f=(f_*,f^*): RX=(\underline{RX},\overline{RX})\to RY=(\underline{RY},\overline{RY})$ be a r- cont. map. If RX is r-comp. space then so is $f(RX)=(f_*(\underline{RX}),f^*(\overline{RX}))$.*

Proof: Suppose RX is r- comp. space, then $\underline{RX}$ is l. comp. set and $\overline{RX}$ is u. comp. set of RX, respectively. Let $\mathcal{M}=\{U_\alpha=(\underline{U_\alpha},\overline{U_\alpha}):U_\alpha\subseteq RY,\alpha\in I\}$ be a r-open covering of the set $f(RX)$. The collection $\{f^{-1}(U_\alpha)=(f_*^{-1}(\underline{U_\alpha}),f^{*-1}(\overline{U_\alpha})),\alpha\in I\}$ is a r-open covering of RX, where $\{f_*^{-1}(\underline{U_\alpha}),\alpha\in I\}$ is open covering of $\underline{RX}$ and $\{f^{*-1}(\overline{U_\alpha}),\alpha\in I\}$ is open covering of $\overline{RX}$. Because of the compactness of $\underline{RX}$ and $\overline{RX}$ then there are finite sets, $\{f_*^{-1}(\underline{U_{\alpha_1}}),\dots,f_*^{-1}(\underline{U_{\alpha_n}})\}$ and $\{f^{*-1}(\overline{U_{\alpha_1}}),\dots,f^{*-1}(\overline{U_{\alpha_n}})\}$ covering $\underline{RX}$ and $\overline{RX}$, respectively. Then the sets $\{f_*^{-1}(\underline{U_{\alpha_1}}),\dots,f_*^{-1}(\underline{U_{\alpha_n}})\}$ and $\{f^{*-1}(\overline{U_{\alpha_1}}),\dots,f^{*-1}(\overline{U_{\alpha_n}})\}$ cover $f_*(\underline{RX})$ and $f^*(\overline{RX})$, respectively. Thus, $f(RX)$ is r-comp. set in RY. ☐

Theorem 3.9 *Let $RX=(\underline{RX},\overline{RX})$ be a r-top space and $K=(\underline{K},\overline{K})$ r- comp. subset of RX. If $F=(\underline{F},\overline{F})$ is r- closed set in RX, then $K\cap F$ is r-comp. set in RX.*

Proof: We have K is r- comp. set, then each of $\underline{K}$ and $\overline{K}$ are comp. sets in $\underline{RX}$ and $\overline{RX}$, respectively. Suppose $\{\underline{V_i}:i\in I\}$ is an open covering of $\underline{K}\cap\underline{F}$. Hence, $\{\underline{V_i}:i\in I\}\cup\underline{F^c}$ is an open cover of $\underline{K}$. But $\underline{K}$ is a comp. set, we can obtain a subcover $\{\underline{V_{i_1}},\dots,\underline{V_{i_n}}\}\cup\underline{F^c}\supset\underline{K}$. Then it is obvious that $\{\underline{V_{i_1}},\dots,\underline{V_{i_n}}\}$ covers $\underline{K}\cap\underline{F}$, so it is comp.. Similarly, we can see that $\overline{K}\cap\overline{F}$ is also comp. This means $K\cap F$ is r-comp. set in RX. ☐

Proposition 3.2 *A r- comp. set in a r- Haus. sp. is r- closed.*

Proof: Suppose $RX=(\underline{RX},\overline{RX})$ is a r-Haus. sp. and let $A=(\underline{A},\overline{A})\subset RX$ to be r- comp.. When $x\in\underline{RX}-\underline{A}$, for any $a\in\underline{A}$, there are disjoint open sets $\underline{U_a},\underline{V_a}$ such that $x\in\underline{U_a}$, $a\in\underline{V_a}$. It is obvious that $\{\underline{V_a}:a\in\underline{A}\}$ covers the set $\underline{A}$. But $\underline{A}$ is comp., so there is a finite set of points $\{a_1,a_2,\dots,a_n\}\subset\underline{A}$ such that $\underline{A}\subseteq\bigcup_{i=1}^n\underline{V_{a_i}}=\underline{V}$. Now if $\underline{U}=\bigcap_{i=1}^n\underline{U_{a_i}}$, then it is clear that, $\underline{U}$ is open and $\underline{V}\cap\underline{U}=\emptyset$. Thus, we have $x\in\underline{U}\subseteq\underline{RX}-\underline{V}\subseteq\underline{RX}-\underline{A}$, and the point x is in the interior of $\underline{RX}-\underline{A}$. Since x can be any point in $\underline{RX}-\underline{A}$, then $\underline{RX}-\underline{A}$ is open and hence, $\underline{A}$ is closed. In similar way we obtain, $\overline{A}$ is closed. Hence, $A=(\underline{A},\overline{A})$ is r-closed. ☐

Lemma 3.1 *If $A=(\underline{A},\overline{A})$ and $B=(\underline{B},\overline{B})$ are r-comp. sets of a r-Haus. sp. $RX=(\underline{RX},\overline{RX})$, such that $A\cap B=\emptyset$ then there are two disjoint r-open sets one containing A and the other containing B.*

Proof: Let b be any point in $\underline{B}$ so there exist disjoint open sets $\underline{U_b}$ and $\underline{V_b}$ in $\underline{RX}$ such that $b \in \underline{U_b}$ and $\underline{A} \subseteq \underline{V_b}$. Clear that, the collection $\{\underline{U_b} : b \in \underline{B}\}$ is an open cover of the comp. set $\underline{B}$. Therefore, we have finitely many points $b_1,\ b_2,\ \ldots,\ b_n \in \underline{B}$ such that $\underline{B} \subseteq \bigcup_{i=1}^{n} \underline{U_{b_i}} = \underline{U}$. Hence, $\underline{V} = \bigcap_{i=1}^{n} \underline{V_{b_i}}$ is an open set and $\underline{A} \subseteq \underline{V}$. But $\underline{U_{b_i}} \cap \underline{V_{b_i}} = \emptyset$ for every i, then we have $\underline{U} \cap \underline{V} = \emptyset$. By same method, we can see $\overline{U} \cap \overline{V} = \emptyset$. This means A and B are in two disjoint r-open sets and this complete the proof.

□

Proposition 3.3 *Let $f = (f_*, f^*) : (RX, \tau) \to (RY, \delta)$ be a r-perfect map. If RX is r- Haus. sp., then so is RY.*

Proof: Suppose y_1 and y_2 are two different points in RY. Because f is r- perfect , then each of $f_*^{-1}(y_1)$, $f^{*-1}(y_1)$ are l. comp. and u. comp. sets of $\underline{RX}$ and $\overline{RX}$ respectively, and $f_*^{-1}(y_2)$, $f^{*-1}(y_2)$ are l. comp. and u. comp. sets of $\underline{RX}$ and $\overline{RX}$ respectively. Clear that, $f_*^{-1}(y_1) \cap f_*^{-1}(y_2) = \emptyset$ and $f^{*-1}(y_1) \cap f^{*-1}(y_2) = \emptyset$. So by Lemma (3.1) , there exist disjoint r- open neighborhoods $M = (\underline{M}, \overline{M})$ of ($\underline{f_*^{-1}(y_1)}$, $\overline{f^{*-1}(y_1)}$) and $N = (\underline{N}, \overline{N})$ of $(\underline{f_*^{-1}(y_2)}, \overline{f^{*-1}(y_2)})$ respectively. Set $V_1 = (\underline{RY} - f_*(\underline{RX} - \underline{M}), \overline{RY} - f^*(\overline{RX} - \overline{M}))$ and $V_2 = (\underline{RY} - f_*(\underline{RX} - \underline{N}), \overline{RY} - f^*(\overline{RX} - \overline{N}))$. Since, $f = (f_*, f^*)$ is r-closed then each of V_1 and V_2 is r- open in $RY = (\underline{RY}, \overline{RY})$ such that, $y_1 \in V_1$ and $y_2 \in V_2$. Also, we have $f^{-1}(V_1) \subseteq M = (\underline{M}, \overline{M})$ and $f^{-1}(V_2) \subseteq N = (\underline{N}, \overline{N})$. Since $M \cap N = \emptyset$ and that f is r- surj. then $V_1 \cap V_2 = \emptyset$. That is, V_1 and V_2 are disjoint r-open neighborhoods of y_1 and y_2, respectively. Thus, $RY = (\underline{RY}, \overline{RY})$ is r- Haus.

□

References

1. Adhikari, A., Adhikari, M. R., *Basic Topology 1*, (1st Edition), Springer Nature Singapore Pte Ltd., (2022).
2. Mathew, B.P., John, S. J. , *On rough topological spaces*, Prentice Hall 3, 9, 3413-3421, (2012).
3. Munkers J. R., *Topology*, (2nd edition). Prentice Hall, (2000).
4. Albayati M., Alhindawe A., Alkhayyat H., *Some Results on Rough k-space.* Central Asian Journal of Mathematical Theory and Computer Sciences, vol.6, 4, 873-877, (2025).
5. Dhivya, M., Dhivya, A. J., *a study of compactness and connectedness in rough topological spaces.* J. Journal of Mathematics 12, 1-7, (2016).
6. Wu, Q. E., Wang, v, Huang, Y.X., Li, J. S., *topology theory on rough sets*, IEEE Transactions on Systems, Man and Cybernetics Part B—Cybernetics 38, 1, 68-77, (2008).
7. Singh, T. B., *introduction to topology*, Springer Nature Singapore Pte Ltd (2019).
8. Singh, P.K., Tiwari, S., *topological structures in rough set theory.* A survey, Hacet J. Math. Stat 49, 4, 1270 – 1294, (2020).
9. Pawlak, Z., *rough sets -theoretical aspects of reasoning about data*, Kluwer Academic Publishers, Boston (1991).
10. Pawlak, Z., *rough sets*, International Journal of Computer and Information Sciences 11, 341-356, (1982).

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