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Laplacian Minimum Independent Dominating Energy of Graphs

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ABSTRACT: If every vertex $v \in V - S$ is adjacent to at least one vertex in S, then the set $S \subset V$ is a dominating set. An independent set of G is a set $S \subset V(G)$ in which no two of its vertices are adjacent. An independent dominating set is a dominating set that is also an independent set. A minimum independent dominating set is an independent dominating set with the least amount of cardinality. The Laplacian minimum independent dominating energy of a graph, denoted as $LE_{D_i}(G)$, is introduced in this article, along with methods for calculating it for various types of graphs. Additionally, the upper and lower bounds of $LE_{D_i}(G)$ are established.

Key Words: Dominating set, independent dominating set, minimum independent dominating energy, Laplacian minimum independent dominating energy.

Contents

1. Introduction

Let $G = (V, E)$ be a graph with n vertices and m edges, and let $A = (a_{i,j})$ denote its adjacency matrix. The eigenvalues of the graph G are defined as the eigenvalues of its adjacency matrix $A(G)$, and are denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. A graph G is said to be singular if at least one of its eigenvalues is zero, which implies that $\det(A) = 0$. Conversely, if none of the eigenvalues are zero, the graph is called nonsingular, in which case $\det(A) > 0$.

A graph G is called k-regular if every vertex in G has degree k. A bull graph is a planar and undirected graph consisting of five vertices and five edges, formed by a triangle with two disjoint pendant edges attached to two of its vertices. A complete graph, denoted by K_n, is a simple graph in which every pair of distinct vertices is connected by an edge. A star graph, denoted by $K_{1,n-1}$, also referred to as a claw or cherry, is a tree with a single central vertex joined to all other vertices.

A subset $D \subseteq V$ is called a dominating set of G if every vertex in $V \setminus D$ is adjacent to at least one vertex in D. A dominating set D is called minimal if no proper subset of D is also a dominating set. The smallest possible size of a minimal dominating set in G is termed the domination number, denoted by $\gamma(G)$.

Let $G = (V, E)$ be a graph. A subset $S \subseteq V(G)$ is called an independent dominating set if for every vertex $v \in V \setminus S$, there exists a vertex $u \in S$ such that the distance between u and v in G, denoted by $d_G(u, v)$, is equal to 2. The minimum cardinality of such a set is called the independent domination number of G, denoted by $\gamma_i(G)$. For a more detailed treatment of independent domination, see [4]. In this work, we initiate a study on the Laplacian minimum independent domination energy of a graph.

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The notion of graph energy was introduced by I. Gutman [7]. Although it initially received limited attention, it has since gained widespread interest and has become a topic of active research. Over time, analogous energy concepts have also been developed for matrices other than the adjacency matrix. The energy $E(G)$ of a graph G is defined as the sum of the absolute values of its eigenvalues:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

I.Gutman and B.Zhou [8] defined the Laplacian energy of a graph G in the year 2006. Let G be a graph with n vertices and m edges. The Laplacian matrix of the graph G, denoted by $L = (L_{ij})$, is a square matrix of order n. The elements of the Laplacian matrix are defined as

$$L_{ij} = \begin{cases} -1, & \text{if } \ v_i \, and \, v_j \, are \, adjacent, \\ 0, & \text{if } \ v_i \, and \, v_j \, are \, not \, adjacent, \\ d_i, & \text{if i=j.} \end{cases}$$

where d_i is the degree of the vertex v_i.
Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of Laplacian matrix G. Laplacian energy of G is defined as

$$LE(G) = \sum_{i=1}^{n} \left| \lambda_i - \frac{2m}{n} \right|.$$

The basic properties Laplacian energy including various upper and lower bounds have been established in [13], [14], [16], [17] and it has found that remarkable chemical application, high resolution satellite image classification and segmentation using Laplacian graph energy and finding semantic structures in image hierarchies using Laplacian graph energy. The Laplacian energy provides a measure of the overall connectivity and robustness of a network. A low Laplacian energy typically indicates a highly connected network (less fragmentation), where as high Laplacian energy might indicate a more fragmented or less connected network. The Laplacian matrix is used to model molecular structures where atoms are represented as vertices and bonds as edges. In this context, Laplacian energy can be used to estimate the stability and reactivity of molecules. A molecule with a high Laplacian energy might be less stable or more reactive, while lower energy could suggest a more stable structure.

2. Minimum Independent Dominating Energy of a Graph

Let G be a simple graph of order n with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. Let D_i be the minimum independent dominating set of a graph G. The minimum independent dominating matrix of G is the $n \times n$ matrix defined by $A_{D_i}(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 2, & \text{if } \ v_i v_j \in E \ and \ either \ v_i \ or \, v_j \in D_i \\ 1, & \text{if } \ v_i v_j \in E \ and \ neither \ v_i \ nor \, , v_j \in D_i \\ 1, & \text{if } \ i = j \, and \, v_i \in D_i \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_{D_i}(G)$ is denoted by $f_n(G, \lambda)$ and is defined by $f_n(G, \lambda) = det(\lambda I - A_{D_i}(G))$. The minimum independent dominating eigenvalues of the graph G are the eigenvalues of the matrix $A_{D_i}(G)$. We note that these eigenvalues are real numbers since $A_{D_i}(G)$ is real and symmetric. So, we can label them in the non-increasing order $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. The minimum independent dominating energy of G is defined to be the sum of the absolute eigenvalues of $A_{D_i}(G)$. For more information about minimum independent dominating energy of a graph refer [9]. In symbols, we write

$$E_{D_i}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

3. The Laplacian Minimum Independent Dominating Energy of a Graph

Let $D(G)$ be the diagonal matrix of vertex degrees of the graph G. Then the Laplacian minimum independent dominating matrix of G is denoted by $LE_{D_i}(G)$ and is defined as follows $LE_{D_i} = D(G) - A_{D_i}(G)$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values $LE_{D_i}(G)$ arranged in non increasing order. These eigen values are called Laplacian minimum independent dominating eigen values of G. The Laplacian minimum independent dominating energy of a graph G is defined as

$$LE_{D_i}(G) = \sum_{i=1}^{n} \left| \lambda_i - \frac{2m}{n} \right|,$$

To prove our results we make use of the Cauchy's - Schwarz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Example 3.1 Let G be a graph on 6 vertices as showm in the Figure 1. The possible minimum independent dominating sets are $D_i^1 = \{v_1, v_3\}, D_i^2 = \{v_3, v_6\}$
The minimum independent dominating matrices of G with respect to above minimum independent dominating sets are

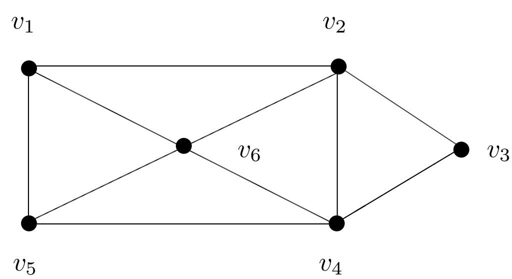


Figure 1. A Simple Graph with six vertices

the independent dominating set is $D_1 = \{v_1, v_3\}$, then

$$A_{D_i^1}(G) = \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & 2 \\ 2 & 0 & 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \text{ and } D(G) = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

If

$$LE_{D_i^1}(G) = D(G) - A_{D_i^1}(G) = \begin{pmatrix} 2 & -2 & 0 & 0 & -2 & -2 \\ -2 & 4 & -2 & -1 & 0 & -1 \\ 0 & -2 & 1 & -2 & 0 & 0 \\ 0 & -1 & -2 & 4 & -1 & -1 \\ -2 & 0 & 0 & -1 & 3 & -1 \\ -2 & -1 & 0 & -1 & -1 & 4 \end{pmatrix}.$$

The characteristic polynomial is given by
$f_n(G, \lambda) = \lambda^6 - 18\lambda^5 + 106\lambda^4 - 158\lambda^3 - 491\lambda^2 + 1296\lambda - 52 = 0.$

$$Spec(G) = \begin{pmatrix} -2.2125 & 0.0408 & 3.400 & 4.5635 & 5.7905 & 6.4178 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Average degree of the graph $=\frac{2m}{n} = \frac{2\times 10}{6} = \frac{10}{3}$
Hence, Laplacian minimum independent dominating energy, $LE_{D_i^1}(G) \approx 11.2505$.
If the independent dominating set is $D_i^2 = \{v_3, v_6\}$, then

$$A_{D_i^2}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 2 \\ 1 & 0 & 2 & 1 & 0 & 2 \\ 0 & 2 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 & 0 & 2 \\ 2 & 2 & 0 & 2 & 2 & 1 \end{pmatrix} \text{ and } D(G) = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

$$LE_{D_i^2}(G) = D(G) - A_{D_i^2}(G) = \begin{pmatrix} 3 & -1 & 0 & 0 & -1 & -2 \\ -1 & 4 & -2 & -1 & 0 & -2 \\ 0 & -2 & 1 & -2 & 0 & 0 \\ 0 & -1 & -2 & 4 & -1 & -2 \\ -1 & 0 & 0 & -1 & 3 & 2 \\ -2 & -2 & 0 & -2 & -2 & 3 \end{pmatrix}.$$

The characteristic polynomial is given by
$f_n(G, \lambda) = \lambda^6 - 18\lambda^5 + 112\lambda^4 - 242\lambda^3 - 162\lambda^2 + 1216\lambda - 1083 = 0.$

$$Spec(G) = \begin{pmatrix} -1.9662 & 1.4927 & 3 & 3.3820 & 6.4735 & 5.6180 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Average degree of the graph $=\frac{2m}{n} = \frac{2\times 10}{6} = \frac{10}{3}$
Hence, Laplacian minimum independent dominating energy, $LE_{D_i^2}(G) \approx 12.6664$.
Note that the Laplacian minimum independent dominating energy of the graph G depends on its minimum independent dominating set.

4. Laplacian Minimum Independent Dominating Energy of Some Standard Graphs

Theorem 4.1 *For $n > 2$, the Laplacian minimum independent dominating energy of a complete graph K_n is $(n-2) + \sqrt{n^2 + 10n - 7}$.*

Proof: Let K_n be the complete graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. For each $1 \leq r \leq n$; $\{v_r\}$ is minimum independent dominating set, hence without loss of generality take $D_i = \{v_1\}$. Then the minimmum independent dominating matrix $A_{D_i}(K_n)$ and its characteristics polynomial are as follows.

$$A_{D_i}(K_n) = \begin{pmatrix} 1 & 2 & 2 & \ldots & 2 & 2 \\ 2 & 0 & 1 & \ldots & 1 & 1 \\ 2 & 1 & 0 & \ldots & 1 & 1 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 2 & 1 & 1 & \ldots & 0 & 1 \\ 2 & 1 & 1 & \ldots & 1 & 0 \end{pmatrix}.$$

and

$$D(K_n) = \begin{pmatrix} n-1 & 0 & 0 & ... & 0 & 0 \\ 0 & n-1 & 0 & ... & 0 & 0 \\ 0 & 0 & n-1 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & n-1 & 0 \\ 0 & 0 & 0 & ... & 0 & n-1 \end{pmatrix}.$$

$$LE_{D_i}(G) = D(K_n) - A_{D_i}(K_n) = \begin{pmatrix} n-2 & -2 & -2 & ... & -2 & -2 \\ -2 & n-1 & -1 & ... & -1 & -1 \\ -1 & -1 & n-1 & ... & -1 & -1 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ -2 & -1 & -1 & ... & n-1 & -1 \\ -2 & -1 & -1 & ... & -1 & n-1 \end{pmatrix}.$$

The characteristic polynomial of $LE_{D_i}(K_n)$ is given by,
$(\lambda - n)^{n-2}(\lambda^2 - (n-1)\lambda - (3n-2)) = 0$.
The Laplacian minimum independent dominating eigen values are:

$$Spec(K_n) = \begin{pmatrix} n & \frac{(n-1)+\sqrt{n^2+10n-7}}{2} & \frac{(n-1)-\sqrt{n^2+10n-7}}{2} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Average degree of $K_n = \frac{2m}{n} = 2\frac{\frac{n(n-1)}{2}}{n} = n-1$.
Hence, the Laplacian minimum independent dominating energy of K_n is
$LE_{D_i}(K_n) = |n-(n-1)|(n-2) + \left|\frac{(n-1)+\sqrt{n^2+10n-7}}{2} - (n-1)\right| + \left|\frac{(n-1)-\sqrt{n^2+10n-7}}{2} - (n-1)\right|$
Therefore, $LE_{D_i}(K_n) = (n-2) + \sqrt{n^2+10n-7}$. □

Theorem 4.2 *For $n \geq 4$, the Laplacian minimum independent dominating energy of a star graph is $\frac{(n-2)^2}{n} + \sqrt{n^2+10n-7}$.*

Proof: Let $K_{1,n-1}$ be a star graph with the vertex set $V = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ having the vertex v_0 at the center then $D_i = \{v_0\}$ is the minimum independent dominating set. The minimum independent dominating matrix $A_{D_i}(K_{1,n-1})$ and its characteristic polynomial are as follows:

$$A_{D_i}(K_{1,n-1}) = \begin{pmatrix} 1 & 2 & 2 & ... & 2 & 2 \\ 2 & 0 & 0 & ... & 0 & 0 \\ 2 & 0 & 0 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 2 & 0 & 0 & ... & 0 & 0 \\ 2 & 0 & 0 & ... & 0 & 0 \end{pmatrix}.$$

and

$$D(K_{1,n-1}) = \begin{pmatrix} n-1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 \end{pmatrix}.$$

$$LE_{D_i}(K_{1,n-1}) = D(K_{1,n-1}) - A_{D_i}(K_{1,n-1}) = \begin{pmatrix} n-2 & -2 & -2 & \dots & -2 & -2 \\ -2 & 1 & 0 & \dots & 0 & 0 \\ -2 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ -2 & 0 & 0 & \dots & 1 & 0 \\ -2 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of $LE_{D_i}(K_{1,n-1})$ is given by,
$f_{D_i}(K_{1,n-1}, \lambda) = (\lambda - 1)^{n-2}(\lambda^2 - (n-1)\lambda - (3n-2))$
The Laplacian minimum independent dominating eigen values are:

$$Spec(K_{1,n-1}) = \begin{pmatrix} 1 & \frac{(n-1)+\sqrt{n^2+10n-7}}{2} & \frac{(n-1)-\sqrt{n^2+10n-7}}{2} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Average degree of $K_{1,n-1} = \frac{2(n-1)}{n}$.

Hence, the Laplacian minimum independent dominating energy is

$$\begin{aligned} LE_{D_i}(K_{1,n-1}) =& \left|1 - \frac{2(n-1)}{n}\right|(n-2) + \left|\frac{(n-1)+\sqrt{n^2+10n-7}}{2} - \frac{2(n-1)}{n}\right| \\ &+ \left|\frac{(n-1)-\sqrt{n^2+10n-7}}{2} - \frac{2(n-1)}{n}\right| \\ =& \left|\frac{(-n+2)}{n}\right|(n-2) + \left|\frac{(n^2-5n+4)+\sqrt{n^2+10n-7}}{2n}\right| + \\ & \left|\frac{(n^2-5n+4)-\sqrt{n^2+10n-7}}{2n}\right|. \\ =& \frac{(n-2)^2}{n} + \sqrt{n^2+10n-7}. \end{aligned}$$

Therefore, $LE_{D_i}(K_{1,n-1}) = \frac{(n-2)^2}{n} + \sqrt{n^2+10n-7}$. □

Theorem 4.3 *For the bull graph G, $LE_{D_i}(G) = 12.1514$*

Proof: Let G be the bull graph and let $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ be the vertex set of G. Then $D = \{v_2, v_5\}$ is a minimum independent dominating set of G. Therefore

$$A_{D_i} = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \text{ and } D(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

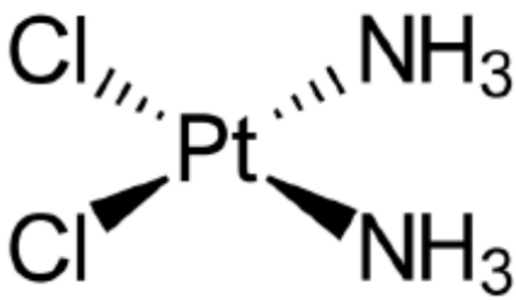
$$LE_{D_i}(G) = D(G) - A_{D_i}(G) = \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ -2 & 2 & -2 & -1 & 0 \\ 0 & -2 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -2 \\ 0 & 0 & 0 & -2 & 1 \end{pmatrix}.$$

The characteristic polynomial is given by
$f_n(G, \lambda) = \lambda^5 - 8\lambda^4 + 9\lambda^3 + 38\lambda^2 - 32\lambda - 12 = 0$.
The Laplacian minimum independent dominating eigen values are

$$Spec(G) = \begin{pmatrix} -1.8761 & -0.2861 & 1.0865 & 4.4085 & 4.6672 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Average degree of the graph $= \frac{2m}{n} = \frac{2\times5}{5} = 2$
Therefore, Laplacian minimum independent dominating energy of G with respect to D is $LE_{D_i} = 12.1514$. □

Cancer is a disease which is caused by an uncontrolled division of abnormal cells in a part of the body. Now a days we can find many types of cancer like, Bladder cancer Lung cancer, Brain cancer, Melanoma, Breast cancer, Non-Hodgkin lymphoma, Cervical cancer, Ovarian cancer. Etc., Cisplatin is a medicine which is widely used against the Cancer disease. In this paper we are calculating the Laplacian minimum independent dominating energy of the Cisplatin which is very much useful for further research and development in the treatment of Cancer.
Structural formula: $Pt(NH_3)_2Cl_2$



The Laplacian minimum independent dominating matrix with the consideration of minimum independent dominating set of $G \cong \{Pt(NH_3)_2Cl_2)\}$ is given by

$$A_{D_i}(G) = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } D(G) = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$LE_{D_i}(G) = D(G) - A_{D_i}(G) = \begin{pmatrix} 3 & -2 & -2 & -2 & -2 \\ -2 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Characteristic equation is $\lambda^5 - 7\lambda^4 + 2\lambda^3 + 26\lambda^2 - 35\lambda + 13 = 0$.
and the minimum independent dominating eigenvalues are, $\lambda_1 = -2.1231, \lambda_2 = 1, \lambda_3 = 1, \lambda_4 = 1, \lambda_5 = 6.1231$
The average degree of the graph G is $\frac{2m}{n} = \frac{8}{5}$
Therefore Laplacian minimum independent dominating energy is $LE_{D_i}(Pt(NH_3)_2Cl_2) = 10.0462$

5. Bounds on Laplacian Minimum Independent Dominating Energy of Graphs

Theorem 5.1 *If D is a minimum independent dominating set of a graph G and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are minimum independent dominating eigen values of $A_{D_i}(G)$ then*

(i) $\sum_{i=1}^{n} \lambda_i = 2 \mid E \mid - \mid D \mid$.

(ii) $\sum_{i=1}^{n} \lambda_i^2 = 2 \mid E \mid + \sum_{i=1}^{n} (d_i - r_i)^2 + 6 \sum_{v \in D_i} d(v)$ *where* $r_i = \begin{cases} 1, & \text{if } v_i \in D_i, \\ 0, & \text{if } v_i \notin D_i. \end{cases}$

Proof: **(i)** By definition, the sum of the principal diagonal elements of $LE_{D_i}(G)$ is equal to $\sum_{i=1}^{n}\lambda_i - \mid D \mid = 2 \mid E \mid - \mid D \mid$.
Also the sum of eigen values of $LE_{D_i}(G)$ is trace of $LE_{D_i}(G)$.
(ii) The sum of squares of eigen values of $LE_{D_i}(G)$ is the trace of $LE_{D_i}(G)^2$
Therefore $\sum_{i=1}^{n}\lambda_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij}l_{ji} = \sum_{i=1}^{n}(l_{ij})^2 + \sum_{j=1}^{n}(l_{ji})^2$

$$= 2\sum_{i<j}(l_{ij})^2 + \sum_{i=1}^{n}(l_{ji})^2$$

$$= 2 \mid E \mid + \sum_{i=1}^{n}(d_i - r_i)^2 + 6\sum_{v \in D_i} d(v) \ where \ r_i = \begin{cases} 1, & \text{if}\, v_i \in D_i \\ 0, & \text{if}\, v_i \notin D_i \end{cases}$$

$$= 2M, \ where \ \ M = \mid E \mid + \frac{1}{2}\left(\sum_{i=1}^{n}(d_i - r_i)^2 + 6\sum_{v \in D_i} d(v)\right)$$

□

6. Upper and Lower Bounds

Theorem 6.1 *Let G be a graph with n vertices, m edges and D is a minimum independent dominating set of a graph G. Then*

$$LE_{D_i}(G) \leq \sqrt{n\left(2m + \sum_{i=1}^{n}(d_i - r_i)^2 + 6\sum_{v \in D_i} d(v)\right)}. \tag{6.1}$$

Proof: Let G be a graph with n vertices and m edges and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of G. By using Cauchy's - Schwarz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \tag{6.2}$$

Put $a_i = 1$, $b_i = \mid \lambda_i \mid$ in equation (6.2) then,

$$\left(LE_{D_i}(G)\right)^2 = \left(\sum_{i=1}^{n} \mid \lambda_i \mid\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \mid \lambda_i \mid^2\right)$$

$$= n\left(\sum_{i=1}^{n} \mid \lambda_i \mid^2\right)$$

$$= n\left(2m + \sum_{i=1}^{n}(d_i - r_i)^2 + 6\sum_{v \in D_i} d(v)\right)$$

$$LE_{D_i}(G) \leq \sqrt{n\left(2m + \sum_{i=1}^{n}(d_i - r_i)^2 + 6\sum_{v \in D_i} d(v)\right)}.$$

□

This leads to inequality (6.1)

Theorem 6.2 *If G be a graph with n vertices, m edges and D is a minimum independent dominating set of a graph G. Then $LE_{D_i}(G) \leq \sqrt{2Mn} + 2m$.*

Proof: Let G be a graph with n vertices and m edges and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of G. By using Cauchy's - Schwarz inequality

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big) \tag{6.3}$$

Put $a_i = 1$, $b_i = \mid \lambda_i \mid$ in equation (6.3) then,

$$\Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2 \leq \Big(\sum_{i=1}^{n} 1\Big)\Big(\sum_{i=1}^{n} \mid \lambda_i \mid^2\Big)$$

$$\Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2 \leq n\ (2M)$$

$$\therefore\ \Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big) \leq \sqrt{2Mn}.$$

By Triangle inequality $\left|\lambda_i - \frac{2m}{n}\right| \leq \mid \lambda_i \mid + \left|\frac{2m}{n}\right|\ \ \forall i = 1, 2, \ldots, n$

$$i.e., \left|\lambda_i - \frac{2m}{n}\right| \leq \mid \lambda_i \mid + \frac{2m}{n}\ \ \forall\, i$$

$$\begin{aligned}\Big(\sum_{i=1}^{n} \left|\lambda_i - \frac{2m}{n}\right|\Big) &\leq \Big(\sum_{i=1}^{n} \lambda_i\Big) + \Big(\sum_{i=1}^{n} \frac{2m}{n}\Big) \\ &\leq \sqrt{2Mn} + 2m \\ \therefore LE_{D_i}(G) &\leq \sqrt{2Mn} + 2m\end{aligned}$$

□

Proposition 6.1 *Let G and H be the two graphs with n vertices, D_i^1 and D_i^2 be the minimum independent dominating sets of G and H, respectively. If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are minimum independent dominating eigenvalues of G and $\lambda_1^{'} \geq \lambda_2^{'} \geq \ldots \lambda_n^{'}$ are minimum independent dominating eigenvalues of H, $m_1 = |E(G)|$ and $m_2 = |E(H)|$, then*

$$\sum_{i=1}^{n} \mid \lambda_i \lambda_i^{'} \mid \leq \sqrt{\Big[2m_1 + \sum_{i=1}^{n}(d_i - r_i)^2 + 6\sum_{v \in D_i^1} d(v)\Big]\Big[2m_2 + \sum_{i=1}^{n}(d_i - r_i)^2 + 6\sum_{v \in D_i^2} d(v)\Big]}$$

Proof: By using Cauchy's - Schwarz inequality

$$\Big(\sum_{i=1}^{n} \mid a_i b_i \mid\Big)^2 \leq \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big) \tag{6.4}$$

Put $a_i = \lambda_i$, $b_i = \lambda_i^{'}$ in equation (6.4) then,

$$\Big(\sum_{i=1}^{n} \mid \lambda_i \lambda_i^{'} \mid\Big)^2 \leq \Big(\sum_{i=1}^{n} (\lambda_i)^2\Big)\Big(\sum_{i=1}^{n} (\lambda_i^{'})^2\Big).$$

$$= \Big(2m_1 + \sum_{i=1}^{n}(d_i - r_i)^2 + 6\sum_{v\in D_i} d(v)\Big)\Big(2m_2 + \sum_{i=1}^{n}(d_i - r_i)^2 + 6\sum_{v\in D_i} d(v)\Big)$$

This completes the proof. □

Theorem 6.3 *Let G be a graph with n vertices and m edges and D is a minimum independent dominating set of G. If $D =\mid detLE_{D_i}(G) \mid$ then*

$$LE_{D_i}(G) \geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}} - 2m.$$

Proof: Consider

$$\Big[\sum_{i=1}^{n} \mid \lambda_i \mid\Big]^2 = \Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big).\Big(\sum_{j=1}^{n} \mid \lambda_j \mid\Big)$$
$$= \sum_{i=1}^{n} \mid \lambda_i \mid^2 + \sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid$$

$$\therefore \sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid = \Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2 - \sum_{i=1}^{n} \mid \lambda_i \mid^2 \qquad (6.5)$$

Applying Arithmtic and Geometric means for $n(n-1)$ terms, we have

$$\frac{\sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid}{n(n-1)} \geq \Big[\prod_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid\Big]^{\frac{1}{n(n-1)}}$$
$$i.e., \sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid \geq n(n-1)\Big[\prod_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid\Big]^{\frac{1}{n(n-1)}}$$

Using (6.5) we get,

$$\Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2 - \sum_{i=1}^{n} \mid \lambda_i \mid^2 \geq n(n-1)\Big[\prod_{i=1}^{n} \mid \lambda_i \mid^{2(n-1)}\Big]^{\frac{1}{n(n-1)}}$$
$$\Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2 - 2M \geq n(n-1)\Big[\prod_{i=1}^{n} \mid \lambda_i \mid\Big]^{\frac{2}{n}}$$
$$\Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2 \geq 2M + n(n-1)\Big[\prod_{i=1}^{n} \mid \lambda_i \mid\Big]^{\frac{2}{n}}$$
$$\therefore \sum_{i=1}^{n} \mid \lambda_i \mid \geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}}$$

We know that

$$| \lambda_i | - \left|\frac{2m}{n}\right| \leq \left|\lambda_i - \frac{2m}{n}\right| \ \forall i$$

$$\sum_{i=1}^{n} | \lambda_i | - \sum_{i=1}^{n} \frac{2m}{n} \leq \sum_{i=1}^{n} \left|\lambda_i - \frac{2m}{n}\right|$$

$$i.e., \sum_{i=1}^{n} \left|\lambda_i\right| - 2m \leq LE_{D_i}(G)$$

$$i.e., LE_{D_i}(G) \geq \sum_{i=1}^{n} \left|\lambda_i\right| - 2m$$

$$\geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}} - 2m$$

$$\therefore LE_{D_i}(G) \geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}} - 2m$$

□

References

1. C.Adiga, A.Bayad, I.Gutman, S.A.Srinivas, *The minimum covering energy of a graph*, Kragujevac J. Sci. 34 (2012) 39-56.

2. C. Adiga and C. S. Shivakumaraswamy, *Bounds on the largest minimum degree eigenvalues of graphs*, Int.Math.Forum., 5(37)(2010), 1823-1831.

3. Akbar Jahanbani, *Some new lower bounds for energy of graphs*, Applied Mathematics and Computation. 296(2017) 233-238.

4. S. T. Bhangale and M. M. Pawar, *Isolate and independent domination number of some classes of graphs*, AKCE Int. J. Graphs and Combinatorics 16 (2019) 110–115.

5. R.B. Bapat, S.Pati, *Energy of a graph is never an odd integer.* Bull. Kerala Math. Assoc. 1 (2011) 129 - 132.

6. C. A. Coulson, *On the calculation of the energy in unsaturated hydrocarbons molecules.* Proc.Cambridge.Phil.Soc, 36(1940), 201-203.

7. I.Gutman, *The energy of a graph.* Ber. Math-Statist. Sekt. Forschungsz.Graz 103 (1978) 1-22.

8. I.Gutman, B.Zhou *Laplacian energy of a graph.* Lin. Algebra Appl. 414, 29-37(2006).

9. Madhukar M. Pawar and Shilpa T. Bhangale, *Minimum independent dominating energy of graphs*, Asian-European Journal of Mathematics (2021) 2150127 (14 pages), World Scientific Publishing Company. DOI: 10.1142/S1793557121501278

10. M.R.Rajesh Kanna, B.N. Dharmendra, G.Sridhara, *Laplacian Minimum Dominating Energy of a Graph*, IJPAM Vol. 89 No.4 (2013), 565-518.

11. E. Sampathkumar and M. A. Sriraj, *Vertex labeled/colored graphs, matrices and signed graphs*, J. of Combinatorics, Information and System Sciences, to appear.

12. H.Wang, H. Hua, *Note on Laplacian energy of graphs.* MATCH Commun. Math. Comput. Chem., 59(2008), 373-380. [28]

13. B. Zhou, *New upper bounds for Laplacian energy.* MATCH Commun. Math. Comput.Chem., 62(2009), 553-560.

14. B. Zhou, I. Gutman, *Nordhaus-Gaddum-type relations for the energy and Laplacian energy of graphs.* Bull. Acad. Serbe Sci. Arts (Cl. Math. Natur.) 134(2007), 1-11.

15. B. Zhou, *On the sum of powers of the Laplacian eigenvalues of graphs.* Lin. Algebra Appl., 429(2008), 2239-2246.

16. B. Zhou, I. Gutman, T. Aleksic, *A note on Laplacian energy of graphs.* MATCH Commun. Math. Comput. Chem., 60(2008), 441-446.

17. B. Zhou, I. Gutman, *On Laplacian energy of graphs.* MATCH Commun. Math. Comput.

18. H. Liu, M. Lu and F. Tian, *Some upper bounds for the energy of graphs*, Journal of Mathematical Chemistry 41(1) (2007).

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