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Unilateral Elliptic Problems with L^1-data in Anisotropic Weighted Sobolev Spaces

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ABSTRACT: In this work, we prove an existence result for a class of strongly nonlinear elliptic equations given by

$$-\operatorname{div}(a(x,\upsilon,\nabla\upsilon))+\Psi(x,\upsilon,\nabla\upsilon)+\Phi(x,\nabla\upsilon)=f \quad \text{in } \Omega$$

where the source term f belongs to $L^1(\Omega)$. The function Ψ is assumed to have critical growth with respect to the gradient $\nabla\upsilon$, without any growth restriction concerning the variable υ, while the function $\Phi(x,\nabla\upsilon)$ grows as $|\nabla\upsilon|^{p_i-1}$..

Keywords: Anisotropic weighted Sobolev spaces, anisotropic degenerate elliptic equations, entropy solutions, L1-data.

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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, where $N \geq 2$, and let p be a real number such that $1 < p < \infty$. Consider a vector of weight functions $w(\cdot)=\{w_i(\cdot): i=0,\dots,N\}$, where each $w_i(\cdot)$ is a measurable function that is strictly positive almost everywhere in Ω and satisfies certain integrability conditions (refer to the Preliminaries section for details). We define the weighted anisotropic Sobolev space $X=W_0^{1,p_i}(\Omega,w)$ corresponding to the weight vector w.

We examine the following nonlinear Dirichlet problem:

$$\begin{cases} \upsilon \in W_0^{1,p_i}(\Omega,w), \quad \Psi(x,\upsilon,\nabla\upsilon)\in L^1(\Omega), \quad \Phi(x,\nabla\upsilon)\in L^1(\Omega) \\ A(\upsilon)+\Psi(x,\upsilon,\nabla\upsilon)+\Phi(x,\nabla\upsilon)=f \quad \text{in } \mathfrak{D}'(\Omega) \end{cases} \tag{1.1}$$

where f is a given element in the space of distributions $\mathfrak{D}'(\Omega)$.

Here, the operator $A(\upsilon)$ is defined as $A(\upsilon)=-\operatorname{div}(a(x,\upsilon,\nabla\upsilon))$, representing a Leray-Lions type operator that maps elements from X into its dual space $X^*=W^{-1,p_i'}(\Omega,w^*)$. The dual weight vector $w^*=\left\{w_i^*=w_i^{1-p_i'}: i=0,1,\dots,N\right\}$, and p_i' denotes the Holder conjugate of p_i, that is, $p_i'=\frac{p_i}{p_i-1}$.

The nonlinear lower-order term $\Psi(x,\upsilon,\nabla\upsilon)$ exhibits at most a growth rate proportional to $|\nabla\upsilon|^{p_i}$. Additionally, it fulfills a sign condition relative to its second argument, and it also verify a coercivity condition as specified below.

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$$|\Psi(x,s,\xi)| \geq \beta \sum_{i=1}^{N} w_i(x)\,|\xi_i|^{p_i}$$

for sufficiently large $|s|$. The term $\Phi(x,\nabla\upsilon)$ grows at most like $|\nabla\upsilon|^{p_i-1}$. We study problem (1.1) in a non-variational framework, where $f \in L^1(\Omega)$.

When $H \equiv 0$ and in the variational setting-specifically when the source term f belongs to $W^{-1,p'}(\Omega, w^*)$ -an existence result for the following unilateral problem has been established in [2]:

$$\begin{cases} \upsilon \in K_\psi, \quad \Psi(x,\upsilon,\nabla\upsilon) \in L^1(\Omega), \quad \Psi(x,\upsilon,\nabla\upsilon)\upsilon \in L^1(\Omega) \\ \langle A(\upsilon), \upsilon - u\rangle + \displaystyle\int_\Omega \Psi(x,\upsilon,\nabla\upsilon)(\upsilon-u)dx \leq \langle f, \upsilon-u\rangle \\ \text{for all } u \in K_\psi \cap L^\infty(\Omega) \end{cases}$$

which corresponds to the equation in problem (1.1). The proof relies on a method involving the strong convergence of the approximated solutions' positive and negative parts, denoted υ_ε^+ and υ_ε^-, respectively.In the non-variational setting, where $f \in L^1(\Omega)$, the term

$$\int_\Omega f(\upsilon-u)dx,$$

is not well-defined, and as a result, the previous formulation is no longer applicable. To address this, the authors in [6] established an existence result for the unilateral problem using a different approach based on the strong convergence of truncations. The problem is formulated as follows:

$$\begin{cases} \upsilon \in K_\psi, \quad \Psi(x,\upsilon,\nabla\upsilon) \in L^1(\Omega) \\ \langle A(\upsilon), T_k(\upsilon-u)\rangle + \displaystyle\int_\Omega \Psi(x,\upsilon,\nabla\upsilon)T_k(\upsilon-u)dx \leq \int_\Omega fT_k(\upsilon-u)dx \\ \text{for all } u \in K_\psi \text{ and all } k > 0 \end{cases}$$

where T_k denotes the truncation function at level k, and the convex set K_ψ is defined by

$$K_\psi = \Big\{\upsilon \in W_0^{1,p}(\Omega,w) : \upsilon \geq \psi \text{ a.e. in } \Omega\Big\}$$

In the weighted case (i.e., when $w \equiv 1$) and for nonzero Φ, Del Vecchio studyied problem (1.1) in [12], under the assumption that Φ depends only on x and υ. When g also depends on $\nabla\upsilon$, existence results for problem (1.1) were first established by Monetti and Randazzo in [23] for the equation case.

Extensive research has been devoted to examining the existence of solutions for parabolic and elliptic problems under different sets of hypotheses. For a comprehensive overview, readers can refer to the extensive studies and publications available on this subject (see [9,10,15,16,17,18,19,4,5,20]).

The primary objective of this paper is to establish an existence result for degenerate unilateral problems related to (1.1), in the setting where $\Phi \neq 0$ and the source term f belongs to $L^1(\Omega)$. Our result extends the work presented in [24] to the framework of anisotropic weighted Sobolev spaces.

The remainder of the paper is structured as follows: Section 2 provides the necessary preliminaries, including notations, assumptions, and several technical lemmas essential for the analysis. In Section 3, we present the main result, and in the final section, we provide the detailed proof of this result.

2. Preliminaries

2.1. Anisotropic weighted Sobolev spaces

In this work, we extend the concept of Sobolev spaces by introducing a new class of anisotropic weighted Sobolev spaces. Let Ω denote a bounded open subset of $\mathbf{R}^N$, $p_0, p_1, \ldots, p_N$ be $N+1$ exponents with $1 < p_i < \infty$ for $i = 0, 1, \ldots, N$ and $w = \{w_i(x), 0 \leq i \leq N\}$ represent a collection of measurable weight functions that are almost everywhere strictly positive on Ω. We also adopt the following assumption: there exist

$$w_i \in L^1_{\text{loc}}(\Omega), \tag{2.1}$$

$$w_i^{\frac{-1}{p_i-1}} \in L^1_{\text{loc}}(\Omega), \tag{2.2}$$

for any $0 \leq i \leq N$. The anisotropic weighted orlicz space $L^{p_i}(\Omega, \gamma)$, where γ is a weight function on Ω will be formulated by the following expression,

$$L^{p_i}(\Omega, \gamma) = \left\{ \upsilon = \upsilon(x), \upsilon \gamma^{\frac{1}{p_i}} \in L^{p_i}(\Omega) \right\}$$

and endowed by the norm

$$\|\upsilon\|_{L^{p_i}(\Omega,\gamma)} = \|\upsilon\|_{p_i,\gamma} = \left(\int_\Omega |\upsilon(x)|^{p_i} \gamma(x) dx \right)^{\frac{1}{p_i}}.$$

We put

$$(p) = (p_0, \ldots, p_N), \quad D^0\upsilon = \upsilon \quad \text{and} \quad D^i\upsilon = \frac{\partial \upsilon}{\partial x_i} \quad \text{for } i = 1, \ldots, N,$$

and we consider that

$$\underline{p} = \min\{p_0, p_1, \ldots, p_N\} \quad \text{then} \quad \underline{p} > 1. \tag{2.3}$$

The anisotropic weighted Sobolev space $W^{1,(p_i)}(\Omega, w)$ is defined as the space of real-valued functions $\upsilon \in L^{p_0}(\Omega, w_0) = \left\{ \upsilon(x), \upsilon w_0^{\frac{1}{p_0}} \in L^{p_0}(\Omega) \right\}$ such that the derivatives in the sense of distributions fulfill

$$D^i\upsilon \in L^{p_i}(\Omega, w_i) \text{ for } i = 1, \ldots, N$$

Which is a Banach space endowed by the following norm

$$\|\upsilon\|_{1,(p_i),w} = \|\upsilon\|_{p_0,w_0} + \sum_{i=1}^{N} \left\| \frac{\partial \upsilon}{\partial x_i} \right\|_{p_i,w_i}. \tag{2.4}$$

The hypothesis (2.1) ensures that $C_0^\infty(\Omega) \subset W^{1,(p_i)}(\Omega, w)$ and consequently, we may define the subspace $V = W_0^{1,(p_i)}(\Omega, w)$ of $W^{1,(p_i)}(\Omega, w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.4).

Moreover, condition (2.2) ensures that $W^{1,(p_i)}(\Omega, w)$ and $W_0^{1,(p_i)}(\Omega, w)$ are reflexive Banach spaces.

We notice that the dual space of weighted Sobolev spaces $W_0^{1,(p)}(\Omega, w)$ is equivalent to $W^{-1,(p')}(\Omega, w^*)$, where $w^* = \left\{ w_i^* = w_i^{1-p'}, i = 0, \ldots, N \right\}$ and $(p_i') = (p_0', p_1', \ldots, p_N')$ where p' is the conjugate of p_i; i.e., $p_i' = \frac{p_i}{p_i-1}$, (see [13] for the isotropic case).

We further define $\mathcal{T}_0^{1,p_i}(\Omega, w)$ as follows

$$\mathcal{T}_0^{1,p_i}(\Omega, w) = \left\{ \upsilon : \Omega \to \mathbb{R} \text{ measurable } : T_k(\upsilon) \in W_0^{1,p_i}(\Omega, w), \quad \forall k \geq 0 \right\},$$

where $T_k : \mathbb{R} \to \mathbb{R}$ is the truncation at height k defined by $T_k(s) = \max(-k, \min(k, s))$.

2.2. Technical results

Lemma 2.1 *[7] Let Ω be a bounded open subset of $\mathbf{R}^N$, w a weight function on Ω, if (2.1) and (2.2) are verified then $L^{p_i}(\Omega, w) \hookrightarrow L^1_{\text{loc}}(\Omega)$.*

Lemma 2.2 *([21]Theorem 13.47) Let $(\upsilon_n)_n$ be a sequence in $L^1(\Omega)$ and $\upsilon \in L^1(\Omega)$ such that*

1. $\upsilon_n \to \upsilon$ a.e. in Ω, *2.* $v_n \geq 0$ *and* $v \geq 0$ *a.e. in* Ω,

3. $\int_\Omega v_n dx \to \int_\Omega v dx$

then $v_n \to v$ *in* $L^1(\Omega)$.

Lemma 2.3 *[1] Let* γ *a weight function,* $f \in L^r(\Omega,\gamma)$, *and* $(f_n)_n \subset L^r(\Omega,\gamma)$ *such that* $\|f_n\|_{r,\gamma} \leq c$, $1 < r < \infty$ *If* $f_n(x) \to f(x)$ *a.e. in* Ω, *then* $f_n \to f$ *weakly in* $L^r(\Omega,\gamma)$.

Lemma 2.4 *(See lemma 8 in [8]) Let* $(u_n)_n$ *be a bounded sequence in* $W_0^{1,(p_i)}(\Omega,\omega)$. *If* $u_n \rightharpoonup u$ *weakly in* $W_0^{1,(p_i)}(\Omega,\omega)$, *therefore* $T_k(u_n) \rightharpoonup T_k(u)$ *weakly in* $W_0^{1,(p_i)}(\Omega,\omega)$ *for any* $k > 0$.

3. Basic Assumptions and Existence Results

We state the following assumptions:
Assumption $(\mathcal{H}_1)$ **:**
- The space $X = W_0^{1,p_i}(\Omega, w)$ is fitted with the norm

$$|||v|||_X = \left(\sum_{i=1}^N \int_\Omega \left|\frac{\partial v}{\partial x_i}\right|^{p_i} w_i(x)dx\right)^{\frac{1}{p_i}}$$

and it is equivalent to the norm (2.4). Note that $(X, |||v|||_X)$ is a uniformly convex and thus reflexive Banach space.
- We can find a weight function σ on Ω such that

$$\sigma \in L^1(\Omega) \text{ and } \sigma^{1-q_i'} \in L^1_{loc}(\Omega) \tag{3.1}$$

for some parameter $1 < q_i < p_i + p_i'$ and $q_i' = q_i/(q_i-1)$, such that the Hardy inequality

$$\left(\int_\Omega |v(x)|^{q_i}\sigma dx\right)^{\frac{1}{q_i}} \leq c\left(\sum_{i=1}^N \int_\Omega \left|\frac{\partial v}{\partial x_i}\right|^{p_i} w_i(x)dx\right)^{\frac{1}{p_i}} \tag{3.2}$$

holds for every $v \in X$ with a constant $c > 0$ independent of v. Furthermore, the imbedding $X \hookrightarrow L^{q_i}(\Omega,\sigma)$ reached by the Hardy inequality (3.2) is compact, i.e.

$$X \hookrightarrow\hookrightarrow L^{q_i}(\Omega,\sigma) \tag{3.3}$$

Assumptions $(\mathcal{H}_2)$ **:**
Let $A(v) = -\operatorname{div} a(x,v,\nabla v)$ be the Leray-Lions operator acting from $W_0^{1,p_i}(\Omega,w)$ into its dual $W^{-1,p_i'}(\Omega,w^*)$, where $a : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying, for a.e $x \in \Omega$, for all $s \in \mathbb{R}$ and all $\xi,\xi^* \in \mathbb{R}^N$ $(\xi \neq \xi^*)$, the following assumptions:

$$|a_i(x,s,\xi)| \leq \alpha_1 w_i^{\frac{1}{p_i}}(x)\left[\delta(x) + \sigma^{\frac{1}{p_i'}}|s|^{\frac{q_i}{p_i'}} + \sum_{j=1}^N w_j^{\frac{1}{p_i'}}(x)\,|\xi_j|^{p_i-1}\right], \quad \text{for } i = 1,\ldots,N, \tag{3.4}$$

$$[a(x,s,\xi) - a(x,s,\xi^*)]\cdot[\xi-\xi^*] > 0, \tag{3.5}$$

$$a(x,s,\xi)\cdot\xi \geq \alpha_2\sum_{i=1}^N w_i(x)\,|\xi_i|^{p_i} \tag{3.6}$$

where $\delta(\cdot)$ is a positive function in $L^{p_i'}(\Omega)$, σ is the weight function already defined in (3.1) and α_1,α_2 are positive constants.
Let $\Psi : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ and $H : \Omega\times\mathbb{R}^N \to \mathbb{R}$ be two Carathéodory functions satisfying, for a.e $x \in \Omega$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$, the following assumption Assumption $(\mathcal{H}_3)$:

$$\Psi(x,s,\xi)s \geq 0 \tag{3.7}$$

$$|\Psi(x,s,\xi)| \leq b(|s|)\left(c(x)+\sum_{i=1}^{N} w_i(x)\,|\xi_i|^{p_i}\right) \tag{3.8}$$

$$|\Psi(x,s,\xi)| \geq \beta \sum_{i=1}^{N} w_i(x)\,|\xi_i|^{p_i} \text{ for } |s|>\kappa, \tag{3.9}$$

$$|\Phi(x,\xi)| \leq h(x)\sum_{i=1}^{N} w_i^{\frac{1}{p_i'}}(x)\,|\xi_i|^{p_i-1} \tag{3.10}$$

where $\beta>0, \kappa>0, b:\mathbb{R}^+\to\mathbb{R}^+$is a continuous increasing function, $c\in L^1(\Omega)$ and $h\in L^r(\Omega)$ with $r>\max(N,p_i)$. Given a measurable function $\psi:\Omega\to\overline{\mathbb{R}}$, called an obstacle function, such that

$$\psi^+ \in W_0^{1,p_i}(\Omega,w)\cap L^\infty(\Omega), \tag{3.11}$$

and consider the set $K_\psi=\left\{v\in W_0^{1,p_i}(\Omega,w): v\geq\psi \text{ a.e. in } \Omega\right\}$ which is convex. We assume that the source term

$$f\in L^1(\Omega). \tag{3.12}$$

We shall prove the following existence result concerning the nonlinear Dirichlet boundary value problem (1.1).

Theorem 3.1 *Under the assumptions* (3.4)-(3.12)*, there exists at least one solution of* (1.1) *in the following sense*

$$\begin{cases} v\in K_\psi, \quad \Psi(x,v,\nabla v)\in L^1(\Omega), \quad \Phi(x,\nabla v)\in L^1(\Omega), \\ \langle A(v),T_k(v-u)\rangle+\displaystyle\int_\Omega(\Psi(x,v,\nabla v)+\Phi(x,\nabla v))T_k(v-u)dx \\ \leq \displaystyle\int_\Omega fT_k(v-u)dx, \forall u\in K_\psi\cap L^\infty(\Omega), \forall k>0. \end{cases}$$

Lemma 3.1 *[3]Assume that $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are fulfilled. For any sequence (v_n) weakly convergent to u in $W_0^{1,p_i}(\Omega,w)$ such that*

$$\int_\Omega \left[a\left(x,v_n,\nabla v_n\right)-a\left(x,v_n,\nabla v\right)\right]\cdot\left[\nabla v_n-\nabla v\right]dx\to 0$$

we have $v_n\to v$ strongly in $W_0^{1,p_i}(\Omega,w)$.

4. Proof of Theorem 3.1

4.1. Step 1: A priori estimates.

Let Ω_n be a sequence of compact subsets of Ω such that Ω_n is increasing to Ω as $n\to\infty$. Let us define

$$\Psi_n(x,s,\xi)=\frac{\Psi(x,s,\xi)}{1+\frac{1}{n}|\Psi(x,s,\xi)|}\chi_{\Omega_n} \text{ and } \Phi_n(x,\xi)=\frac{\Phi(x,\xi)}{1+\frac{1}{n}|\Phi(x,\xi)|}\chi_{\Omega_n}$$

where χ_{Ω_n} is the characteristic function of Ω_n.

Consider the sequence of approximate problems

$$\begin{cases} v_n\in K_\psi, \Psi\left(x,v_n,\nabla v_n\right)\in L^1(\Omega), \quad \Phi_n\left(x,\nabla v_n\right)\in L^1(\Omega), \\ \Psi\left(x,v_n,\nabla v_n\right)v_n\in L^1(\Omega), \quad \Phi_n\left(x,\nabla v_n\right)v_n\in L^1(\Omega), \\ \left\langle A\left(v_n\right),v_n-u\right\rangle+\displaystyle\int_\Omega\left(\Psi\left(x,v_n,\nabla v_n\right)+\Phi_n\left(x,\nabla v_n\right)\right)\left(v_n-u\right)dx \\ \leq \displaystyle\int_\Omega f_n\left(v_n-u\right)dx, \quad \forall v\in K_\psi\cap L^\infty(\Omega), \end{cases} \tag{4.1}$$

where (f_n) is a sequence of smooth functions which converges strongly to f in $L^1(\Omega)$ with $\|f_n\|_{L^1(\Omega)} \leq C_f$.

$$\Psi_n(x,s,\xi) = \frac{\Psi(x,s,\xi)}{1+\frac{1}{n}|\Psi(x,s,\xi)|} \text{ and } \Phi_n(x,\xi) = \frac{\Phi(x,\xi)}{1+\frac{1}{n}|\Phi(x,\xi)|}$$

Note that $\Psi_n(x,s,\xi)$ and $\Phi_n(x,\xi)$ are satisfying the following conditions

$$|\Psi_n(x,s,\xi)| \leq n \quad \text{and} \quad |\Phi_n(x,\xi)| \leq n$$

We define the operator $\Psi_n : W_0^{1,p_i}(\Omega,w) \to W^{-1,p_i'}(\Omega,w^*)$ by

$$\langle \Psi_n \upsilon, u\rangle = \int_\Omega \left(\Psi(x,\upsilon,\nabla\upsilon) + \Phi_n(x,\nabla\upsilon)\right) u dx.$$

Thanks to the classical result of Theorem 8.2 of [22] and by using the following lemma which we can be proved by the same way as Lemma 4.2 of [22], the problem (4.1) has at least one solution υ_n.

Lemma 4.1 *The operator $B_n = A + \Psi_n$ from K_ψ into $W^{-1,p_i'}(\Omega,w^*)$ is pseudomonotone. Moreover, B_n is coercive in the following sense*

$$\frac{\langle B_n u, u-u_0\rangle}{\|u\|} \to +\infty \textit{ if } \|u\| \to +\infty, u \in K_\psi, \textit{ where } u_0 \in K_\psi$$

Taking $u \in K_\psi$ and choosing $h \geq \|\psi^+\|_\infty$ so as $\widetilde{u} = T_h(\upsilon_n - T_k(\upsilon_n - u)) \in K_\psi \cap L^\infty(\Omega)$. Using the test function $\tilde{u}$ in (4.1) and letting $h \to +\infty$, we obtain

$$\begin{aligned} &\langle A(\upsilon_n), T_k(\upsilon_n - u)\rangle + \int_\Omega [\Psi(x,\upsilon_n,\nabla\upsilon_n) + \Phi_n(x,\nabla\upsilon_n)] T_k(\upsilon_n - u)\, dx + \\ &\leq \int_\Omega f_n T_k(\upsilon_n - u)\, dx, \quad \text{for all } u \in K_\psi \quad \text{and for all } k > 0. \end{aligned} \tag{4.2}$$

For $k \geq \kappa + \|\psi^+\|_\infty$, where ρ is defined in (3.9), taking $u = \psi^+$as a test function in (4.1) we get

$$\begin{aligned} &\left\langle A(\upsilon_n), T_k\left(\upsilon_n - \psi^+\right)\right\rangle + \int_\Omega [\Psi(x,\upsilon_n,\nabla\upsilon_n) + \Phi_n(x,\nabla\upsilon_n)] T_k\left(\upsilon_n - \psi^+\right) dx \\ &\leq \int_\Omega f_n T_k\left(\upsilon_n - \psi^+\right) \end{aligned} \tag{4.3}$$

which implies by using (3.12) and Young's inequality

$$\begin{aligned} &\int_\Omega a(x,\upsilon_n,\nabla\upsilon_n)\cdot\nabla T_k\left(\upsilon_n - \psi^+\right) dx + \int_\Omega \Psi(x,\upsilon_n,\nabla\upsilon_n) T_k\left(\upsilon_n - \psi^+\right) dx \\ &\leq kC_f + k\sum_{i=1}^N \int_\Omega \Phi(x) w_i^{\frac{1}{p'}}(x) \left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i-1} dx \\ &\leq kC_f + C(k,p_i,N,\beta)\int_\Omega |\Phi(x)|^{p_i} dx + \frac{\beta}{k}\sum_{i=1}^N \int_\Omega w_i(x)\left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx \\ &\leq C_k + \frac{\beta}{k}\sum_{i=1}^N \int_{\{|\upsilon_n - \psi^+| \leq k\}} w_i(x)\left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx + \frac{\beta}{k}\sum_{i=1}^N \int_{\{|\upsilon_n - \psi^+| > k\}} w_i(x)\left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx, \end{aligned}$$

where C_k is a constant not depending on n and may be different at each occurrence. Using (3.11) together with the fact that $|\upsilon_n| \geq k - \|\psi^+\|_\infty \geq \kappa$ on the set $\{|\upsilon_n - \psi^+| > k\}$, then we have

$$\begin{aligned}\frac{\beta}{k}\sum_{i=1}^{N}\int_{\{|\upsilon_n-\psi^+|>k\}} w_i(x)\left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx &\leq \frac{1}{k}\int_{\{|\upsilon_n-\psi^+|>k\}} |\Psi(x,\upsilon_n,\nabla\upsilon_n)|\, dx\\ &= \frac{1}{k^2}\int_{\{|\upsilon_n-\psi^+|>k\}} \Psi(x,\upsilon_n,\nabla\upsilon_n)\, T_k\left(\upsilon_n-\psi^+\right) dx\\ &\leq \int_\Omega \Psi(x,\upsilon_n,\nabla\upsilon_n)\, T_k\left(\upsilon_n-\psi^+\right) dx.\end{aligned}$$

Hence, we have

$$\int_\Omega a(x,\upsilon_n,\nabla\upsilon_n)\cdot\nabla T_k\left(\upsilon_n-\psi^+\right) dx \leq C_k + \frac{\beta}{k}\sum_{i=1}^{N}\int_{\{|\upsilon_n-\psi^+|\leq k\}} w_i(x)\left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx.$$

This implies that

$$\begin{aligned}\int_{\{|\upsilon_n-\psi^+|\leq k\}} a(x,\upsilon_n,\nabla\upsilon_n)\cdot\nabla\upsilon_n dx \leq & C_k + \frac{\beta}{k}\sum_{i=1}^{N}\int_{\{|\upsilon_n-\psi^+|\leq k\}} w_i(x)\left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx\\ &+ \int_{\{|\upsilon_n-\psi^+|\leq k\}} \left|a(x,\upsilon_n,\nabla\upsilon_n)\cdot\nabla\psi^+\right| dx.\end{aligned}$$

By using Young's inequality we obtain for a positive constant λ

$$\begin{aligned}\int_{\{|\upsilon_n-\psi^+|\leq k\}} a(x,\upsilon_n,\nabla\upsilon_n)\cdot\nabla\upsilon_n dx \leq & C_k + \frac{\beta}{k}\sum_{i=1}^{N}\int_{\{|\upsilon_n-\psi^+|\leq k\}} w_i(x)\left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx\\ &+ \sum_{i=1}^{N}\int_{\{|\upsilon_n-\psi^+|\leq k\}} \frac{\lambda^{p_i'}}{p_i'}\left|a_i(x,\upsilon_n,\nabla\upsilon_n)\right|^{p_i'} w_i^{1-p_i'}(x) dx\\ &+ \sum_{i=1}^{N}\int_{\{|\upsilon_n-\psi^+|\leq k\}} \frac{1}{p\lambda^{p_i}} w_i(x)\left|\frac{\partial \psi^+}{\partial x_i}\right|^{p_i} dx.\end{aligned}$$

By virtue of (3.4), we get

$$\begin{aligned}&\int_{\{|\upsilon_n-\psi^+|\leq k\}} a(x,\upsilon_n,\nabla\upsilon_n)\cdot\nabla\upsilon_n dx\\ \leq & C_k + \frac{\beta}{k}\sum_{i=1}^{N}\int_{\{|\upsilon_n-\psi^+|\leq k\}} w_i(x)\left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx\\ &+ \frac{\lambda^{p_i'}}{p_i'}\alpha_1^{p_i'}N\int_\Omega \delta^{p_i'}(x)dx + \frac{\lambda^{p_i'}}{p_i'}\alpha_1^{p_i'}N\int_{\{|\upsilon_n-\psi^+|\leq k\}}\sigma(x)|\upsilon_n|^{q_i} dx\\ &+ \frac{\lambda^{p_i'}}{p_i'}\alpha_1^{p_i'}N\sum_{i=1}^{N}\int_{\{|\upsilon_n-\psi^+|\leq k\}} w_i(x)\left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx\\ \leq & C_k + \frac{\beta}{k}\sum_{i=1}^{N}\int_{\{|\upsilon_n-\psi^+|\leq k\}} w_i(x)\left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx\\ &+ \frac{\lambda^{p_i'}}{p_i'}\alpha_1^{p_i'}N\int_{\left\{|\upsilon_n|\leq k+\|\psi^+\|_\infty\right\}}\sigma(x)|\upsilon_n|^{q_i} dx\\ &+ \frac{\lambda^{p_i'}}{p_i'}\alpha_1^{p_i'}N\sum_{i=1}^{N}\int_{\{|\upsilon_n-\psi^+|\leq k\}} w_i(x)\left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx\\ \leq & C_k + \left(\frac{\beta}{k} + \frac{\lambda^{p_i'}}{p_i'}\alpha_1^{p_i'}N\right)\sum_{i=1}^{N}\int_{\{|\upsilon_n-\psi^+|\leq k\}} w_i(x)\left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx.\end{aligned}$$

Using the coercivity condition (3.6) we obtain

$$\begin{aligned}&\alpha_2 \sum_{i=1}^{N} \int_{\{|\upsilon_n-\psi^+|\leq k\}} w_i(x) \left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx \\ &\leq C_k + \left(\frac{\beta}{k} + \frac{\lambda^{p_i'}}{p_i'}\alpha_1^{p_i'} N\right) \sum_{i=1}^{N} \int_{\{|\upsilon_n-\psi^+|\leq k\}} w_i(x) \left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx.\end{aligned}$$

Choosing $\lambda > 0$ small enough such that $\alpha_2 > \frac{\beta}{k} + \frac{\lambda^{p_i'}}{p_i'}\alpha_1^{p_i'} N$ for $k > \frac{\beta}{\alpha_2}$, then

$$\sum_{i=1}^{N} \int_{\{|\upsilon_n-\psi+|\leq k\}} w_i(x) \left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx \leq C_1. \tag{4.4}$$

On the other hand, from (4.3) we have

$$\begin{aligned}\int_\Omega \Psi\left(x, \upsilon_n, \nabla \upsilon_n\right) T_k\left(\upsilon_n - \psi^+\right) dx \leq & kC_f + k \int_\Omega \left|\Phi_n\left(x, \nabla \upsilon_n\right)\right| dx \\ & - \int_{\{|\upsilon_n-\psi^+|\leq k\}} a\left(x, \upsilon_n, \nabla \upsilon_n\right) \cdot \nabla\left(\upsilon_n - \psi^+\right) dx\end{aligned}$$

which implies by using (3.12), (3.6) and Young's inequality, that

$$\begin{aligned}&\int_\Omega \Psi\left(x, \upsilon_n, \nabla \upsilon_n\right) T_k\left(\upsilon_n - \psi^+\right) dx \\ &\leq kC_f + k\sum_{i=1}^{N} \int_\Omega \Phi(x) w_i^{\frac{1}{p_i'}}(x) \left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i-1} dx + \int_{\{|\upsilon_n-\psi^+|\leq k\}} a\left(x, \upsilon_n, \nabla \upsilon_n\right) \cdot \nabla \psi^+ dx \\ &- \int_{\{|\upsilon_n-\psi^+|\leq k\}} a\left(x, \upsilon_n, \nabla \upsilon_n\right) \cdot \nabla \upsilon_n dx \\ &\leq kC_f + C(k, p_i, N, \beta, \lambda) \int_\Omega |\Phi(x)|^{p_i} dx + \lambda\beta \sum_{i=1}^{N} \int_\Omega w_i(x) \left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx \\ &+ \int_{\{|\upsilon_n-\psi^+|\leq k\}} \left|a\left(x, \upsilon_n, \nabla \upsilon_n\right) \cdot \nabla \psi^+\right| dx.\end{aligned} \tag{4.5}$$

In view of (4.4), the last term of the right-hand side of (4.5) is uniformly bounded in n, then

$$\begin{aligned}\int_\Omega \Psi\left(x, \upsilon_n, \nabla \upsilon_n\right) T_k\left(\upsilon_n - \psi^+\right) dx \leq & C_k + \lambda\beta \sum_{i=1}^{N} \int_{\{|\upsilon_n-\psi^+|\leq k\}} w_i(x) \left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx \\ & + \lambda\beta \sum_{i=1}^{N} \int_{\{|\upsilon_n-\psi^+|> k\}} w_i(x) \left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx.\end{aligned}$$

By using (3.11) we have for $k > \kappa + \|\psi^+\|_\infty$

$$\begin{aligned}k\beta \sum_{i=1}^{N} \int_{\{|\upsilon_n-\psi^+|> k\}} w_i(x) \left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx &\leq k \int_\Omega \left|\Psi\left(x, \upsilon_n, \nabla \upsilon_n\right)\right| dx \\ &\leq \int_\Omega \Psi\left(x, \upsilon_n, \nabla \upsilon_n\right) T_k\left(\upsilon_n - \psi^+\right) dx.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}(k-\lambda)\beta \sum_{i=1}^{N} \int_{\{|\upsilon_n-\psi^+|> k\}} w_i(x) \left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx &\leq C_k + \lambda\beta \sum_{i=1}^{N} \int_{\{|\upsilon_n-\psi^+|\leq k\}} w_i(x) \left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i} dx \\ &\leq C_k + \lambda\beta C_1.\end{aligned}$$

Consequently

$$|||\upsilon_n|||_X \le C \tag{4.6}$$

where C is a constant not depending on n. The boundedness of the sequence (υ_n) in X with (3.3) implies the existence of a function u in $W_0^{1,p_i}(\Omega, w)$ and a therefore subsequence, still denoted by (υ_n), such that

$$\upsilon_n \rightharpoonup \upsilon \text{ weakly in } W_0^{1,p_i}(\Omega, w), \text{ strongly in } L^{q_i}(\Omega, \sigma) \text{ and a.e. in } \Omega. \tag{4.7}$$

4.2. Step 2: Almost everywhere convergence of the gradients.

For $k \ge \|\psi^+\|$and $\theta \ge \left(\frac{b(k)}{2\alpha_2}\right)^2$ let $\varphi(s) = se^{\theta s^2}$. It is well known that for all $s \in \mathbb{R}$ one has

$$\varphi'(s) - \frac{b(k)}{\alpha_2}|\varphi(s)| \ge \frac{1}{2} \tag{4.8}$$

where k is a fixed real number which will be used as a level of truncation. Let $\eta = e^{-4\theta k^2}$ and $z_n = T_k(\upsilon_n) - T_k(\upsilon)$. Using $v_n = \upsilon_n - \eta\varphi(z_n)$ as a test function in (4.1) to get

$$\begin{aligned}&\langle A(\upsilon_n), T_{3k}(\eta\varphi(z_n))\rangle + \int_\Omega \Psi(x, \upsilon_n, \nabla\upsilon_n) T_{3k}(\eta\varphi(z_n))\,dx + \int_\Omega \Phi_n(x, \nabla\upsilon_n) T_{3k}(\eta\varphi(z_n))\,dx \\ &\le \int_\Omega f_n T_{3k}(\eta\varphi(z_n))\,dx.\end{aligned}$$

We have $|\eta\varphi(z_n)| \le |z_n| \le 2k < 3k$, so

$$\begin{aligned}&\langle A(\upsilon_n), \varphi(z_n)\rangle + \int_\Omega \Psi(x, \upsilon_n, \nabla\upsilon_n)\varphi(z_n)\,dx + \int_\Omega \Phi_n(x, \nabla\upsilon_n)\varphi(z_n)\,dx \\ &\le \int_\Omega f_n\varphi(z_n)\,dx.\end{aligned} \tag{4.9}$$

Since $\varphi(z_n) \rightharpoonup 0$ weakly in $L^\infty(\Omega)$ and $f_n \to f$ strongly in $L^1(\Omega)$, the right-hand side of (4.9) converges to zero when n tends to ∞. As for the last term of the left-hand side of (4.9) we have

$$\begin{aligned}\int_\Omega |\Phi_n(x, \nabla\upsilon_n)\varphi(z_n)|\,dx &\le \int_\Omega |\Phi(x)\varphi(z_n)| \sum_{i=1}^N w_i^{\frac{1}{p_i'}}(x)\left|\frac{\partial\upsilon_n}{\partial x_i}\right|^{p_i-1} dx \\ &\le \|h\varphi(z_n)\|_{L^{p_i}(\Omega)} |||\,\upsilon_n\|_X^{p_i-1}.\end{aligned}$$

The Lebesgue dominated convergence theorem easily implies

$$\Phi(x)\varphi(z_n) \to 0 \text{ strongly in } L^{p_i}(\Omega)$$

and from (4.6) we conclude that

$$\int_\Omega |\Phi_n(x, \nabla\upsilon_n)\varphi(z_n)|\,dx \to 0, \text{ as } n \to \infty.$$

By using the fact that $\Psi(x, \upsilon_n, \nabla\upsilon_n)\varphi(z_n) \ge 0$ on the subset $\{x \in \Omega : |\upsilon_n| > k\}$, we can write (4.9) as follows

$$\langle A(\upsilon_n), \varphi(z_n)\rangle + \int_{\{|\upsilon_n| \le k\}} \Psi(x, \upsilon_n, \nabla\upsilon_n)\varphi(z_n)\,dx \le \varepsilon_1(n) \tag{4.10}$$

where $\varepsilon_i(n), (i = 1, 2, \ldots)$, denote various sequences of real numbers which converge to zero when n tends to ∞. On the one hand

$$\begin{aligned}\langle A(\upsilon_n), \varphi(z_n)\rangle =& \int_{\{|\upsilon_n|\le k\}} a(x,\upsilon_n,\nabla\upsilon_n)\cdot(\nabla T_k(\upsilon_n)-\nabla T_k(\upsilon))\varphi'(z_n)\,dx\\ &+\int_{\{|\upsilon_n|>k\}} a(x,\upsilon_n,\nabla\upsilon_n)\cdot(\nabla T_k(\upsilon_n)-\nabla T_k(\upsilon))\varphi'(z_n)\,dx\\ =&\int_\Omega a(x,T_k(\upsilon_n),\nabla T_k(\upsilon_n))\cdot(\nabla T_k(\upsilon_n)-\nabla T_k(\upsilon))\varphi'(z_n)\,dx\\ &-\int_{\{|\upsilon_n|>k\}} a(x,\upsilon_n,\nabla\upsilon_n)\cdot\nabla T_k(\upsilon)\varphi'(z_n)\,dx\\ =&\int_\Omega[a(x,T_k(\upsilon_n),\nabla T_k(\upsilon_n))-a(x,T_k(\upsilon_n),\nabla T_k(\upsilon))]\\ &\cdot[\nabla T_k(\upsilon_n)-\nabla T_k(\upsilon)]\varphi'(z_n)\,dx\\ &+\int_\Omega a(x,T_k(\upsilon_n),\nabla T_k(\upsilon))\cdot(\nabla T_k(\upsilon_n)-\nabla T_k(\upsilon))\varphi'(z_n)\,dx\\ &-\int_{\{|\upsilon_n|>k\}} a(x,\upsilon_n,\nabla\upsilon_n)\cdot\nabla T_k(\upsilon)\varphi'(z_n)\,dx.\end{aligned}$$

Concerning the second term of the right-hand side of the above equality, by using (4.7) and Lemma 2.4, we have $T_k(\upsilon_n)\rightharpoonup T_k(\upsilon)$ weakly in $W_0^{1,p_i}(\Omega,w)$. Thus

$$\nabla T_k(\upsilon_n)\rightharpoonup\nabla T_k(\upsilon)\text{ weakly in }\Pi_{i=1}^N L^{p_i}(\Omega,w_i). \tag{4.11}$$

Moreover, by the growth condition (3.4) we can show the equi-integrability of the sequence $(a(x,T_k(\upsilon_n),\nabla T_k(\upsilon))\varphi'(z_n))_n$. Then, from (4.7) we have

$$a(x,T_k(\upsilon_n),\nabla T_k(\upsilon))\varphi'(z_n)\to a(x,T_k(\upsilon),\nabla T_k(\upsilon))\text{ a.e. in }\Omega.$$

Therefore by using Vitali's theorem one has

$$a(x,T_k(\upsilon_n),\nabla T_k(\upsilon))\varphi'(z_n)\to a(x,T_k(\upsilon),\nabla T_k(\upsilon))\text{ strongly in }\Pi_{i=1}^N L^{p_i'}(\Omega,w_i^*). \tag{4.12}$$

Since (4.11) and (4.12) we deduce that

$$\int_\Omega a(x,T_k(\upsilon_n),\nabla T_k(\upsilon))\cdot(\nabla T_k(\upsilon_n)-\nabla T_k(\upsilon))\varphi'(z_n)\,dx\to 0\text{ when }n\to\infty.$$

Thanks to (3.2), (4.6) and (4.7), the sequence $(a(x,\upsilon_n,\nabla\upsilon_n)\varphi'(z_n))_n$ is bounded in $\Pi_{i=1}^N L^{p_i'}(\Omega,w_i^*)$. Then, since $\nabla T_k(\upsilon)\chi_{\{|\upsilon_n|>k\}}\to 0$ strongly in $\Pi_{i=1}^N L^{p_i}(\Omega,w_i)$, we have

$$\int_{\{|\upsilon_n|>k\}} a(x,\upsilon_n,\nabla\upsilon_n)\cdot\nabla T_k(\upsilon)\varphi'(z_n)\,dx\to 0\text{ when }n\to\infty.$$

Consequently we can write

$$\begin{aligned}\langle A(\upsilon_n),\varphi(z_n)\rangle =& \int_\Omega[a(x,T_k(\upsilon_n),\nabla T_k(\upsilon_n))-a(x,T_k(\upsilon_n),\nabla T_k(\upsilon))]\\ &\cdot[\nabla T_k(\upsilon_n)-\nabla T_k(\upsilon)]\varphi'(z_n)\,dx+\varepsilon_2(n).\end{aligned} \tag{4.13}$$

By using (3.6) and (3.10), the second term in the left hand-side of (4.10), yields

$$
\begin{aligned}
&\left|\int_{\{|\upsilon_n|\leq k\}} \Psi\left(x, \upsilon_n, \nabla \upsilon_n\right) \varphi\left(z_n\right) dx\right| \\
&\leq \int_{\{|\upsilon_n|\leq k\}} b(k)\left(\sum_{i=1}^N w_i(x)\left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i}+|c(x)|\right)\left|\varphi\left(z_n\right)\right| dx \\
&\leq b(k) \int_\Omega |c(x)|\left|\varphi\left(z_n\right)\right| dx+b(k) \int_{\{|\upsilon_n|\leq k\}} \sum_{i=1}^N w_i(x)\left|\frac{\partial \upsilon_n}{\partial x_i}\right|^{p_i}\left|\varphi\left(z_n\right)\right| dx \\
&\leq \varepsilon_3(n)+b(k) \int_\Omega \sum_{i=1}^N w_i(x)\left|\frac{\partial T_k\left(\upsilon_n\right)}{\partial x_i}\right|^{p_i}\left|\varphi\left(z_n\right)\right| dx \\
&\leq \varepsilon_3(n)+\frac{b(k)}{\alpha_2} \int_\Omega a\left(x, T_k\left(\upsilon_n\right), \nabla T_k\left(\upsilon_n\right)\right) \cdot \nabla T_k\left(\upsilon_n\right)\left|\varphi\left(z_n\right)\right| dx \\
&=\varepsilon_3(n)+\frac{b(k)}{\alpha_2} \int_\Omega\left[a\left(x, T_k\left(\upsilon_n\right), \nabla T_k\left(\upsilon_n\right)\right)-a\left(x, T_k\left(\upsilon_n\right), \nabla T_k(\upsilon)\right)\right] \\
&\cdot\left[\nabla T_k\left(\upsilon_n\right)-\nabla T_k(\upsilon)\right]\left|\varphi\left(z_n\right)\right| dx \\
&+\frac{b(k)}{\alpha_2} \int_\Omega a\left(x, T_k\left(\upsilon_n\right), \nabla T_k\left(\upsilon_n\right)\right) \cdot \nabla T_k(\upsilon)\left|\varphi\left(z_n\right)\right| dx \\
&+\frac{b(k)}{\alpha_2} \int_\Omega a\left(x, T_k\left(\upsilon_n\right), \nabla T_k(\upsilon)\right) \cdot\left(\nabla T_k\left(\upsilon_n\right)-\nabla T_k(\upsilon)\right)\left|\varphi\left(z_n\right)\right| dx .
\end{aligned}
$$

For the last term we have

$$
\begin{aligned}
&\left|\int_\Omega a\left(x, T_k\left(\upsilon_n\right), \nabla T_k(\upsilon)\right) \cdot\left(\nabla T_k\left(\upsilon_n\right)-\nabla T_k(\upsilon)\right)\right| \varphi\left(z_n\right)|dx| \\
&\quad \leq \varphi(2k) \int_\Omega\left|a\left(x, T_k\left(\upsilon_n\right), \nabla T_k(\upsilon)\right)\right|\left|\nabla T_k\left(\upsilon_n\right)-\nabla T_k(\upsilon)\right| dx .
\end{aligned}
$$

As in (4.12), we use (3.4), (4.7) and Vitali's theorem to obtain

$$
a\left(x, T_k\left(\upsilon_n\right), \nabla T_k(\upsilon)\right) \rightarrow a\left(x, T_k(\upsilon), \nabla T_k(\upsilon)\right) \text{ strongly in } \Pi_{i=1}^N L^{p_i'}\left(\Omega, w_i^*\right).
$$

By using (4.11) we get

$$
\int_\Omega a\left(x, T_k\left(\upsilon_n\right), \nabla T_k(\upsilon)\right) \cdot\left(\nabla T_k\left(\upsilon_n\right)-\nabla T_k(\upsilon)\right)\left|\varphi\left(z_n\right)\right| dx \rightarrow 0 \text{ as } n \rightarrow 0 .
$$

In addition, the sequence $\left(a\left(x, T_k\left(\upsilon_n\right), \nabla T_k\left(\upsilon_n\right)\right)\right)_n$ converges weakly in $\Pi_{i=1}^N L^{p_i'}\left(\Omega, w_i^*\right)$ to a function l_k thanks to (3.4) and (4.6). Thus

$$
\begin{aligned}
&\int_\Omega a\left(x, T_k\left(\upsilon_n\right), \nabla T_k\left(\upsilon_n\right)\right) \cdot \nabla T_k(\upsilon)\left|\varphi\left(z_n\right)\right| dx \\
&\quad=\int_\Omega\left(a\left(x, T_k\left(\upsilon_n\right), \nabla T_k\left(\upsilon_n\right)\right)-l_k\right) \cdot \nabla T_k(\upsilon)\left|\varphi\left(z_n\right)\right| dx+\int_\Omega l_k \cdot \nabla T_k(\upsilon)\left|\varphi\left(z_n\right)\right| dx .
\end{aligned}
$$

By using $\varphi\left(z_n\right) \rightarrow 0$ weakly in $L^{\infty}(\Omega)$, we deduce that

$$
\int_\Omega a\left(x, T_k\left(\upsilon_n\right), \nabla T_k\left(\upsilon_n\right)\right) \cdot \nabla T_k(\upsilon)\left|\varphi\left(z_n\right)\right| dx \rightarrow 0 \text{ as } n \rightarrow \infty .
$$

$$
\begin{aligned}
&\left|\int_{\{|\upsilon_n|\leq k\}} \Psi\left(x, \upsilon_n, \nabla \upsilon_n\right) \varphi\left(z_n\right) dx\right| \\
&\quad \leq \varepsilon_4(n)+\frac{b(k)}{\alpha_2} \int_\Omega\left[a\left(x, T_k\left(\upsilon_n\right), \nabla T_k\left(\upsilon_n\right)\right)-a\left(x, T_k\left(\upsilon_n\right), \nabla T_k(\upsilon)\right)\right] \\
&\quad \cdot\left[\nabla T_k\left(\upsilon_n\right)-\nabla T_k(\upsilon)\right]\left|\varphi\left(z_n\right)\right| dx
\end{aligned}
$$

which with (4.10) and (4.13) give

$$\int_\Omega \left[a\left(x, T_k\left(\upsilon_n\right), \nabla T_k\left(\upsilon_n\right)\right)-a\left(x, T_k\left(\upsilon_n\right), \nabla T_k(\upsilon)\right)\right] \cdot\left[\nabla T_k\left(\upsilon_n\right)-\nabla T_k(\upsilon)\right] \left(\varphi^{\prime}\left(z_n\right)-\frac{b(k)}{\alpha_2}\left|\varphi\left(z_n\right)\right|\right) d x \leq \varepsilon_5(n).$$

Using (3.5) and (4.8)to obtain

$$0 \leq \frac{1}{2} \int_\Omega\left[a\left(x, T_k\left(\upsilon_n\right), \nabla T_k\left(\upsilon_n\right)\right)-a\left(x, T_k\left(\upsilon_n\right), \nabla T_k(\upsilon)\right)\right] \cdot\left[\nabla T_k\left(\upsilon_n\right)-\nabla T_k(\upsilon)\right] d x \leq \varepsilon_5(n).$$

Then

$$\int_\Omega\left[a\left(x, T_k\left(\upsilon_n\right), \nabla T_k\left(\upsilon_n\right)\right)-a\left(x, T_k\left(\upsilon_n\right), \nabla T_k(\upsilon)\right)\right] \cdot\left[\nabla T_k\left(\upsilon_n\right)-\nabla T_k(\upsilon)\right] d x \rightarrow 0 \text{ as } n \rightarrow \infty.$$

By virtue of Lemma 3.1 and the fact that $T_k\left(\upsilon_n\right) \rightharpoonup T_k(\upsilon)$ weakly in $W_0^{1,p_i}(\Omega, w)$, we conclude that

$$T_k\left(\upsilon_n\right) \rightarrow T_k(\upsilon) \text{ strongly in } W_0^{1,p_i}(\Omega, w) \text{ for any fixed } k \geq\left\|\psi^{+}\right\|. \tag{4.14}$$

So that

$$\nabla T_k\left(\upsilon_n\right) \rightarrow \nabla T_k(\upsilon) \text{ strongly in } \prod_{i=1}^N L_i^{p_i}\left(\Omega, w_i\right).$$

As in [11], we deduce that there exists a subsequence still denoted by υ_n such that

$$\nabla \upsilon_n \rightarrow \nabla \upsilon \text{ a.e. in } \Omega. \tag{4.15}$$

4.3. Step 3: Equi-integrability of the non-linearities $\Psi\left(x, \upsilon_n, \nabla \upsilon_n\right)+\Phi_n\left(x, \nabla \upsilon_n\right)$.

By using Vitali's theorem we will show that

$$\Psi\left(x, \upsilon_n, \nabla \upsilon_n\right)+\Phi_n\left(x, \nabla \upsilon_n\right) \rightarrow \Psi(x, \upsilon, \nabla \upsilon)+\Phi(x, \nabla \upsilon) \text{ strongly in } L^1(\Omega). \tag{4.16}$$

Thanks to (4.7) and (4.15) we have

$$\Psi\left(x, \upsilon_n, \nabla \upsilon_n\right)+\Phi_n\left(x, \nabla \upsilon_n\right) \rightarrow \Psi(x, \upsilon, \nabla \upsilon)+\Phi(x, \nabla \upsilon) \text{ a.e. in } \Omega,$$

so it suffices to prove that $\Psi\left(x, \upsilon_n, \nabla \upsilon_n\right)+\Phi_n\left(x, \nabla \upsilon_n\right)$ is uniformly equi-integrable in Ω. For any measurable subset E of Ω and any $m>0$ we have

$$\begin{aligned}
& \int_E\left|\Psi\left(x, \upsilon_n, \nabla \upsilon_n\right)+\Phi_n\left(x, \nabla \upsilon_n\right)\right| d x \\
& =\int_{E \cap\left\{\left|\upsilon_n\right| \leq m\right\}}\left|\Psi\left(x, \upsilon_n, \nabla \upsilon_n\right)+\Phi_n\left(x, \nabla \upsilon_n\right)\right| d x \\
& \quad+\int_{E \cap\left\{\left|\upsilon_n\right|>m\right\}}\left|\Psi\left(x, \upsilon_n, \nabla \upsilon_n\right)+\Phi_n\left(x, \nabla \upsilon_n\right)\right| d x \\
& \leq \int_E b(m)\left(c(x)+\sum_{i=1}^N w_i(x)\left|\frac{\partial T_m\left(\upsilon_n\right)}{\partial x_i}\right|^{p_i}\right) d x \\
& \quad+\left(\int_E h_i^{p_i}(x) d x\right)^{\frac{1}{p_i}} \sum_{i=1}^N\left(\int_E w_i(x)\left|\frac{\partial T_m\left(\upsilon_n\right)}{\partial x_i}\right|^{p_i} d x\right)^{\frac{1}{p_i^{\prime}}} \\
& \quad+\int_{E \cap\left\{\left|\upsilon_n\right|>m\right\}}\left|\Psi\left(x, \upsilon_n, \nabla \upsilon_n\right)+\Phi_n\left(x, \nabla \upsilon_n\right)\right| d x .
\end{aligned} \tag{4.17}$$

In view of (4.14) for any $\varepsilon>0$ there exists $\mu(\varepsilon,m)>0$ such that for all E satisfying $|E|<\mu(\varepsilon,m)$ we have

$$\begin{aligned}&\int_E b(m)\left(c(x)+\sum_{i=1}^N w_i(x)\left|\frac{\partial T_m(\upsilon_n)}{\partial x_i}\right|^{p_i}\right)dx\\&+\left(\int_E h_i^{p_i}(x)dx\right)^{\frac{1}{p_i}}\sum_{i=1}^N\left(\int_E w_i(x)\left|\frac{\partial T_m(\upsilon_n)}{\partial x_i}\right|_i^{p_i}dx\right)^{\frac{1}{p_i'}}<\frac{\varepsilon}{2}\quad\forall n.\end{aligned}\tag{4.18}$$

Now, for $m\geq 2+\|\psi^+\|_\infty$, we define a function ϕ_m satisfying for all $s\in\mathbb{R}$

$$\begin{cases}\phi_m(s)=0 & \text{if } |s|\leq m-1,\\ \phi_m'(s)=1 & \text{if } m-1\leq|s|\leq m,\\ \phi_m(s)=\frac{s}{|s|} & \text{if } |s|\geq m.\end{cases}$$

Note that $\upsilon_n-\phi_m(\upsilon_n)\in K_\psi$, then by using it as test function in (4.2) we get

$$\langle A(\upsilon_n),T_k(\phi_m(\upsilon_n))\rangle+\int_\Omega(\Psi(x,\upsilon_n,\nabla\upsilon_n)+\Phi_n(x,\nabla\upsilon_n))T_k(\phi_m(\upsilon_n))\,dx$$

$$\leq\int_\Omega f_nT_k(\phi_m(\upsilon_n))\,dx$$

which implies, for $k\geq 1$

$$\begin{aligned}&\int_\Omega a(x,\upsilon_n,\nabla\upsilon_n)\cdot\nabla\upsilon_n\phi_m'(\upsilon_n)\,dx+\int_\Omega(\Psi(x,\upsilon_n,\nabla\upsilon_n)+\Phi_n(x,\nabla\upsilon_n))\,\phi_m(\upsilon_n)\,dx\\&\leq\int_\Omega f_n\phi_m(\upsilon_n)\,dx.\end{aligned}$$

Since (3.6) and by using the fact that $\phi_m(\upsilon_n)$ and υ_n have the same sign we conclude that

$$\int_{\{|\upsilon_n|>m\}}|\Psi(x,\upsilon_n,\nabla\upsilon_n)|\,dx\leq\int_{\{|\upsilon_n|>m-1\}}|\Phi_n(x,\nabla\upsilon_n)|\,dx+\int_{\{|\upsilon_n|>m-1\}}|f_n|\,dx.$$

The right-hand side of the above inequality converges to 0 uniformly in n when m tends to ∞ by using (3.12), Hölder's inequality, $f_n\to f$ strongly in $L^1(\Omega)$ and the fact that $|\{|\upsilon_n|>m\}|\to 0$ uniformly in n when $m\to\infty$. Hence there exists $m(\varepsilon)>1$ such that

$$\int_{\{|\upsilon_n|>m\}}|\Psi(x,\upsilon_n,\nabla\upsilon_n)|\,dx\leq\frac{\varepsilon}{2}\quad\forall n.\tag{4.19}$$

Finally from (4.17), (4.18) and (4.19) we have

$$\int_E|\Psi(x,\upsilon_n,\nabla\upsilon_n)+\Phi_n(x,\nabla\upsilon_n)|\,dx<\varepsilon\quad\forall n,$$

if $|E|<\mu(\varepsilon)$ for some $\mu(\varepsilon)>0$, which gives the uniform equi-integrability in Ω of $\Psi(x,\upsilon_n,\nabla\upsilon_n)+\Phi_n(x,\nabla\upsilon_n)$.

4.4. Step 4: Passage to the limit.

We can write (4.2) as follows

$$\begin{aligned}&\int_\Omega a(x,\upsilon_n,\nabla\upsilon_n)\cdot\nabla T_k(\upsilon_n-u)\,dx+\int_\Omega(\Psi(x,\upsilon_n,\nabla\upsilon_n)+\Phi_n(x,\nabla\upsilon_n))\,T_k(\upsilon_n-u)\,dx\\&\leq\int_\Omega f_nT_k(\upsilon_n-u)\,dx.\end{aligned}\tag{4.20}$$

for all $u \in K_\psi \cap L^\infty(\Omega)$ and all $k > 0$. From (3.4) and (4.6) we have $a\left(x, \upsilon_n, \nabla \upsilon_n\right)$ is bounded in $\Pi_{i=1}^N L^{p_i'}\left(\Omega, w_i^*\right)$ and since (4.7) and (4.15) we have $a\left(x, \upsilon_n, \nabla \upsilon_n\right) \rightarrow a(x, \upsilon, \nabla \upsilon)$ a.e. in Ω. Therefore by Lemma 2.3 we obtain

$$a\left(x, \upsilon_n, \nabla \upsilon_n\right) \rightarrow a(x, \upsilon, \nabla \upsilon) \text{ weakly in } \Pi_{i=1}^N L^{p_i'}\left(\Omega, w_i^*\right).$$

Let E be a measurable subset in Ω and for $i = 1, \ldots, N$ we have

$$\begin{aligned}
\int_E w_i^{\frac{1}{p_i}}(x)\left|\frac{\partial T_k\left(\upsilon_n-u\right)}{\partial x_i}\right| dx &= \int_E w_i^{\frac{1}{p_i}}(x)\left|\frac{\partial\left(\upsilon_n-u\right)}{\partial x_i}\right| \chi_{\{|\upsilon_n-u| \leq k\}} dx \\
&\leq \int_E w_i^{\frac{1}{p_i}}(x)\left(\left|\frac{\partial \upsilon_n}{\partial x_i}\right|+\left|\frac{\partial u}{\partial x_i}\right|\right) \chi_{\{|\upsilon_n| \leq k+\|u\|_\infty\}} dx \\
&\leq \int_E w_i^{\frac{1}{p_i}}(x)\left|\frac{\partial u}{\partial x_i}\right| dx+\int_E w_i^{\frac{1}{p_i}}(x)\left|\frac{\partial T_{k+\|u\|_\infty}\left(\upsilon_n\right)}{\partial x_i}\right| dx .
\end{aligned}$$

By using (4.7), (4.14) and Vitali's theorem we get $\nabla T_k\left(\upsilon_n-u\right) \rightarrow \nabla T_k(\upsilon-u)$ strongly in $\Pi_{i=1}^N L^{p_i}\left(\Omega, w_i\right)$. So that

$$\int_\Omega a\left(x, \upsilon_n, \nabla \upsilon_n\right) \cdot \nabla T_k\left(\upsilon_n-u\right) dx \rightarrow \int_\Omega a(x, \upsilon, \nabla \upsilon) \cdot \nabla T_k(\upsilon-u) dx \text{ as } n \rightarrow \infty.$$

Finally we use (4.16) and the fact that $f_n \rightarrow f$ strongly in $L^1(\Omega)$ for passing to the limit in (4.20) and this complete the proof of Theorem (3.1).

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