Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 2** : 1–13.
ISSN-0037-8712
doi:10.5269/bspm.79091

Sliding Contact Problem with Wear for Thermoviscoelastic Material

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ABSTRACT: We study a quasistatic sliding contact problem with wear between a viscoelastic with long memory and a rigid moving foundation. We model the wear with a version of Archard's law. Thermal effects is taken into account. We establish a variational formulation of the model and we prove the existence and uniqueness of the weak solution. The proofs are based on the nonlinear equations involving the monotone operators, the classical result of nonlinear first order evolution inequalities, and the fixed-point arguments. We also establish the dependent results with respect to certain data.

Key Words: Thermo-viscoelastic material, long memory, sliding contact, wear, evolution equation, weak solution, fixed point.

Contents

1. Introduction

Problems related to thermoelastic contact arise naturally in many situations, especially those involving industrial processes when two or more materials may come into contact or may lose contact as a result of thermoelastic expansion or contraction. Such thermoelastic phenomena can be divided into three parts: static, quasistatic and dynamic.

The quasistatic case with various boundary conditions has been widely studied by [14, 15]. A quasistatic contact problem for viscoelastic material involving the thermal effects was studied in [2, 3, 4, 5].

Contact problems using viscoelastic with long memory have been studied in [1, 11, 8] and contact problems for viscoelastic material with long memory and with electric effects was studied in [10, 12, 13]. There are various models of contact with thermoviscoelastic and thermo-elastic-viscoplastic materials studied, associated with a large number of the boundary conditions, in this work, we are interested of the contact with one of the boundary conditions which is the wear, it is one of the processes that reduces the lifetime of modern machine elements. It represents the unwanted removal of materials from surfaces of contacting bodies occurring in relative motion. Wear arises when a hard rough surface digs into it, and its asperities plough a series of grooves.

The aim of this paper is to make a coupling of a thermo-viscoelastic problem with long memory and a sliding contact problem with wear. The studied problem is in process quasistatic and we model the material behavior with a thermo-viscoelastic constitutive law with long memory and we assume the contact is maintained during the movement and it is a sliding contact with wear which is modeled by a version of Archard's law. We drive a variational formulation and prove the existence and uniqueness of the weak solution, and finally we study the dependence of the solution with respect to the data.

The paper is structed as follows. In section 2 we present notation and somme preliminaires. The model is described in section 3 where the variational formulaion is given. In section 4, we presente our

* Corresponding autor.
2010 *Mathematics Subject Classification:* 74M15, 74F05, 74H20, 74H25.
Submitted September 21, 2025. Published December 19, 2025

existence and uniqueness result and the proof is based on the nonlinear equation involving the monotone operator, on classical result of nonlinear first order evolution inequalities and the fixed point arguments. In section 5, we study the dependence of the solution with respect to certain data.

2. Notation and Preliminaries

In this section we present some notation which we shall use in the study of a sliding contact problem and preliminary material.

We denote by $\mathbb{S}^d$ the space of second order symmetric tensor on $\mathbb{R}^d$. We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\begin{aligned} u.v &= u_i v_i, \qquad \|v\| = (v.v)^{\frac{1}{2}} \qquad \forall u, v \in \mathbb{R}^d, \\ \sigma.\tau &= \sigma_{ij}\tau_{ij}, \qquad \|\tau\| = (\tau.\tau)^{\frac{1}{2}} \qquad \forall \sigma, \tau \in \mathbb{S}^d \end{aligned}$$

Here and everyhere in this paper the indices i, j run between 1 to d, the summation over repeated indices is used and the index which follows a comma represents the partial derivative. We use the classical notation for L^p and Sobolev spaces associated to Ω and Γ. We use the following spaces :

$$\begin{aligned} H &= L^2(\Omega)^d = \left\{ v = (v_i) \ / \ v_i \in L^2(\Omega) \right\}, \\ \mathcal{H} &= \left\{ \sigma = (\sigma_{ij}) \ / \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \right\}, \\ H_1 &= \left\{ u = (u_i) \ / \ \varepsilon(u) \in \mathcal{H} \right\}, \\ \mathcal{H}_1 &= \left\{ \sigma \in \mathcal{H} \ / \ Div\sigma \in H \right\}, \end{aligned}$$

where, the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the following canonical inner products

$$\begin{aligned} (u,v)_H &= \int_\Omega u.v dx, \quad (\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma.\tau dx, \\ (u,v)_{H_1} &= (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \quad (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (Div\sigma, Div\tau)_H, \end{aligned}$$

and the associated norms $\|.\|_H$, $\|.\|_{\mathcal{H}}$, $\|.\|_{H_1}$ and $\|.\|_{\mathcal{H}_1}$, respectively. Here and below we use the notation

$$\begin{aligned} \varepsilon(v) &= (\varepsilon_{ij}(v)), \quad \varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i}) \quad \forall v \in H^1(\Omega)^d, \\ Div\tau &= (\tau_{ij,j}) \qquad \forall \tau \in \mathcal{H}_1. \end{aligned}$$

For every element $v \in H_1$ we also write v for the trace of v on Γ and we denote by v_ν and v_τ the normal and tangential components of v on Γ given by $v_\nu = v.\nu$, $v_\tau = v - v_\nu \nu$. We also denote by $\sigma_\nu$and σ_τ the normal and the tangential traces of a function $\sigma \in \mathcal{H}_1$, and we recall that when σ is a regular function then $\sigma_\nu = (\sigma\nu).\nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$.

Let $T > 0$. For every real Banach space X we use the notation $C(0,T;X)$ and $C^1(0,T;X)$ for the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively; $C(0,T;X)$ is a real Banach space with the norm $\|f\|_{C(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X$, while $C^1(0,T;X)$ is a real Banach space with the norm $\|f\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X + \max_{t\in[0,T]} \left\|\dot{f}(t)\right\|_X$. Finally, for $k \in \mathbb{N}$ and $p \in [1,\infty]$, we use the standard notation for the Lebesgue spaces $L^P(0,T;X)$ and for the Sobolev spaces $W^{k,p}(0,T;X)$. Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1\times X_2}$.

Let X be a real Hilbert space, and let $A : X \to X$ and $B : X \to X$ be given nonlinear operators, let $f : [0,T] \to X$ be a continuous function and let u_0 be the initial data. we have the following nonlinear equation: find a function $u : [0,T] \to X$ such that

$$A\dot{u}(t) + Gu(t) = f_\eta(t) \tag{2.1}$$

$$u(0) = u_0. \tag{2.2}$$

We have the following existence and uniqueness result, can be found in [18] .

Lemma 2.1 *Let X be a real Hilbert space and assume that $A : X \to X$ is a strongly monotone Lipschitz continous operator and $B : X \to X$ is a Lipschitz continuous operator. Then, for each $f \in C(0, T; X)$ and $u_0 \in X$ there exists a unique solution $u \in C^1([0, T] ; X)$ which satisfies* (2.1) − (2.2).

3. Mechanical and Variational Formulations

We consider a thermo-viscoelastic body which ocuppies a bounded domain $\Omega \subset \mathbb{R}^d\,(d = 2.3)$ with a Lipschitz continous boundary Γ that is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that meas $\Gamma_1 > 0$. Let $T > 0$ and let $[0, T]$ be time interval of interest.The body is clamped on $\Gamma_1 \times (0, T)$, so the displacement field vanishes there. A surface tractions of density f_2 act on $\Gamma_2 \times (0, T)$, and a body force of density f_0 acts in $\Omega \times (0, T)$. The contact between the body and the moving rigid foundation, over the contact surface Γ_3, is a sliding contact with wear and is maintained during the movement. Moreover the process is quasistatic, i.e. the intertial terms are negleted in the equation of motion. We use a thermo-viscoelastic constitutive law with long memory given by

$$\sigma = \mathcal{A}\varepsilon\left(\dot{u}\right) + \mathcal{G}\varepsilon\left(u\right) + \int_0^t M\left(t-s\right)\varepsilon\left(u\left(s\right)\right)ds - \mathcal{M}\theta. \tag{3.1}$$

where σ denotes the stress tensor, u represents the displacement field, $\dot{u}$ is the velocity, $\varepsilon\left(u\right)$ is the small strain tensor, and M is relaxation fourth order tenseur. Here $\mathcal{A}$ and $\mathcal{G}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively. θ is the temperature field and $\mathcal{M} = (m_{ij})_{i,j=1}^d$ is the thermal expension tensor. We use dots for derivatives with respect to the time variable t. The constitutive law (3.1) became thermo-viscoelastic constitutive law when $M = 0$ and its given by

$$\sigma = \mathcal{A}\varepsilon\left(\dot{u}\right) + \mathcal{G}\varepsilon\left(u\right) - \mathcal{M}\theta.$$

The evolution of the temperature field θ is governed by the heat equation (*see* [2]), obtained from the conservation of energy which has the following differential equation

$$\dot{\theta} - div(K\nabla\theta) = q - \mathcal{M}\nabla\dot{u},$$

where $K = (k_{ij})$ represents the thermal conductivity tensor, $div(K\nabla\theta) = (k_{ij}\theta_{,i})_{,i}$ and q represents the density of volume heat sources.

The associated temperature boundary condition on Γ_3 is given by

$$k_{ij}\theta_{,i}n_j = -k_e\left(\theta - \theta_F\right) \qquad \text{on } \Gamma_3 \times (0, T),$$

where θ_F is the temperature of the foundation, k_e is the heat exchange coefficient between the body and the obstacle.

Thus, the classical formulation of the mechanical problem corresponding to the quasistatic contact of a viscoelastic material with a long memory, taking into account the heat generation is as follows.

Problème P. Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0, T] \to S^d$ and a temperature field $\theta : \Omega \times [0, T] \to \mathbb{R}$ such that

$$\sigma = \mathcal{A}\varepsilon\left(\dot{u}\right) + \mathcal{G}\left(\varepsilon\left(u\right)\right) + \int_0^t M\left(t-s\right)\varepsilon\left(u\right)ds - \mathcal{M}\theta \tag{3.2}$$

$$Div\sigma + f_0 = 0 \quad \text{in } \Omega \times (0, T)\,, \tag{3.3}$$

$$\dot{\theta} - div(K\nabla\theta) = q - \mathcal{M}\nabla\dot{u} \text{ in } \Omega \times (0, T)\,, \tag{3.4}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T)\,, \tag{3.5}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0, T)\,, \tag{3.6}$$

$$-\sigma_\nu = \beta \left| \dot{u}_\nu \right| \quad \text{on } \Gamma_3 \times (0,T), \tag{3.7}$$

$$u_\tau = 0 \text{ on } \Gamma_3 \times (0,T), \tag{3.8}$$

$$k_{ij}\theta_{,i}n_j = -k_e\,(\theta - \theta_F) \text{ on } \Gamma_3 \times (0,T), \tag{3.9}$$

$$\theta = 0 \qquad \text{on } (\Gamma_1 \cup \Gamma_2) \times (0,T), \tag{3.10}$$

$$u\,(0) = u_0 \text{ and } \theta\,(0) = \theta_0 \quad \text{in } \Omega. \tag{3.11}$$

We now provide some comments on equations and conditions (3.2) − (3.11).

First, (3.2) represents the thermo-viscoelastic constitutive law with long memory. (3.3) represents the equilibrium equations for the stress field. (3.4) represents the evolution differential equation for the temperature. (3.5) and (3.6) represent the displacement and traction boundary conditions, respectively. Conditions (3.7) and (3.8) are the sliding contact conditions with the wear. Condition (3.9) represents the temperature boundary condition. (3.10) means that the temperature vanishes on $(\Gamma_1 \cup \Gamma_2) \times (0,T)$. Denoting by u_0, θ_0 the given initial displacement field and initial temperature field, respectively. To simplify the notation, we do not indicate explicitely the dependence of various functions on the variables $x \in \Omega \cup \Gamma$ and $t \in [0,T]$. To obtain a variational formulation of the problem (3.2) − (3.11), we need additional notations. Let us consider the closed subspace of H_1 defined by

$$V = \left\{ v \in H^1\,(\Omega)^d \,/\, v = 0 \quad \text{on } \Gamma_1, v_\tau = 0 \quad \text{on } \Gamma_3 \right\},$$

and

$$E = \left\{ \gamma \in H^1\,(\Omega) \,/\, \gamma = 0 \qquad on \;\; \Gamma_1 \cup \Gamma_2 \right\}.$$

denote the closed subspace of $H^1\,(\Omega)$.

Since $meas\,(\Gamma_1) \succ 0$, the following Korn's inequality holds:

$$\|\varepsilon\,(v)\|_{\mathcal{H}} \geqslant C_k\,\|v\|_{H_1} \qquad \forall v \in V, \tag{3.12}$$

where $C_k > 0$ is a constant which depends only on Ω and Γ_1. On the space V, we consider the inner product and the associated norm given by

$$(u,v)_V = (\varepsilon\,(u)\,, \varepsilon\,(v))_{\mathcal{H}}\,, \qquad \|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}} \qquad \forall u,v \in V. \tag{3.13}$$

It follows from Korn's inequality that $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent norms on V. Therefore $(V, \|.\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3.13), there exists a constant $C_0 > 0$ depending only on the domain Ω , $\Gamma_1$and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leqslant C_0\,\|v\|_V \qquad \forall v \in V. \tag{3.14}$$

In the study of the mechanical problem (3.2) − (3.11), we assume that the viscosity operator $\mathcal{A}$ $: \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{A}} > 0 \text{ such that} \\ \|\mathcal{A}\,(x,\varepsilon_1) - \mathcal{A}\,(x,\varepsilon_2)\| \leqslant L_{\mathcal{A}}\,\|\varepsilon_1 - \varepsilon_2\| \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, a.e. x \in \Omega, \\ (b) \text{ There exists a constant } m_{\mathcal{A}} > 0 \text{ such that} \\ (\mathcal{A}\,(x,\varepsilon_1) - \mathcal{A}\,(x,\varepsilon_2))\,.\,(\varepsilon_1 - \varepsilon_2) \geqslant m_{\mathcal{A}}\,\|\varepsilon_1 - \varepsilon_2\|^2\,,\ \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, a.e. x \in \Omega, \\ (c)\ x \mapsto \mathcal{A}\,(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \quad \varepsilon \in \mathbb{S}^d, \\ (d)\ x \mapsto \mathcal{A}\,(x,0) \in \mathcal{H}. \end{cases} \tag{3.15}$$

The elasticity operator $\mathcal{G} : \Omega \times S^d \to S^d$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{G}} > 0 \text{ such that} \\ \|\mathcal{G}\,(x,\varepsilon_1) - \mathcal{G}\,(x,\varepsilon_2)\| \leqslant L_{\mathcal{G}}\,\|\varepsilon_1 - \varepsilon_2\| \\ \forall \varepsilon_1, \varepsilon_2 \in S^d, \text{ a .e. } x \in \Omega. \\ (b) \text{ The mapping } x \to \mathcal{G}\,(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega, \text{ for any } \varepsilon \in S^d\,. \\ (c) \text{ The mapping } x \to \mathcal{G}\,(x,0) \in \mathcal{H}. \end{cases} \tag{3.16}$$

The relaxation operator satisfies

$$M \in C\left(0, T, \mathcal{H}_{\infty}\right). \tag{3.17}$$

Where $\mathcal{H}_{\infty}$ is the space of fourth order tensor fields

$$\mathcal{H}_{\infty} = \{\zeta = \left(\zeta_{ijkl}\right)/\zeta_{ijkl} = \zeta_{jikl} = \zeta_{klij} \in L^{\infty}\left(\Omega\right) 1 \leq i, j, k, l \leq d\},$$

which is a real Banach space with the norm

$$\|\zeta\|_{\mathcal{H}_{\infty}} = \max_{1 \leq i,j,k,l \leq d} \|\zeta_{ijkl}\|_{L^{\infty}(\Omega)}$$

and, moreover,

$$\|\zeta\tau\|_{\mathcal{H}} \leqslant d\, \|\zeta\|_{\mathcal{H}_{\infty}} \|\tau\|_{\mathcal{H}} \qquad \forall \zeta \in \mathcal{H}_{\infty},\ \tau \in \mathcal{H}$$

The densities of body forces and surface tractions satisfy

$$f_0 \in C\left(0, T; H\right), \qquad f_2 \in C\left(0, T; L^2(\Gamma_2)^d\right), \tag{3.18}$$

The function β has the following properties

$$\beta \in L^{\infty}(\Gamma_3) \quad \beta(x) \geqslant \beta_* > 0 \text{ a.e. on } \Gamma_3. \tag{3.19}$$

The thermal tensors and the heat source density satisfy

$$\begin{cases} \mathcal{M} = (m_{ij}), & m_{ij} = m_{ji} \in L^{\infty}(\Omega). \\ K = (k_{ij}), & k_{ij} = k_{ji} \in L^{\infty}(\Omega), k_{ij}\zeta_i\zeta_i \geqslant c_k\zeta_i\zeta_i, \\ & \text{for some } c_k > 0, \text{ for all } (\zeta_i) \in R^d. \\ q \in L^2(0, T; L^2(\Omega)). & \end{cases} \tag{3.20}$$

Finally, the boundary and initial data verify

$$\begin{array}{cc} u_0 \in V, & \theta_0 \in E, \\ \theta_F \in L^2(0, T; L^2(\Gamma_3)), & k_e \in L^{\infty}(\Omega, R^+). \end{array} \tag{3.21}$$

We define the functions $f : [0, T] \to V$ and $q : [0, T] \to W$ by

$$(f(t), v) = \int_{\Omega} f_0(t).v dx + \int_{\Gamma_3} f_2(t).v da. \quad \forall v \in V, \forall t \in [0, T], \tag{3.22}$$

Next, we denote by $j : V \times V \to \mathbb{R}$ the functional defined by

$$j(u, v) = \int_{\Gamma_3} \beta \left| u_{\nu} \right| .v_{\nu} da \qquad \forall u, v \in V. \tag{3.23}$$

We note that condition (3.18) implies

$$f \in C\left(0, T; V\right). \tag{3.24}$$

Using standard arguments, we obtain the variational formulation of the mechanical problem (3.2) – (3.11).

Problem PV**.** Find a displacement field $u : [0, T] \to V$ and a temperature field $\theta : [0, T] \to E$ such that for all $t \in [0, T]$,

$$\begin{aligned} &\left(\mathcal{A}\varepsilon\left(\dot{u}(t)\right), \varepsilon(v)\right)_{\mathcal{H}} + \left(\mathcal{G}\varepsilon\left(u(t)\right), \varepsilon(v)\right)_{\mathcal{H}} + \left(\int_0^t M\left(t-s\right)\varepsilon\left(u(t)\right) ds, \varepsilon(v)\right)_{\mathcal{H}} \\ &- \left(\mathcal{M}\theta\left(t\right), \varepsilon(v)\right)_{\mathcal{H}} + j\left(\dot{u}(t), v\right) = \left(f(t), v\right)_V. \end{aligned} \tag{3.25}$$

$$\dot{\theta}(t) + K\theta(t) = R\dot{u}(t) + Q(t) \quad \text{ in } E' \tag{3.26}$$

$$u(0) = u_0, \quad \theta(0) = \theta_0, \tag{3.27}$$

where $K : E \to E'$, $R : V \to E'$ and $Q : [0,T] \to E'$ are given by

$$(K\tau,\omega)_{E'\times E} = \sum_{i,j=1}^{d} \int_{\Omega} k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \omega}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau \omega da,$$

$$(Rv,\omega)_{E'\times E} = -\int_{\Omega} m_{ij} \frac{\partial v_i}{\partial x_j} \omega dx,$$

$$(Q(t),\omega)_{E'\times E} = \int_{\Gamma_3} k_e \theta_F(t) \omega da + \int_{\Omega} q(t) \omega dx,$$

for all $v \in V, \tau, \omega \in E$.

4. Existence and Uniqueness Result

Now, we propose our existence and uniqueness result.

Theorem 4.1 *Assume that* $(3.15)-(3.21)$ *hold. Then, if* $\|\beta\|_{L^\infty(\Gamma_3)} < \frac{m_{\mathcal{A}}}{C_0^2}$ *the Problem* PV *has a unique solution which satisfies*

$$u \in C^1\left([0,T], V\right), \tag{4.1}$$

$$\theta \in W^{1,2}(0,T;E') \cap L^2(0,T;E) \cap C(0,T;L^2(\Omega)). \tag{4.2}$$

The functions u and θ which satisfy $(3.25)-(3.27)$ are called a weak solution of the contact prolem P. We conclude that, under the assumptions $(3.15)-(3.21)$,the mechanical problem $(3.2)-(3.11)$ has a unique weak solution satisfying $(4.1)-(4.2)$. The regularity of the weak solution is given by $(4.1)-(4.2)$ and in term of stresses,

$$\sigma \in C\left(0,T;\mathcal{H}_1\right). \tag{4.3}$$

Indeed, it follows from (3.25) that $Div\sigma\left(t\right) + f_0 = 0$ for all $t \in [0,T]$ and therefore the regularity (4.1) and (4.2) of u and θ, combined with $(3.15)-(3.21)$ implies (4.3).

The proof of Theorem 4.1 is carried out in several steps that we prove in what follows, everywhere in this section we suppose that assumptions of Theorem 4.1 hold, and we consider that C is a generic positive constant which is independent of time and whose value may change from one occurrence to another.

Let $\eta \in C\left(0,T;\mathcal{H}\right)$ be given, in the first step we consider the following variational problem.

Problem PV_η. Find a displacement field $u_\eta : [0,T] \to V$ such that

$$\begin{aligned}\left(\mathcal{A}\varepsilon\left(\dot{u}_\eta(t)\right), \varepsilon(v)\right)_{\mathcal{H}} + \left(\mathcal{G}\varepsilon\left(u_\eta(t)\right), \varepsilon(v)\right)_{\mathcal{H}} + \left(\eta\left(t\right), \varepsilon(v)\right)_{\mathcal{H}} \\ + j\left(\dot{u}_\eta(t), v\right) = \left(f(t), v - \dot{u}_\eta\right)_V.\end{aligned} \tag{4.4}$$

$$u_\eta(0) = u_0. \tag{4.5}$$

We have the following result for the problem.

Lemma 4.1 *There exists* β_0 *depending only on* $\Omega, \Gamma_1, \Gamma_3$ *and* $\mathcal{A}$ *such that if* $\|\beta\|_{L^\infty(\Gamma_3)} < \beta_0$*, the problem* PV_η *has a unique solution* $u_\eta \in C^1\left([0,T], V\right)$.

Proof: We define the operators $A : V \to V$, $G : V \to V$ and the function $f_\eta : [0,T] \to V$ by

$$(Au,v)_V = \left(\mathcal{A}\varepsilon\left(u\right), \varepsilon(v)\right)_{\mathcal{H}} + j\left(u,v\right) \tag{4.6}$$

$$(Gu,v)_V = \left(\mathcal{G}\varepsilon\left(u\right), \varepsilon(v)\right)_{\mathcal{H}}, \tag{4.7}$$

$$(f_\eta, v)_V = (f(t), v)_V - \left(\eta, \varepsilon(v)\right)_{\mathcal{H}}. \tag{4.8}$$

for all $u, v \in V$ et $t \in [0,T]$.

We use (4.6) and (3.15)(a) to find that

$$\|Au_1 - Au_2\| \leqslant \left(L_{\mathcal{A}} + C_0^2 \|\beta\|_{L^\infty(\Gamma_3)}\right) \|u_1 - u_2\|_V , \tag{4.9}$$

which shows that A is Lipschitz continous.

It follows from (4.6) and (3.15) (b) that

$$(Au_1 - Au_2, u_1 - u_2)_V \geqslant \left(m_{\mathcal{A}} - C_0^2 \|\beta\|_{L^\infty(\Gamma_3)}\right) \|u_1 - u_2\|_V^2 , \tag{4.10}$$

which shows that A is a strongly monotone operator on V, if $\|\beta\|_{L^\infty(\Gamma_3)} < \frac{m_{\mathcal{A}}}{C_0^2}$.

From (3.16)(a) and (4.7), we have

$$\|Gu_1 - Gu_2\| \leqslant L_{\mathcal{G}} \|u_1 - u_2\|_V \tag{4.11}$$

i.e, that G is a Lipschitz continuous operator on V

From (4.9) and (4.10) A is a strongly monotone and Lipschitz continous operator then from (4.11) G is a Lipschitz continous operator.We use (3.18), we find that the fonction f defined by (3.22) satisfies $f \in C([0,T], V)$ and keeping in mind that $\eta \in C([0,T], \mathcal{H})$, we deduce by (4.8) that $f_\eta \in C([0,T], V)$, and $u_0 \in V$ and it follows from Theorem 2.1 that there exists a unique function $u_\eta \in C^1([0,T], V)$ such that

$$A\dot{u}_\eta(t) + Gu_\eta(t) = f_\eta(t) \tag{4.12}$$

$$u_\eta(0) = u_0. \tag{4.13}$$

We use (4.6), (4.7), (4.12) and (4.13) to see that u_η is the unique solution to PV_η. □

Let $u_\eta : [0,T] \to V$ be the fonction defined by

$$u = \int_0^t v_\eta(s) ds + u_0, \qquad \forall t \in [0,T]. \tag{4.14}$$

In the second step, let $\eta \in C([0,T], \mathcal{H})$, we use the displacement field u_η obtained in lemma 4.1 and we consider the following variational problem.

Problem PV_θ. Find the temperature field $\theta_\eta : [0,T] \to E$. such that

$$\dot{\theta}_\eta(t) + K\theta_\eta(t) = R\dot{u}_\eta(t) + Q(t), \tag{4.15}$$

$$\theta_\eta(0) = \theta_0. \tag{4.16}$$

We have the following result.

Lemma 4.2 *PV_θ has a unique solution θ_η which satisfies the regularity* (4.2). *Moreover, there exists $C > 0$, such that*

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 \leqslant C \int_0^t \left\|\dot{u}_1(s) - \dot{u}_2(s)\right\|_V^2 ds. \tag{4.17}$$

.

Proof: The proof of Lemma 4.2 is done using a classical result on first order evolution equation given in [17], as in [5].

We denote $\theta_{\eta_i} = \theta_i$ and $\dot{u}_{\eta_i} = \dot{u}_i$for $i = 1, 2$, we use (4.15) to get

$$\begin{aligned} &\left(\dot{\theta}_1(t) \;-\; \dot{\theta}_2(t), \theta_1(t) - \theta_2(t)\right)_{E'\times E} + (K\theta_1(t) - K\theta_2(t), \theta_1(t) - \theta_2(t))_{E'\times E} \\ &\quad = \left(R\dot{u}_1(t) - R\dot{u}_2(t), \theta_2(t) - \theta_2(t)\right)_{E'\times E}, \end{aligned} \tag{4.18}$$

We integrate (4.18) with respect to time and we use the coercivity of K and the Lipschitz continuity of $R : V \to E'$ to deduce that (4.17) is satisfies. □

Finally, we consider the operator $\Lambda : C(0,T;\mathcal{H}) \to C(0,T;\mathcal{H})$ defined by

$$\Lambda\eta(t) = \int_0^t M(t-s)\,\varepsilon(u_\eta(s))\,ds - \mathcal{M}\theta_\eta(t). \tag{4.19}$$

Here, for every $\eta \in C(0,T;\mathcal{H})$, u_η and θ_η represent the displacement field and the temperature field which obtained in Lemma 4.1 and Lemma 4.2 respectively. We have the following result.

Lemma 4.3 *The operateur Λ has a unique fixed point $\eta^* \in C(0,T;\mathcal{H})$ such that $\Lambda\eta^* = \eta^*$.*

Proof: Let now $\eta_1, \eta_2 \in C(0,T;\mathcal{H})$. We use the notation $u_{\eta_i} = u_i$, $\dot{u}_{\eta_i} = v_{\eta_i} = \dot{v}_i$ and $\theta_{\eta_1} = \theta_i$ for $i = 1,2$. Using (3.17), (3.13), (3.20) and (4.19) to deduce that

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_{\mathcal{H}}^2 \leqslant C\left(\int_0^t \|u_1(s) - u_2(s)\|_V^2\,ds + \|\theta_1(t) - \theta_2(t)\|_E^2\right) \tag{4.20}$$

since

$$u_i(t) = \int_0^t v_i(s)ds + u_0 \qquad \forall t \in [0,T],$$

we have

$$\|u_1(t) - u_2(t)\|_V^2 \leqslant C\int_0^t \|v_1(s) - v_2(s)\|_V^2\,ds. \tag{4.21}$$

Moreover, from (4.4) we obtain

$$\begin{aligned}&(\mathcal{A}\varepsilon(v_1) - \mathcal{A}\varepsilon(v_2), \varepsilon(v_1) - \varepsilon(v_2))_{\mathcal{H}} + (\mathcal{G}\varepsilon(u_1) - \mathcal{G}\varepsilon(u_2), \varepsilon(v_1) - \varepsilon(v_2))_{\mathcal{H}}\\ &+ (\eta_1 - \eta_2, \varepsilon(v_1) - \varepsilon(v_2))_{\mathcal{H}} = j(v_2, v_1 - v_2) - j(v_1, v_1 - v_2).\end{aligned}$$

We use the previous equality, the assumptions (3.15) (a), (3.16) (a), (3.25) and (3.14) to find that

$$\|v_1 - v_2\|_V^2 \leqslant C\left(\|u_1 - u_2\|_V^2 + \|\eta_1 - \eta_2\|_{\mathcal{H}}^2\right).$$

Integrating this inequality with respect to time, we find

$$\int_0^t \|v_1(s) - v_2(s)\|_V^2\,ds \leqslant C\int_0^t \left(\|u_1(s) - u_2(s)\|_V^2 + \|\eta_1(s) - \eta_2(s)\|_{\mathcal{H}}^2\right)ds \qquad \forall t \in [0,T]. \tag{4.22}$$

Next, we use (4.21), to deduce

$$\|u_1(t) - u_2(t)\|_V^2 \leqslant C\int_0^t \|u_1(s) - u_2(s)\|_V^2 + C\int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathcal{H}}^2\,ds \qquad \forall t \in [0,T]. \tag{4.23}$$

Then, we apply Gronwall's inequality to the previous inequality yields

$$\|u_1(t) - u_2(t)\|_V^2 \leqslant C\int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathcal{H}}^2\,ds \qquad \forall t \in [0,T].$$

Now, it follows from (4.22) that

$$\int_0^t \|v_1(s) - v_2(s)\|_V^2\,ds \leqslant C\int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathcal{H}}^2\,ds \qquad \forall t \in [0,T]. \tag{4.24}$$

and from (4.17) and (4.24) we have

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 \leqslant \int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathcal{H}}^2\,ds. \tag{4.25}$$

We now conclude from (4.21), (4.24) and (4.25) that

$$\|\Lambda\eta_1(t)-\Lambda\eta_2(t)\|_{\mathcal{H}}^2 \leqslant C\int_0^t \|\eta_1(s)-\eta_2(s)\|_{\mathcal{H}}^2\,ds$$

Reiterating this inequality m times leads to

$$\|\Lambda^m\eta_1-\Lambda^m\eta_2\|_{C(0,T;\mathcal{H})}^2 \leqslant \frac{C^mT^m}{m!}\|\eta_1-\eta_2\|_{C(0,T;\mathcal{H})}^2.$$

For m sufficiently large, Λ^m is a contraction on the Banach space $C(0,T;\mathcal{H})$, and so Λ has a unique fixed point. □

Now, we have all the ingredients needs to prove Theorem 4.1.

Proof: Existence. Let $\eta^*\in C(0,T;\mathcal{H})$ be the fixed point of Λ defined by (4.18), and let u_{η^*} and θ_{η^*} the solutions of the problems PV_η and PV_θ respectively for $\eta=\eta^*$ and denote

$$u=u_{\eta^*},\quad \dot{u}=\dot{u}_{\eta^*},\quad \theta=\theta_{\eta^*}, \tag{4.26}$$

Let $\sigma:[0,T]\to\mathcal{H}$ defined by

$$\sigma(t)=\mathcal{A}\varepsilon\left(\dot{u}(t)\right)+\mathcal{G}\varepsilon\left(u(t)\right)+\int_0^t M\left(t-s\right)\varepsilon\left(u(t)\right)ds-\mathcal{M}\theta(t), \tag{4.27}$$

We prove that (u,θ) satisfies (3.25) – (3.27) and the regularity (4.1) – (4.2). Indeed, we write (4.4) for $\eta=\eta^*$ and use (4.26) to obtain (3.25). Indeed, we write (4.4) for $\eta=\eta^*$ and we use (4.26) to find

$$\begin{aligned}\left(\mathcal{A}\varepsilon\left(\dot{u}(t)\right),\varepsilon(v)\right)_{\mathcal{H}}+\left(\mathcal{G}\varepsilon\left(u(t)\right),\varepsilon(v)\right)_{\mathcal{H}}+\left(\eta^*\left(t\right),\varepsilon(v)\right)_{\mathcal{H}}\\ +j\left(\dot{u}(t),v\right)=\left(f(t),v-\dot{u}\right)_V.\end{aligned} \tag{4.28}$$

Equality $\Lambda\eta^*=\eta^*$ combined with (4.18) show that

$$\eta^*=\int_0^t M\left(t-s\right)\varepsilon\left(u\right)ds-\mathcal{M}\theta. \tag{4.29}$$

We substitute (4.29) in (4.28) to conclude that (3.25) is satisfied.

Now, we write (4.15) for $\eta=\eta^*$ and use (4.26) to find that (3.26) is also satisfies. Next, (3.27) and the regularities (4.1) – (4.2) follow from Lemmas 4.1 and 4.2, since (u,θ) satisfies (4.1) – (4.2), it follows from (4.27) that

$$\sigma\in C\left(0,T;\mathcal{H}\right), \tag{4.30}$$

we choose $v=\pm\varphi\in C_0^\infty\left(\Omega\right)^d$ in (3.25), we use (4.27) and (3.22) to obtain

$$Div\sigma\left(t\right)=-f_0\left(t\right)\qquad\forall t\in[0,T],$$

then, we use (3.18) and (4.30) to find

$$\sigma\in C\left(0,T;\mathcal{H}_1\right).$$

Uinqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.18) and the unique solvability of the problems PV_η and PV_θ . □

5. Convergence Results

In this section we study the dependence of the solution to problem PV when we introduce the perturbation of certain data. We suppose that the assumptions (3.15) – (3.21) satisfay. Moreover, we assume that $\|\beta\|_{L^\infty(\Gamma_3)} < \beta_0$, where $\beta_0 = \frac{m_{\mathcal{A}}}{C_0^2}$. Let (u,θ) the solution of PV which obtained by the Theorem 4.1, for every $\rho > 0$, let M_ρ and f_ρ the perturbations of M and f, respectively, which satisfay (3.17) and (3.24).

Under these assumptions, we consider the following variational problem.

Problem PV_ρ. Find a displacement field $u_\rho : [0,T] \to V$ and a temperature fild $\theta_\rho : [0,T] \to E$ such thatfor all $t \in [0,T]$,

$$\begin{aligned}&\left(\mathcal{A}\varepsilon\left(\dot{u}_\rho(t)\right), \varepsilon(v)\right)_{\mathcal{H}} + \left(\mathcal{G}\varepsilon\left(u_\rho(t)\right), \varepsilon(v)\right)_{\mathcal{H}} + \left(\int_0^t M_\rho\left(t-s\right)\varepsilon\left(u_\rho(t)\right) ds, \varepsilon(v)\right)_{\mathcal{H}} \\ &- \left(\mathcal{M}\theta_\rho\left(t\right), \varepsilon(v)\right)_{\mathcal{H}} + j\left(\dot{u}_\rho(t), v\right) = \left(f_\rho(t), v\right)_V .\end{aligned} \tag{5.1}$$

$$\dot{\theta}_\rho(t) + K\theta_\rho(t) = R\dot{u}_\rho(t) + Q(t), \quad \forall\varphi, \phi \in W, \tag{5.2}$$

$$u_\rho(0) = u_0, \quad \theta_\rho(0) = \theta_0. \tag{5.3}$$

Assume that

$$\|\beta\|_{L^\infty(\Gamma_3)} < \frac{m_{\mathcal{A}}}{C_0^2} .$$

We deduce from Theorem 4.1 that for each $\rho > 0$, the problem PV_ρ has a unique solution (u_ρ, θ_ρ) satisfying $u_\rho \in C^1\left([0,T], V\right)$ and $\theta \in W^{1,2}(0,T;E') \cap L^2(0,T;E) \cap C(0,T;L^2(\Omega))$.

Let consider M_ρ, M and f_ρ , f satisfy the following assumptions

$$M_\rho \to M \text{ in } C\left(0,T,\mathcal{H}_\infty\right) \text{ as } \rho \to 0, \tag{5.4}$$

$$f_\rho \to f \quad \text{in} \quad C\left(0,T,V\right) \quad \text{as } \rho \to 0. \tag{5.5}$$

We have the following convergence result.

Theorem 5.1 *Assume that (5.4) – (5.5) hold. Then the solution (u_ρ, θ_ρ) of the problem PV_ρ converges to the solution (u,θ) of problem PV.*

$$u_\rho \to u \textit{ in } C^1(0,T;V) \quad \textit{as } \rho \to 0, \tag{5.6}$$

$$\theta_\rho \to \theta \textit{ in } C(0,T;L^2(\Omega)) \quad \textit{as } \rho \to 0. \tag{5.7}$$

.

Proof: Let $\rho > 0$ and $t \in [0,T]$,we use $v = \dot{u}_\rho(t) - \dot{u}(t)$ in (5.1) and $v = \dot{u}(t) - \dot{u}_\rho(t)$ in (3.25), then we additional the tow inequalities, we get

$$\begin{aligned}&\left(\mathcal{A}\varepsilon\left(\dot{u}_\rho(t)\right) - \mathcal{A}\varepsilon\left(\dot{u}(t)\right), \varepsilon(\dot{u}_\rho(t)) - \varepsilon\left(\dot{u}(t)\right)\right)_{\mathcal{H}} + \left(\mathcal{G}\varepsilon\left(u_\rho(t)\right) - \mathcal{G}\varepsilon\left(u(t)\right), \varepsilon(\dot{u}_\rho(t)) - \varepsilon\left(\dot{u}(t)\right)\right)_{\mathcal{H}} \\ &+ \quad \left(\int_0^t \left(M_\rho\left(t-s\right)\varepsilon\left(u_\rho(t)\right) - M\left(t-s\right)\varepsilon\left(u(t)\right)\right) ds, \varepsilon(\dot{u}_\rho(t)) - \varepsilon\left(\dot{u}(t)\right)\right)_{\mathcal{H}} \\ &- \quad \left(\mathcal{M}\theta_\rho(t) - \mathcal{M}\theta(t), \varepsilon(\dot{u}_\rho(t)) - \varepsilon\left(\dot{u}(t)\right)\right)_{\mathcal{H}} - j\left(\dot{u}(t), \dot{u}_\rho(t) - \dot{u}(t)\right) + j\left(\dot{u}_\rho(t), \dot{u}_\rho(t) - \dot{u}(t)\right) \\ &\qquad\qquad = \left(f_\rho(t) - f\left(t\right), \dot{u}_\rho(t) - \dot{u}(t)\right) .\end{aligned} \tag{5.8}$$

Moreover, from (3.15), it follows that for a.e. $t \in [0,T]$

$$\left(\mathcal{A}\varepsilon\left(\dot{u}_\rho(t)\right) - \mathcal{A}\varepsilon\left(\dot{u}(t)\right), \varepsilon(\dot{u}_\rho(t)) - \varepsilon\left(\dot{u}(t)\right)\right)_{\mathcal{H}} \geqslant m_{\mathcal{A}} \left\|\dot{u}_\rho(t) - \dot{u}(t)\right\|_V^2 . \tag{5.9}$$

Using (3.16), to obtain

$$-\left(\mathcal{G}\varepsilon\left(u_\rho(t)\right) - \mathcal{G}\varepsilon\left(u(t)\right), \varepsilon(\dot{u}_\rho(t)) - \varepsilon\left(\dot{u}(t)\right)\right)_{\mathcal{H}} \leqslant L_{\mathcal{G}} \left\|u_\rho(t) - u(t)\right\|_V \left\|\dot{u}_\rho(t) - \dot{u}(t)\right\|_V , \tag{5.10}$$

Now, we write

$$\begin{aligned}
&\left(\int_0^t \left(M_\rho\left(t-s\right)\varepsilon\left(u_\rho\right)-M\left(t-s\right)\varepsilon\left(u\right)\right)ds, \varepsilon(\dot{u}_\rho(t))-\varepsilon\left(\dot{u}(t)\right)\right)_{\mathcal{H}} \\
&=\left(\int_0^t M_\rho\left(t-s\right)\left(\varepsilon\left(u_\rho\right)-\varepsilon\left(u\right)\right)ds, \varepsilon(\dot{u}_\rho(t))-\varepsilon\left(\dot{u}(t)\right)\right)_{\mathcal{H}} \\
&+\left(\int_0^t \left(M_\rho\left(t-s\right)-M\left(t-s\right)\right)\varepsilon\left(u\right)ds, \varepsilon(\dot{u}_\rho(t))-\varepsilon\left(\dot{u}(t)\right)\right)_{\mathcal{H}}.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\left(\int_0^t \left(M_\rho\left(t-s\right)\varepsilon\left(u_\rho(t)\right)-M\left(t-s\right)\varepsilon\left(u(t)\right)\right)ds, \varepsilon(\dot{u}_\rho(t))-\varepsilon\left(\dot{u}(t)\right)\right)_{\mathcal{H}} \leqslant \\
&\max_{s\in[0,T]}\left\|M_\rho\left(s\right)\right\|_{\mathcal{H}_\infty}\left(\int_0^t \left\|u_\rho\left(s\right)-u\left(s\right)\right\|_V ds\right)\left\|\dot{u}_\rho(t)-\dot{u}(t)\right\|_V \\
&+\max_{s\in[0,T]}\left\|M_\rho\left(s\right)-M\left(s\right)\right\|_{\mathcal{H}_\infty}\left(\int_0^t \left\|u\left(s\right)\right\|_V ds\right)\left\|\dot{u}_\rho(t)-\dot{u}(t)\right\|_V.
\end{aligned} \tag{5.11}$$

We use the definition of j, we find

$$\begin{aligned}
&j\left(\dot{u}, \dot{u}_\rho-\dot{u}\right)-j\left(\dot{u}_\rho, \dot{u}_\rho-\dot{u}\right) \\
&=\int_{\Gamma_3}\beta\left|\dot{u}_\nu\right|\left(\dot{u}_{\rho\nu}-\dot{u}_\nu\right)da-\int_{\Gamma_3}\beta\left|\dot{u}_{\rho\nu}\right|\left(\dot{u}_{\rho\nu}-\dot{u}_\nu\right)da \\
&=\int_{\Gamma_3}\beta\left(\left|\dot{u}_\nu\right|-\left|\dot{u}_{\rho\nu}\right|\right)\left(\dot{u}_{\rho\nu}-\dot{u}_\nu\right)da \\
&\leqslant C_0^2\left\|\beta\right\|_{L^\infty(\Gamma_3)}\left\|\dot{u}_\rho-\dot{u}\right\|_V^2
\end{aligned} \tag{5.12}$$

Finally, we note that

$$\left(f_\rho(t)-f\left(t\right), \dot{u}_\rho(t)-\dot{u}(t)\right)\leqslant\delta\left(\rho\right)\left\|\dot{u}_\rho(t)-\dot{u}(t)\right\|_V, \tag{5.13}$$

where

$$\delta\left(\rho\right)=\max_{t\in[0,T]}\left\|f_\rho\left(t\right)-f\left(t\right)\right\|_V \tag{5.14}$$

Substituting $(5.9), (5.10), (5.11), (5.12)$ and (5.13) in (5.8), we obtain

$$\begin{aligned}
\left(m_{\mathcal{A}}-C_0^2\left\|\beta\right\|_{L^\infty(\Gamma_3)}\right)\left\|\dot{u}_\rho(t)-\dot{u}(t)\right\|_V \leqslant\ & L_{\mathcal{G}}\left\|u_\rho(t)-u(t)\right\|_V \\
&+\max_{s\in[0,T]}\left\|M_\rho\left(s\right)\right\|_{\mathcal{H}_\infty}\left(\int_0^t \left\|u_\rho\left(s\right)-u\left(s\right)\right\|_V ds\right) \\
&+\max_{s\in[0,T]}\left\|M_\rho\left(s\right)-M\left(s\right)\right\|_{\mathcal{H}_\infty}\left(\int_0^t \left\|u\left(s\right)\right\|_V ds\right) \\
&+c\left\|\theta_\rho(t)-\theta(t)\right\|_{L^2(\Omega)}+\delta\left(\rho\right).
\end{aligned}$$

From $(3.26), (5,2)$ and $(3,20)$, we have

$$\left\|\theta_\rho(t)-\theta(t)\right\|_{L^2(\Omega)}^2\leqslant C\int_0^t\left\|\dot{u}_\rho(s)-\dot{u}(s)\right\|_V^2 ds. \tag{5.15}$$

Then

$$\begin{aligned}
\left(m_{\mathcal{A}}-C_0^2\left\|\beta\right\|_{L^\infty(\Gamma_3)}\right)\left\|\dot{u}_\rho(t)-\dot{u}(t)\right\|_V \leqslant\ & d\left(\left\|u_\rho(t)-u(t)\right\|_V+\int_0^t \left\|u_\rho\left(s\right)-u\left(s\right)\right\|_V ds\right) \\
&+\max_{s\in[0,T]}\left\|M_\rho\left(s\right)-M\left(s\right)\right\|_{\mathcal{H}_\infty}\left(\int_0^t \left\|u\left(s\right)\right\|_V ds\right) \\
&+C\int_0^t\left\|\dot{u}_\rho(s)-\dot{u}(s)\right\|_V ds+\delta\left(\rho\right).
\end{aligned}$$

We use the notation

$$\zeta\left(\rho\right)=\max_{s\in[0,T]}\left\|M_\rho\left(s\right)-M\left(s\right)\right\|_{\mathcal{H}_\infty}\int_0^T\left\|u\left(s\right)\right\|_V ds. \tag{5.16}$$

So

$$\begin{aligned}\left\|\dot{u}_\rho(t)-\dot{u}(t)\right\|_V \leqslant & \tfrac{\zeta(\rho)}{m_{\mathcal{A}}-C_0^2\|\beta\|_{L^\infty(\Gamma_3)}}+\tfrac{\delta(\rho)}{m_{\mathcal{A}}-C_0^2\|\beta\|_{L^\infty(\Gamma_3)}}\\ &+\tfrac{C}{m_{\mathcal{A}}-C_0^2\|\beta\|_{L^\infty(\Gamma_3)}}\int_0^t\left\|\dot{u}_\rho(s)-\dot{u}(s)\right\|_V ds,\end{aligned}$$

this inequality implies that

$$\begin{aligned}\left\|\dot{u}_\rho-\dot{u}\right\|_V \leqslant & \frac{1}{m_{\mathcal{A}}-C_0^2\|\beta\|_{L^\infty(\Gamma_3)}}\left(\zeta(\rho)+\delta(\rho)\right)\\ &+\frac{C}{m_{\mathcal{A}}-C_0^2\|\beta\|_{L^\infty(\Gamma_3)}}\int_0^t\left\|\dot{u}_\rho(s)-\dot{u}(s)\right\|_V ds.\end{aligned}$$

Using a Gronwall inequality, we find

$$\left\|\dot{u}_\rho-\dot{u}\right\|_V \leqslant c\left(\zeta(\rho)+\delta(\rho)\right). \tag{5.17}$$

We integrate (5.17) over $[0,t]$, using $(4.14),(3.27)$ and (5.3) we get

$$\left\|u_\rho-u\right\|_V \leqslant c\int_0^t\left\|\dot{u}_\rho(s)-\dot{u}(s)\right\|_V ds \leqslant c\left(\zeta(\rho)+\delta(\rho)\right). \tag{5.18}$$

The assumptions $(5.5),(5.4)$ and the definitions $(5.14),(5.16)$ imply

$$\zeta(\rho)\to 0,\qquad \delta(\rho)\to 0\quad \text{as } \rho\to 0. \tag{5.19}$$

We result from (5.18) and (5.19) that (5.6) is satisfied.
We conclude that (5.7) is a consequence of $(5.18),(5.15)$ and (5.19). □

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