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Specializability of Continued Fractions of Infinite Series Involving Some Recurrence Sequences

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ABSTRACT: In this paper, we describe the continued fraction expansion for the series $\sum_k T^{-a(k)}$, where $(a(n))_n$ is the A248098 sequence, or the A213967 sequence in the OEIS, defined by the same recurrence formula $a(n) = 1 + a(n-1) + a(n-2) + a(n-3)$ for $n \geq 3$ with different initial conditions. We prove that these two series are specializable and converge to transcendental numbers when we specialize T by an even integer ≥ 4 for the first one and by any integer ≥ 2 for the second one. We also describe the continued fraction expansion for the series $\sum_k T^{-\mathcal{L}_{2,k}}$, where $(\mathcal{L}_{2,k})_k$ is the sequence of modified Leonardo 2-number. We prove that it is specializable and converges to a transcendental number when we specialize T by an integer ≥ 2. We then show that the arguments suggested by van der Poorten and Shallit in [9], that the phenomenon of specializability for series of the kind that they studied, is not reserved to just this kind, but it is included in a large class of series.

Keywords: Continued fraction, Folding Lemma.

Contents

1. Introduction

The continued fraction expansion of a real number α is a very efficient process for finding the best rational approximations of α. Moreover, continued fractions are a very versatile tool for solving problems related to the theoretical questions of number theory, complex analysis and dynamical systems.

Every irrational number or irrational formal power series can be expanded as an infinite continued fraction $\alpha = [a_0, a_1, a_2, \ldots]$ where the partial quotients a_n are positive integers or polynomials in T of positive degree (except perhaps for a_0). The rationals $p_k/q_k = [a_0, a_1, a_2, \ldots, a_n]$ are called the convergents of α. The partial quotients a_n are obtained by the continued fraction algorithm (extracting the integer part of the complete quotient), which never terminates due to the irrationality of the element. For $k \geq 0$, the integers (or the polynomials) p_k and q_k are defined by the recurrence formulas:

$$\begin{aligned} p_k &= a_k p_{k-1} + p_{k-2} \ (k \geq 1), \\ q_k &= a_k q_{k-1} + q_{k-2} \ (k \geq 1), \end{aligned} \tag{1.1}$$

with $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = a_0$ and $q_0 = 1$. Moreover

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^{k+1}, \quad for\ k \geq 1. \tag{1.2}$$

Further, we have the following interesting property of continued fraction

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1} q_n^2}. \tag{1.3}$$

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The book by Perron [8] reveals a lot of fundamental continued fraction properties to the reader.
Once see easily that if the partial quotients a_k are polynomials of degree at least 1 then the convergents p_k/q_k converge to a formal series α in T^{-1}, see [6]. For this case, we define the non-Archimedean absolute value by $|\alpha| = |T|^{-t}$, where $|T| > 1$ a fixed real number. Thus for a rational function P/Q, $|\frac{P}{Q}| = |T|^{\deg P - \deg Q}$ and $|0| = 0$. The non-Archimedean absolute value yields to the following improvement of the inequality (1.3)

$$\left|\alpha - \frac{P_n}{Q_n}\right| = \frac{1}{|a_{n+1}||Q_n|^2}, \tag{1.4}$$

where the rational function $\frac{P_n}{Q_n} = [a_0, a_1, a_2, \dots, a_n]$ is the sequence of convergent of α. In addition, if the partial quotients a_n in the continued fraction of α are polynomials in $\mathbb{Z}[T]$, then in the terminology of [9], the continued fraction is specializable. We can specialize T to any positive integer greater than 1, and get a simple continued fraction.

The foundation of our subsequent observations is the following practical lemma.

Lemma 1.1 *(Folding Lemma) Let y be an integer and $\frac{p_n}{q_n} = [a_0, a_1, \dots, a_n]$. Then*

$$\frac{p_n}{q_n} + \frac{(-1)^n}{yq_n^2} = \left[a_0, \overrightarrow{w}, y - \frac{q_{n-1}}{q_n}\right] = \left[a_0, \overrightarrow{w}, y, -\overleftarrow{w}\right]. \tag{1.5}$$

Here $\overrightarrow{w}$ is a convenient abbreviation for the word $a_1, a_2, \cdots, a_n$ and, accordingly, $-\overleftarrow{w}$ denotes the word $-a_n, -a_{n-1}, \cdots, -a_1$.

The proof of this lemma, appeared initially in the paper [5], can be found in many other papers. In fact, this lemma is the main tool to obtain the continued fraction of many series as $\sum_n T^{-F_n}$ [9], $\sum_n T^{-2^n}$ and $\sum_n T^{-2^n-1}$ [3], $T\sum_n T^{-F_n}$ and $T\sum_n T^{-L_n}$ [2], $\sum_n T^{-\mathcal{L}_n}$ [1], $\sum_{n=0}^{\infty} \frac{1}{f^n(T)}$ [4] (F_n, L_n and $\mathcal{L}_n$ are respectively the well known Fibonacci, Lucas and Leonardo numbers, and f^n stands for the n^{th} iterate of a polynomial f). Nevertheless the application is not as simple as one might think. We can directly apply this lemma for the series of the form $\sum_n T^{-u_n}$, when $u_{n+1} \geq 2u_n$, but actually, it is not so for the case $1 < u_{n+1} < 2u_n$, see [6,7].
Our significant goal is to identify series that are distinguished by the fact that only rational integer coefficients occur in their partial quotients. In that particular instance, the variable T may be substituted with an integer ≥ 2; in other words, one may "specialise" and as a result, achieve the series 'values' consistent, continued fraction expansion. It is important to note that it is typically challenging to obtain the explicit continued fraction expansion of a number that is presented in a different shape. It may happen that some partial quotients belong to $\mathbb{Z}_{<0}[T]$, then when we specialise, we obtain negative partial quotients. To fix this problem, it suffice to apply the following properties of continued fraction of a real number

$$[\dots, a, 0, b, \dots] = [\dots, a + b, \dots] \tag{1.6}$$

$$[\dots, a, -b, c, \dots] = [\dots, a - 1, 1, b - 2, 1, c - 1, \dots] \tag{1.7}$$

where a, b and c are positive integers.

We note for the reader that we used the Pari/gp program for the first computation of the sums, which, although it seems long in certain cases, are necessary to understand the regularity of the partial quotients.

2. Observations

In [9], the authors have stated the following conjecture.

Conjecture 1 *Let $a(n)$ be an increasing sequence of nonnegative integers satisfying a recurrence relation*

$$a(h+d) = a(h+d-1) + a(h+d-2) + \cdots + a(h) \quad \textit{with} \quad d > 1,$$

and set

$$s_n = T^{-a(d)} + T^{-a(d+1)} + \cdots + T^{-a(n)}; s_n = [0, t_n].$$

Then, subject to appropriate initial conditions on the $a(h)$, the words t_h consist of polynomials with integer coefficients, which is to say that s_∞ has a specializable continued fraction expansion.

They have also pointed out that the valuable identities $2a(h) = a(h+1) + a(h-d)$ and $a(h-2) + a(h-1) \le a(h)$ are the main supports to the conjecture. Moreover, the experimental data shows that for small $d = 3, 4, 5, 6, \ldots$ and initial values $0, \ldots, 0, 1$ the commencing partial quotients are specializable.

We are interested here to $d = 3$ which corresponds to Tribonacci sequence. The Tribonacci sequence are defined in On-Line Encyclopedia of Integer Sequences (the OEIS), for $n \ge 3$, by the same recurrence formula $\mathcal{T}(n) = \mathcal{T}(n-1) + \mathcal{T}(n-2) + \mathcal{T}(n-3)$ with different initial conditions for $n \le 3$.

Consider the sequence A000073 in OEIS, of Tribonacci numbers with initial condition $\mathcal{T}(0) = \mathcal{T}(1) = 0$ and $\mathcal{T}(2) = 1$. The computation of the commencing partial quotients (at the stage where regularity begins) gives

$$\begin{aligned}\sum_{k=3}^{12} T^{-\mathcal{T}(k)} = [0, &T-1, T+2, T, -T^2+T-2, -T^2-T, -T+2, -T,\\
&-T^2-T-2, T^4-T^2-T+1, T^2, T^4+2T, -T^5+T^2, -T^7+T^2-T,\\
&-T^3, -T^2+T-1, T+1, T^9-T^3, -T, -T^{11}+T^2-1, T, T^5, -T,\\
&-T^2+1, T+1, T, T-1, T^{16}-T^5, -T+1, -T-1, T^{20}-T^3+1,\\
&T+1, T-1, T^{10}, -T+1, -T-1, T^3-1, T+1, -T^2+T-2,\\
&-T^2+1, -T^{30}+T^{10}, T^2-1, T^2+1, -T^{36}+T^5, -T+1, -T,\\
&-T-2, -T+1, -T^{18}, T-1, T+2, T, T-1, -T^5, -T^2-1,\\
&-T^2-T, -T+2, -T, -T^2-T-2, -T^2+1, T^{18}, T^2-1,\\
&T^2+T+2, T, T-2, T^2+T, T^2-T+2, -T, -T-2, -T+1]\end{aligned} \tag{2.1}$$

which is specializable.

However, for the sequence A000213 in OEIS, of Tribonacci numbers with initial condition $\mathcal{T}(0) = \mathcal{T}(1) = \mathcal{T}(2) = 1$, and with first terms $1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, 355$, the computation of the beginning partial quotients gives

$$\begin{aligned}\sum_{k=2}^{10} T^{-\mathcal{T}(k)} = \big[0, &T, -T, -T^3-2T, -T, -T^3, -T, \tfrac{1}{2}T, 4T, -\tfrac{1}{2}T,\\
&-T, T^5-T^3, T^3, T^5+2T, -T^7+T^3, -T^9+T^3-T, -T^3, -T^3+T,\\
&-T, -T, -T^{11}+T^3, T, T^{15}-T^3+T, -T, -T^7, T, T^3-T, -T^3-T,\\
&-T^{21}+T^7, T^3, T^{25}-T^3, T, T, -T, -T^{13}, T, -T, -T, T^3, -T^3-T,\\
&-T, -T^3+T, T, T^{13}, T^3-T, T, T^3+2T, T, -T\big].\end{aligned} \tag{2.2}$$

The surprising fact here is the appearance of the partial quotient $\frac{1}{2}T$. This improve the effect of the initial conditions in the kind of first partial quotients. Remark that here we shall consider $\sum_{k=2}^{10}(2T)^{-\mathcal{T}(k)}$ to obtain specialized continued fraction expansion.

An other surprising situation for the sequence A081172 in OEIS, of Tribonacci numbers with initial condition $\mathcal{T}(0) = 1, \mathcal{T}(1) = 1, \mathcal{T}(2) = 0$ and first beginning terms $1, 1, 0, 2, 3, 5, 10, 18, 33, 61, 112, 206, 379$. Our first computation gives that

$$\sum_{k=3}^{7} T^{-\mathcal{T}(k)} = [0, T^2 - T + 1, -\frac{1}{2}T - \frac{3}{4}, -8T + 12, -\frac{1}{16}T, -16T - 32, -\frac{1}{32}T + \frac{3}{64}, -128T - 192, -\frac{1}{256}T^3 + \frac{1}{256}T^2 + \frac{1}{256}, -256T, -\frac{1}{256}T - \frac{1}{256}6, -256T + 256, -\frac{1}{256}T, -256T^3 + 256T].$$

However, when we add $T^{-\mathcal{T}(1)}$, we obtain the following expansion.

$$T^{-1} + \sum_{k=3}^{12} T^{-\mathcal{T}(k)} = \big[0, T - 1, T^2 + 2T + 2, -T^2 + T - 2, -T^2 + T, T^2 + 2T + 1, -T + 1, -T^3 - T + 1, -T^3 - T - 1, T^3 - T, T + 1, T - 1, T^3 - 1, -T, -T, T + 1, T - 2, T, T + 1, T^6 - T, -T, -T^8 + T - 1, T, T^4, -T, -T + 1, T^2 + T, T^{12} - T^4, -T^2, -T^{14} + T, -T - 1, -T + 1, -T^8, T - 1, T + 1, -T, T^2 + T + 1, -T^2 + T - 1, T^8, T^2 - T + 1, -T^2 - 2T - 2, -T + 1\big]. \tag{2.3}$$

This improve again that the initial condition of the sequence determines the nature of the expansion and that it is not necessary to be $0, \ldots, 0, 1$.

The continued fraction expansions of the three series related to the sums (2.1), (2.2) and (2.3) above are fully described in [9]. Similar proof will fairly closely used to the infinite series that will be studied in the following section.

3. Continued Fraction of Series Involving A248098 Numbers

The sequence A248098 is defined by

$$a(n) = \begin{cases} 1, \ if \ n \leq 3 \\ a(n) = 1 + a(n-1) + a(n-2) + a(n-3), \ for \ all \ n \geq 4 \end{cases}$$

The first value of this sequence are:

$$1, 1, 1, 4, 7, 13, 25, 46, 85, 157, 289, 532, 979, 1801, 3313, 6094, \ldots$$

Set $s_n = T^{-1} + T^{-4} + T^{-7} + T^{-13} + T^{-25} + \cdots + T^{-a(n)}$. Computation gives that

$$\begin{aligned}
s_3 &= [0, T] \\
s_4 &= [0, T, -T^2, -T] \\
s_5 &= [0, T, -T^2, -T^4 - T] \\
s_6 &= [0, T, -T^2, -T^4 - 2T, -T^2, -T^4 + T] \\
s_7 &= \big[0, T, -T^2, -T^4 - 2T, -T^2, -T^4, -T^2, \frac{1}{2}T, 4T^2, -\frac{1}{2}T, -T^2, -T^4 + T\big] \\
s_8 &= \big[0, T, -T^2, -T^4 - 2T, -T^2, -T^4, -T^2, \frac{1}{2}T, 4T^2, -\frac{1}{2}T, -T^2, T^7 - T^4, T^5, T^7 + 2T, T^5, -T\big] \\
s_9 &= \big[0, T, -T^2, -T^4 - 2T, -T^2, -T^4, -T^2, \frac{1}{2}T, 4T^2, -\frac{1}{2}T, -T^2, T^7 - T^4, T^5, T^7 + 2T, -T^{11} + T^5, -T^{13} + T^4 - T, -T^5, -T^4 + T, -T^2, -T, T^5, T, T^2, -T\big]
\end{aligned}$$

$$\begin{aligned}
s_{10} =&[0, T, -T^2, -T^4 - 2T, -T^2, -T^4, -T^2, \frac{1}{2}T, 4T^2, -\frac{1}{2}T, -T^2, T^7 - T^4, T^5, T^7 + 2T,\\
&-T^{11} + T^5, -T^{13} + T^4 - T, -T^5, -T^4 + T, -T^2, -T, -T^{17} + T^5, T, T^{23} - T^5 + T^2,\\
&-T, -T^{11}, T, T^5 - T^2, -T^4 - T, T^{11}, T^4 + T, T^2, -T]\\
s_{11} =&[0, T, -T^2, -T^4 - 2T, -T^2, -T^4, -T^2, \frac{1}{2}T, 4T^2, -\frac{1}{2}T, -T^2, T^7 - T^4, T^5, T^7 + 2T,\\
&-T^{11} + T^5, -T^{13} + T^4 - T, -T^5, -T^4 + T, -T^2, -T, -T^{17} + T^5, T, T^{23} - T^5 + T^2,\\
&-T, -T^{11}, T, T^5 - T^2, -T^4 - T, -T^{32} + T^{11}, T^4, T^{38} - T^5, T, T^2, -T, -T^{20}, T, -T^2,\\
&-T, T^5, -T^4 - T, -T^2, -T^4 + T, T^{20}, T^4 - T, T^2, T^4 + 2T, T^2, -T]\\
s_{12} =&[0, T, -T^2, -T^4 - 2T, -T^2, -T^4, -T^2, \frac{1}{2}T, 4T^2, -\frac{1}{2}T, -T^2, T^7 - T^4, T^5,\\
&T^7 + 2T, -T^{11} + T^5, -T^{13} + T^4 - T, -T^5, -T^4 + T, -T^2, -T, -T^{17} + T^5, T,\\
&T^{23} - T^5 + T^2, -T, -T^{11}, T, T^5 - T^2, -T^4 - T, -T^{32} + T^{11}, T^4, T^{38} - T^5, T,\\
&T^2, -T, -T^{20}, T, -T^2, -T, T^5, -T^4 - T, -T^2, -T^4 + T, -T^{59} + T^{20}, T^4 - T,\\
&T^2, T, T^{71} - T^{11}, T^4 + T, T^2, -T, -T^{35}, T, -T^2, -T^4 - T, T^{11}, -T, -T^2, -T^4,\\
&-T^2, \frac{1}{2}T, 4T^2, -\frac{1}{2}T, -T^2, -T^4 + T, T^{35}, T^4 - T, T^2, \frac{1}{2}T, -4T^2, -\frac{1}{2}T, T^2, T^4,\\
&T^2, T^4 + 2T, T^2, -T]\\
s_{13} =&[0, T, -T^2, -T^4 - 2T, -T^2, -T^4, -T^2, \frac{1}{2}T, 4T^2, -\frac{1}{2}T, -T^2, T^7 - T^4, T^5,\\
&T^7 + 2T, -T^{11} + T^5, -T^{13} + T^4 - T, -T^5, -T^4 + T, -T^2, -T, -T^{17} + T^5, T,\\
&T^{23} - T^5 + T^2, -T, -T^{11}, T, T^5 - T^2, -T^4 - T, -T^{32} + T^{11}, T^4, T^{38} - T^5, T,\\
&T^2, -T, -T^{20}, T, -T^2, -T, T^5, -T^4 - T, -T^2, -T^4 + T, -T^{59} + T^{20}, T^4 - T,\\
&T^2, T, T^{71} - T^{11}, T^4 + T, T^2, -T, -T^{35}, , -T^2, -T^4 - T, T^{11}, -T, -T^2, -T^4,\\
&-T^2, \frac{1}{2}T, 4T^2, -\frac{1}{2}T, -T^2, -T^4 + T, -T^{107} + T^{35}, T^4 - T, T^2, \frac{1}{2}T, -4T^2,\\
&-\frac{1}{2}T, T^2, T, T^{131} - T^{20}, T^4 - T, T^2, T^4 + 2T, T^2, -T, -T^{65}, T, -T^2, -T^4 - 2T,\\
&-T^2, -T^4 + T, T^{20}, -T, -T^2, \frac{1}{2}T, 4T^2, -\frac{1}{2}T, -T^2, T^7 - T^4, T^5, T^7 + 2T, T^5,\\
&-T, T^{65}, T, -T^5, -T^7 - 2T, -T^5, -T^7 + T^4, T^2, \frac{1}{2}T, -4T^2, -\frac{1}{2}T, T^2, T^4,\\
&T^2, T^4 + 2T, T^2, -T]\\
s_{14} =&[0, T, -T^2, -T^4 - 2T, -T^2, -T^4, -T^2, \frac{1}{2}T, 4T^2, -\frac{1}{2}T, -T^2, T^7 - T^4, T^5,\\
&T^7 + 2T, -T^{11} + T^5, -T^{13} + T^4 - T, -T^5, -T^4 + T, -T^2, -T, -T^{17} + T^5, T,\\
&T^{23} - T^5 + T^2, -T, -T^{11}, T, T^5 - T^2, -T^4 - T, -T^{32} + T^{11}, T^4, T^{38} - T^5, T, T^2,\\
&-T, -T^{20}, T, -T^2, -T, T^5, -T^4 - T, -T^2, -T^4 + T, -T^{59} + T^{20}, T^4 - T, T^2, T,\\
&T^{71} - T^{11}, T^4 + T, T^2, -T, -T^{35}, T, -T^2, -T^4 - T, T^{11}, -T, -T^2, -T^4, -T^2, \frac{1}{2}T,\\
&4T^2, -\frac{1}{2}T, -T^2, -T^4 + T, -T^{107} + T^{35}, T^4 - T, T^2, \frac{1}{2}T, -4T^2, -\frac{1}{2}T, T^2, T,\\
&T^{131} - T^{20}, T^4 - T, T^2, T^4 + 2T, T^2, -T, -T^{65}, T, -T^2, -T^4 - 2T, -T^2, -T^4 + T,\\
&T^{20}, -T, -T^2, \frac{1}{2}T, 4T^2, -\frac{1}{2}T, -T^2, T^7 - T^4, T^5, T^7 + 2T, T^5, -T, -T^{197} + T^{65},
\end{aligned}$$

$$T, -T^5, -T^7 - 2T, -T^5, -T^7 + T, T^{239} - T^{35}, T^4 - T, T^2, \frac{1}{2}T, -4T^2, -\frac{1}{2}T, T^2,$$
$$T^4, T^2, T^4 + 2T, T^2, -T, -T^{119}, T, -T^2, -T^4 - 2T, -T^2, -T^4, -T^2, \frac{1}{2}T, 4T^2,$$
$$-\frac{1}{2}T, -T^2, -T^4 + T, T^{35}, T^7 - T, T^5, T^7 + 2T, -T^{11} + T^5, -T^{13} + T^4 - T,$$
$$-T^5, -T^4 + T, -T^2, -T, T^5, T, T^2, -T, T^{119}, T, -T^2, -T, -T^5, T, T^2, T^4 - T, T^5,$$
$$T^{13} - T^4 + T, T^{11} - T^5, -T^7 - 2T, -T^5, -T^7 + T^4, T^2, \frac{1}{2}T, -4T^2, -\frac{1}{2}T, T^2, T^4,$$
$$T^2, T^4 + 2T, T^2, -T].$$

It is noted that

$$s_{11} = s_{10}, s_5, -T^{32}, -\overleftarrow{s_4}, s_4, T^{38}, -\overleftarrow{s_4}, s_4, -T^5, -\overleftarrow{s_4}, -T^{20}, s_5, T^5, -\overleftarrow{s_4}, s_6, T^{20}, -\overleftarrow{s_6}$$
$$s_{12} = s_{11}, s_6, -T^{59}, -\overleftarrow{s_6}, s_5, T^{71}, -\overleftarrow{s_5}, s_5, -T^{11}, -\overleftarrow{s_5}, -T^{35}, s_5, T^{11}, -\overleftarrow{s_5}, s_7, T^{35}, -\overleftarrow{s_7}$$
$$s_{13} = s_{12}, s_7, -T^{107}, -\overleftarrow{s_7}, s_6, T^{131}, -\overleftarrow{s_6}, s_6, -T^{20}, -\overleftarrow{s_6}, -T^{65}, s_6, T^{20}, -\overleftarrow{s_6}, s_8, T^{65}, -\overleftarrow{s_8}$$
$$s_{14} = s_{13}, s_8, -T^{197}, -\overleftarrow{s_8}, s_7, T^{239}, -\overleftarrow{s_7}, s_7, -T^{35}, -\overleftarrow{s_7}, -T^{119}, s_7, T^{35}, -\overleftarrow{s_7}, s_9, T^{119}, -\overleftarrow{s_9}$$

We can show the following conclusion thanks to these initial findings:

Theorem 3.1 *Set for $h \geq 3$*

$$s_h = T^{-1} + T^{-4} + T^{-7} + T^{-13} + \cdots + T^{-a(h)} = [0, f_h].$$

Then the words f_h, $3 \leq h \leq 10$, are given by $s_h = [0, f_h]$ as listed above. Let for $h \geq 11$

$$f_{h+1} = 0, f_h, 0, f_{h-5}, -T^{a(h-2)-2a(h-5)}, -\overleftarrow{f_{h-5}}, 0, f_{h-6}, -\overleftarrow{f_{h-6}}, -T^{a(h-2)-2a(h-4)}, f_{h-6},$$
$$T^{a(h-4)-2a(h-6)}, -T^{a(h-2)-2a(h-6)}, -\overleftarrow{f_{h-6}}, 0, f_{h-4}, T^{a(h-2)-2a(h-4)}, -\overleftarrow{f_{h-4}}.$$

Then

$$s_\infty = \sum_{n=3}^{\infty} T^{-a(n)} = \lim_{h\to\infty} [0, f_h] \tag{3.1}$$

Proof: The proof is induction on h. $h = 13$ is the base case. It is assumed that the length of s_h is odd, as it does appear to be for $h \geq 8$ (this will also be established during the proof). We have that $s_h = [0, f_h] = p/q$ and the facts point to the polynomial q being of type $T^{-a(h)}$. Noting that $a(h+1) - 2a(h) = -a(h-3)$, so

$$s_{h+1} = s_h + T^{-a(h+1)} = \frac{p}{q} - \frac{1}{-T^{-a(h-3)}q^2}$$

and the Folding lemma yields

$$s_{h+1} = [0, f_h, -T^{-a(h-3)}, -\overleftarrow{f_h}]$$

which is an illegal expansion. We continue to hold out hope that the Folding Lemma will get the result we desire. Obviously,

$$s_{h+1} = s_h + T^{-a(h+1)} = \frac{p}{q} - \frac{1}{-T^{a(h-3)}q^2} = [0, f_h, -T^{-a(h-3)} - q'/q], \tag{3.2}$$

where q' is the denominator of the next-to-last convergent to s_h, and we can take $q = T^{-a(h)}$ as previously mentioned. As p'/q' is the penultimate convergent of the last convergent, p/q, we know that $pq' - p'q = 1$, which results in

$$s_h(q'/q) = 1/T^{2a(h)} + (p'/q) \tag{3.3}$$

The most plausible assumption for our computation is that

$$-(s_{h-1}+T^{-a(h)})(s_{h-1}-T^{-a(h)}-T^{-a(h-4)})$$

is of the shape (3.3), given the presence of $T^{-2a(h)}$ and $T^{-a(h+1)}$ to the right of (3.3).

Thus

$$q'/q=-(s_{h-1}-T^{-a(h)}-T^{-a(h-4)}). \tag{3.4}$$

So $-T^{-a(h-3)}-q'/q=s_{h-1}-T^{-a(h)}-T^{a(h-3)}-T^{-a(h-4)}=s_{h-5}+T^{-a(h-2)}+T^{-a(h-1)}-T^{-a(h)}$. But

$$s_{h-5}+T^{-a(h-2)}=[0,f_{h-5},-T^{a(h-2)-2a(h-5)},-\overleftarrow{f_{h-5}}],$$

by the application of the Folding lemma.

Now note that $s_{h-5}+T^{-a(h-2)}-T^{-a(h-5)}=s_{h-6}+T^{a(h-2)}$ and its expansion is

$$[0,f_{h-6},T^{a(h-2)-2a(h-6)},-\overleftarrow{f_{h-6}}].$$

Thankfully, by using the formula

$$-(s_{h-5}+T^{-a(h-2)}+T^{-a(h-1)})(s_{h-6}+T^{a(h-2)}-T^{a(h-1)})=T^{-2a(h-1)}+p'/q,$$

, with the operative observation that $2a(h-2)=a(h-1)+a(h-5)$, it only requires appending the expansion of

$$s_{h-6}+T^{-a(h-2)}-T^{-a(h-1)}+T^{-a(h-4)},$$

since $a(h)-2a(h-1)=-a(h-4)$. But

$$\begin{aligned}s_{h-6}+T^{-a(h-4)}+T^{-a(h-2)}=&[0,f_{h-6},-T^{a(h-4)-2a(h-6)},-\overleftarrow{f_{h-6}},\\&-T^{a(h-2)-2a(h-4)},f_{h-6},T^{a(h-4)-2a(h-6)},-\overleftarrow{f_{h-6}}].\end{aligned}$$

Now by

$$-(s_{h-6}+T^{-a(h-4)}+T^{-a(h-2)})(s_{h-6}+T^{-a(h-4)}-T^{-a(h-2)})=T^{-2a(h-2)}+p'/q$$

and $a(h-1)-2a(h-2)=-a(h-5)$ we have only to append the continued fraction expansion of

$$s_{h-6}+T^{-a(h-4)}-T^{-a(h-2)}+T^{a(h-5)}=s_{h-4}-T^{-a(h-2)}$$

Then

$$\begin{aligned}s_{h+1}=&[0,f_h,0,f_{h-5},-T^{a(h-2)-2a(h-5)},-\overleftarrow{f_{h-5}},0,f_{h-6},-T^{a(h-4)-2a(h-6)},\\&-\overleftarrow{f_{h-6}},-T^{a(h-2)-2a(h-4)},f_{h-6},T^{a(h-4)-2a(h-6)},-\overleftarrow{f_{h-6}},0,f_{h-4},\\&T^{a(h-2)-2a(h-4)},-\overleftarrow{f_{h-4}}].\end{aligned}$$

□

Corollary 3.1 *The series*

$$\mathcal{S}_\infty=\sum_{n=3}^{\infty}(2T)^{-a(n)}$$

is specializable.

Proof: We replace T by $2T$ in the continued fraction $[0, f_h]$. The partial quotient $\frac{1}{2}T$ changes to T. Hence, all the coefficients of the partial quotients belongs to $\mathbb{Z}$. □

Corollary 3.2
$4^{-1}+4^{-4}+4^{-7}+4^{-13}+\cdots+4^{-a(h)}+\cdots = [0,5,6,2,1,120,1,29,2,1,2,1016,5,2,1,32751,2,4,$
$33554303,1,136,\cdots]$.

Proof: We have that
$\mathcal{S}_\infty = [0, 2T, -4T^2, -16T^4-4T, -4T^2, -16T^4, -4T^2, T, 16T^2, -T, -4T^2, 128T^7-16T^4, 32T^5,$
$128T^7+4T, -2048T^{11}+32T^5, \ldots]$
We replace T by 2 and we apply the identities (1.6) and (1.7) we obtain

$$\begin{aligned}\mathcal{S}_\infty &= [0,4,-16,-264,-16,-256,-16,2,64,-2,-16,16128,1024,16392,-4193280,\ldots]\\ &= [0,3,1,15,264,16,256,15,1,1,63,1,1,15,1,16127,1024,\ldots].\end{aligned}$$

□

Proposition 3.1 *For any even $x>2$, the number $\sum_{n=3}^{\infty} x^{-a(n)}$ is transcendental.*

Proof: We have that the product

$$(T^{-a(h-3)}+T^{-a(h-2)}+T^{-a(h-1)}-T^{-a(h)})s_\infty$$

consists of terms with exponents $\geq -a(h-5)-a(h)$, there then is a gap, and the remaining terms have degree $\leq -a(h-2)-a(h+1)$. According we set

$$q_h = T^{a(h)+a(h-5)}(T^{-a(h-3)}+T^{-a(h-2)}+T^{-a(h-1)}-T^{-a(h)})$$

and note that $\deg q_h = a(h)+a(h-5)-a(h-3)$. Thus, there is a polynomial p_h so that

$$\begin{aligned}\deg(q_h s_\infty - p_h) &= -((a(h-2)+a(h+1))-(a(h-5)+a(h)))\\ &= -\deg q_h - ((a(h-2)+a(h+1))-(2a(h-5)+2a(h)-a(h-3)))\\ &= -\deg q_h - (a(h-2)-2a(h-5)).\end{aligned}$$

Hence p_h/q_h is a convergent to s_∞ and the next partial quotient has degree $a(h-2)-2a(h-5)$. As $h\to+\infty$, we have

$$(a(h-2)-2a(h-5))/(a(h)+a(h-5)-a(h-3)) \to (\tau^3-2)/(\tau^5-\tau^2+1) =$$

$$\lim_{n\to+\infty}\frac{\mathcal{T}(n+1)}{\mathcal{T}(n)} = \lim_{n\to+\infty}\frac{a(n+1)}{a(n)} = \tau \approx 1.8392867552141,$$

the Tribonacci constant. It follows that on specializing T to x we have a sequence of rational approximations $p_h(x)/q_h(x)$ so that for sufficiently large h

$$\left|\sum_{n=3}^{\infty} x^{-a(n)} - p_h(x)/q_h(x)\right| < q_h(x)^{-3.839},$$

proving our assertion by Roth's Theorem [11]. □

Consider also the sequence A213967 which is defined by

$$a(n) = \begin{cases} n, \ if\ n\leq 3\\ a(n) = 1+a(n-1)+a(n-2)+a(n-3), \ for\ all\ n\geq 4\end{cases}$$

The first value of this sequence are $0,1,2,3,7,13,24,45,83,153,282,519,955,1757,3232,5945,\ldots$.

Set $s_n = \sum_{i=1}^{n} T^{-a(i)} = T^{-1} + T^{-2} + T^{-3} + T^{-7} + T^{-13} + \cdots + T^{-a(n)}$. Computation gives that

$\mathbf{s_1} = [0, T]$

$\mathbf{s_2} = [0, T-1, T+1]$

$\mathbf{s_3} = [0, T-1, T^2+T+1]$

$\mathbf{s_4} = [0, T-1, T^2+T+1, T, -T^2-T-1, -T+1]$

$\mathbf{s_5} = [0, T-1, T^2+T+1, T, -T^2-T-1, T^2-2T+2, -T-1, -T, T+1, -T^2+T-1]$

$\mathbf{s_6} = [0, T-1, T^2+T+1, T, -T^2-T-1, T^2-2T+2, -T-1, -T, T+1, T^3-T^2+T-1, -T, -T^3+T, -T, -T, T, T, T, -T]$

$\mathbf{s_7} = [0, T-1, T^2+T+1, T, -T^2-T-1, T^2-2T+2, -T-1, -T, T+1, T^3-T^2+T-1, -T, -T^3+T, -T, -T, T, T, -T^5+T, -T^7+T^2-T, -T^3, -T^2+T-1, T+1, -T^3, -T-1, -T+1]$

$\mathbf{s_8} = [0, T-1, T^2+T+1, T, -T^2-T-1, T^2-2T+2, -T-1, -T, T+1, T^3-T^2+T-1, -T, -T^3+T, -T, -T, T, T, -T^5+T, -T^7+T^2-T, -T^3, -T^2+T-1, T+1, T^9-T^3, -T, -T^{11}+T-1, T, T^7, -T, -T+1, T^2+T, -T^7, -T^2-T-1, -T+1]$

$\mathbf{s_9} = [0, T-1, T^2+T+1, T, -T^2-T-1, T^2-2T+2, -T-1, -T, T+1, T^3-T^2+T-1, -T, -T^3+T, -T, -T, T, T, -T^5+T, -T^7+T^2-T, -T^3, -T^2+T-1, T+1, T^9-T^3, -T, -T^{11}+T-1, T, T^7, -T, -T+1, T^2+T, T^{18}-T^7, -T^2, -T^{20}+T^3, -T-1, -T+1, -T^{10}, T-1, T+1, -T^3, T^2, T, -T^2-T-1, -T+1, T^{10}, T-1, T^2+T+1, -T, -T^2-T-1, -T+1]$

$\mathbf{s_{10}} = [0, T-1, T^2+T+1, T, -T^2-T-1, T^2-2T+2, -T-1, -T, T+1, T^3-T^2+T-1, -T, -T^3+T, -T, -T, T, T, -T^5+T, -T^7+T^2-T, -T^3, -T^2+T-1, T+1, T^9-T^3, -T, -T^{11}+T-1, T, T^7, -T, -T+1, T^2+T, T^{18}-T^7, -T^2, -T^{20}+T^3, -T-1, -T+1, -T^{10}, T-1, T+1, -T^3, T^2, T, -T^2-T-1, -T+1, -T^{31}+T^{10}, T-1, T^2+T+1, -T^{39}+T^7-T, -T^2-T-1, -T+1, -T^{19}, T-1, T^2+T+1, -T^7+T, -T^2-T-1, T^2-2T+2, -T-1, -T, T+1, -T^2+T-1, T^{19}, T^2-T+1, -T-1, T, T+1, -T^2+2T-2, T^2+T+1, -T, -T^2-T-1, -T+1]$

$\mathbf{s_{11}} = [0, T-1, T^2+T+1, T, -T^2-T-1, T^2-2T+2, -T-1, -T, T+1, T^3-T^2+T-1, -T, -T^3+T, -T, -T, T, T, -T^5+T, -T^7+T^2-T, -T^3, -T^2+T-1, T+1, T^9-T^3, -T, -T^{11}+T-1, T, T^7, -T, -T+1, T^2+T, T^{18}-T^7, -T^2, -T^{20}+T^3, -T-1, -T+1, -T^{10}, T-1, T+1, -T^3, T^2, T, -T^2-T-1, -T+1, -T^{31}+T^{10}, T-1, T^2+T+1, -T^{39}+T^7-T, -T^2-T-1, -T+1, -T^{19}, T-1, T^2+T+1, -T^7+T, -T^2-T-1, T^2-2T+2, -T-1, -T, T+1, -T^2+T-1, -T^{57}+T^{19}, T^2-T+1, -T-1, T, T+1, -T^2+T-1, T^{69}-T^{10}, T-1, T^2+T+1, -T, -T^2-T-1, -T+1, -T^{35}, T-1, T^2+T+1, T, -T^2-T-1, -T+1, T^{10}, T^2-T+1, -T-1, -T, T+1, T^3-T^2+T-1, -T, -T^3+T, -T, -T, T, T, T, -T, T^{35}, T, -T, -T, -T, T, T, T^3-T, T, -T^3+T^2-T+1, -T-1, T, T+1, -T^2+2T-2, T^2+T+1, -T, -T^2-T-1, -T+1]$.

It should be noted that

$$s_{10} = s_9, s_4, -T^{31}, -\overleftarrow{s_4}, s_3, T^{39}, -\overleftarrow{s_3}, s_3, -T^7, -\overleftarrow{s_3}, -T^7, s_3, T^{19}, -\overleftarrow{s_3}, s_5, T^{19}, -\overleftarrow{s_5}$$
$$s_{11} = s_{10}, s_5, -T^{57}, -\overleftarrow{s_5}, s_4, T^{69}, -\overleftarrow{s_4}, s_4, -T^{10}, -\overleftarrow{s_4}, -T^{35}, s_4, T^{10}, -\overleftarrow{s_4}, s_6, T^{35}, -\overleftarrow{s_6}$$

We can conclude equivalently to the previous case that

Theorem 3.2 *Set for $h \geq 1$*

$$s_h = T^{-1} + T^{-2} + T^{-3} + T^{-7} + \cdots + T^{-a(h)} = [0, f_h].$$

Then the words f_h, $2 \leq h \leq 9$, are given by $s_h = [0, f_h]$ as listed above. Let for $h \geq 10$ $f_{h+1} = 0, f_h, 0, f_{h-5}, -T^{a(h-2)-2a(h-5)}, -\overleftarrow{f_{h-5}}, 0, f_{h-6}, -T^{a(h-2)-2a(h-6)}, -\overleftarrow{f_{h-6}}, -T^{a(h-2)-2a(h-4)}, f_{h-6}, T^{a(h-4)-2a(h-6)}, -\overleftarrow{f_{h-6}}, 0, f_{h-4}, T^{a(h-2)-2a(h-4)}, -\overleftarrow{f_{h-4}}$.
Then

$$s_\infty = \sum_{n=2}^{\infty} T^{-a(n)} = \lim_{h \to \infty} [0, f_h] \tag{3.5}$$

Further, for any fixed integer $x \geq 2$, the number $x^{-1}+x^{-2}+x^{-3}+x^{-7}+\cdots+x^{-a(h)}+\cdots$ is transcendental.

Consider now the following sequence:

$$a(n) = \begin{cases} 0, \ if \ n \leq 2 \\ 1, \ if \ n = 3 \\ a(n) = 1 + a(n-1) + a(n-2) + a(n-3), \ for \ all \ n \geq 4 \end{cases}$$

The first value of this sequence are $0, 0, 0, 1, 2, 4, 8, 15, 28, 52, 96, 178, 327, \dots$
A computation gives that in this case $\sum_{n=3}^{\infty} T^{-a(n)}$ is not specializable!
We state an analogue of the Conjecture 1 in our case.

Conjecture 2 *Let $a(n)$ be an increasing sequence of nonnegative integers satisfying a recurrence relation*

$$a(n) = \begin{cases} d_n, \ if \ n \leq d \\ a(n+d) = a(n+d-1) + a(n+d-2) + \cdots + a(n) + 1, \ for \ all \ n > d \end{cases}$$

and set

$$s_n = T^{-a(d_{n_0})} + T^{-a(d_{n_0}+1)} + \cdots + T^{-a(n)}; s_n = [0, f_n].$$

Then, subject to appropriate initial conditions on the $a(h)$, the words f_h consist of polynomials with integer coefficients, which is to say that s_∞ has a specializable continued fraction expansion.

4. Continued Fraction of Series Involving Modified Leonardo 2-Numbers

The Leonardo numbers are a sequence of numbers given by the recurrence:

$$\mathcal{L}_n = \begin{cases} 1 & \text{if } n \leq 1 \\ \mathcal{L}_{n-1} + \mathcal{L}_{n-2} + 1 & \text{if } n > 1 \end{cases}$$

This sequence is referenced as A001595 in the OEIS.
The modified Leonardo numbers, related to Leonardo numbers, and defined for the first time in [13], is a sequence of numbers given by the recurrence:

$$\mathcal{L}_{1,n} = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \mathcal{L}_{1,n-1} + \mathcal{L}_{1,n-2} + 1 & \text{if } n > 1 \end{cases}$$

The first few modified Leonardo numbers are: $0, 1, 2, 4, 7, 12, 20, 33, 54, 88, 143, 232, 376$.
A computation gives
$\sum_{k=1}^{11} T^{-\mathcal{L}_{2,k}} = [0, T-1, T+2, T, -T^2+T-2, T^4-T^2+T+1, -T^5+T^2-1, -T, -T^8+T^3-1, T+1, T-1, T^{12}-T^4, -T^2-1, -T^2+1, -T^{19}+T^6, T^4+T, T^2, -T, -T^{30}+T^9, T, -T^5-1, -T, T^3, T+1, T-1, T^{48}-T^{14}, -T+1, -T-1, -T^8-1, T+1, T-1, -T^4, -T+1, -T, -T-2, -T+1, T^{22}, T-1, T+2, T, T-1, T^4, -T+1, -T-1, T^8-T^3+1, T, T^5-T^2+1, -T^4+T^2-T-1, T^2-T+2, -T, -T-2, -T+1]$.
So, the partial sum has specializable continued fraction.
The Leonardo 2-numbers are a sequence of numbers given by the recurrence:

$$\mathcal{L}_{2,n} = \begin{cases} 1 & \text{if } n \leq 1 \\ \mathcal{L}_{2,n-1} + \mathcal{L}_{2,n-2} + 2 & \text{if } n > 1 \end{cases}$$

The terms of Leonardo 2-numbers begins with: $1, 1, 1, 4, 7, 10, 16, 25, 37, 55$. A computation of continued fraction expansion of a partial sum involving this numbers gives that
$\sum_{k=2}^{9} T^{-\mathcal{L}_{2,k}} = [0, T, -T^2, -T^7-2T^4-2T, -T^8+T^5-2T^2, \frac{1}{2}T, 4T^2, \frac{1}{6}T, -\frac{9}{2}T^2, -\frac{2}{3}T, \frac{1}{2}T^2, T, 4T^2, \frac{1}{3}T, -\frac{3}{2}T^2, -\frac{2}{3}T, -\frac{3}{2}T^2, T, \frac{3}{2}T^2, -\frac{1}{3}T, -12T^2, -\frac{1}{21}T, \frac{49}{2}T^2, \frac{1}{21}T, 9T^5-18T^2, \frac{1}{6}T, 4T^2, -\frac{1}{2}T]$,
which is not specializable.

The modified Leonardo 2-numbers are a sequence of numbers given by the recurrence:

$$\mathcal{L}_{2,n} = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \mathcal{L}_{2,n-1} + \mathcal{L}_{2,n-2} + 2 & \text{if } n > 1 \end{cases}$$

It's related to the Fibonacci numbers by the following relation established in [13]

$$\mathcal{L}_{2,n} = 2F_{n+1} + F_n - 2 \tag{4.1}$$

The first twenty terms of this sequence are

$$0, 1, 3, 6, 11, 19, 32, 53, 87, 142, 231, 375, 608, 985, 1595, 2582, 4179, 6763, 10944.$$

We are now interested to the continued fraction expansion of the series involving this sequence. A computation gives us the following.

$\mathbf{s_1} = [0, T]$

$\mathbf{s_2} = [0, T, -T, -T]$

$\mathbf{s_3} = [0, T, -T, -T-1, -T+1, -T, T]$

$\mathbf{s_4} = [0, T, -T, -T-1, -T+1, -T, -T^3+T-1, -T^3+1]$

$\mathbf{s_5} = [0, T, -T, -T-1, -T+1, -T, -T^3+T-1, T^6-T^3+T+1, T^4, -T]$

$\mathbf{s_6} = [0, T, -T, -T-1, -T+1, -T, -T^3+T-1, T^6-T^3+T+1, -T^9+T^4-T, -T, T^5, T, T, -T]$

$\mathbf{s_7} = [0, T, -T, -T-1, -T+1, -T, -T^3+T-1, T^6-T^3+T+1, -T^9+T^4-T, -T, -T^{13}+T^5-1, T, T, -T, T^7, T, -T, -T+1, -T-1, -T, T]$

$\mathbf{s_8} = [0, T, -T, -T-1, -T+1, -T, -T^3+T-1, T^6-T^3+T+1, -T^9+T^4-T, -T, -T^{13}+T^5-1, T, T, -T, -T^{20}+T^7, T^3+1, T^3-1, -T^{10}, -T^3+1, -T^3+T-1, -T, -T+1, -T-1, -T, T]$

$\mathbf{s_9} = [0, T, -T, -T-1, -T+1, -T, -T^3+T-1, T^6-T^3+T+1, -T^9+T^4-T, -T, -T^{13}+T^5-1, T, T, -T, -T^{20}+T^7, T^3+1, T^3-1, T^{31}-T^{10}, -T^6-T, -T^4, T, -T^{15}, -T, T^4, T^6-T^3+T+1, -T^3+T-1, -T, -T+1, -T-1, -T, T]$

$\mathbf{s_{10}} = [0, T, -T, -T-1, -T+1, -T, -T^3+T-1, T^6-T^3+T+1, -T^9+T^4-T, -T, -T^{13}+T^5-1, T, T, -T, -T^{20}+T^7, T^3+1, T^3-1, T^{31}-T^{10}, -T^6-T, -T^4, T, T^{49}-T^{15}, -T, T^9+T, T, -T^5, -T, -T, T, -T^{23}, -T, T, T, T^5, -T, -T^9+T^4-T, T^6-T^3+T+1, -T^3+T-1, -T, -T+1, -T-1, -T, T]$

$\mathbf{s_{11}} = [0, T, -T, -T-1, -T+1, -T, -T^3+T-1, T^6-T^3+T+1, -T^9+T^4-T, -T, -T^{13}+T^5-1, T, T, -T, -T^{20}+T^7, T^3+1, T^3-1, T^{31}-T^{10}, -T^6-T, -T^4, T, T^{49}-T^{15}, -T, T^9+T, T, -T^5, -T, -T, T, T^{78}-T^{23}, -T, T, T, T^{13}+1, -T, -T, T, -T^7, -T, T, T-1, T+1, T, -T, T^{36}, T, -T, -T-1, -T+1, -T, T, T^7, -T, T, T, -T^{13}+T^5-1, -T, -T^9+T^4-T, T^6-T^3+T+1, -T^3+T-1, -T, -T+1, -T-1, -T, T]$

$\mathbf{s_{12}} = [0, T, -T, -T-1, -T+1, -T, -T^3+T-1, T^6-T^3+T+1, -T^9+T^4-T, -T, -T^{13}+T^5-1, T, T, -T, -T^{20}+T^7, T^3+1, T^3-1, T^31-T^{10}, -T^6-T, -T^4, T, T^{49}-T^{15}, -T, T^9+T, T, -T^5, -T, -T, T, T^{78}-T^{23}, -T, T, T, T^{13}+1, -T, -T, T, -T^7, -T, T, T-1, T+1, T, -T, -T^{125}+T^{36}, T, -T, -T-1, -T+1, -T, T, T^{20}, -T^3-1, -T^3+1, T^{10}, T^3-1, T^3-T+1, T, T-1, T+1, T, -T, T^{57}, T, -T, -T-1, -T+1, -T, -T^3+T-1, -T^3+1, -T^{10}, T^3-1, T^3+1, -T^{20}+T^7, -T, T, T, -T^{13}+T^5-1, -T, -T^9+T^4-T, T^6-T^3+T+1, -T^3+T-1, -T, -T+1, -T-1, -T, T]$

$\mathbf{s_{13}} = [0, T, -T, -T-1, -T+1, -T, -T^3+T-1, T^6-T^3+T+1, -T^9+T^4-T, -T, -T^13+T^5-1, T, T, -T, -T^{20}+T^7, T^3+1, T^3-1, T^{31}-T^{10}, -T^6-T, -T^4, T, T^{49}-T^{15}, -T, T^9+T, T, -T^5, -T, -T, T, T^{78}-T^{23}, -T, T, T, T^{13}+1, -T, -T, T, -T^7, -T, T, T-1, T+1, T, -T, -T^{125}+T^{36}, T, -T, -T-1, -T+1, -T, T, T^{20}, -T^3-1, -T^3+1, T^{10}, T^3-1, T^3-T+1, T, T-1, T+1, T, -T, -T^{201}+T^{57}, T, -T, -T-1, -T+1, -T, -T^3+T-1, -T^3+1, -T^{31}, T^6+T, T^4, -T, T^{15}, T, -T^4, -T^6+T^3-T-1, T^3-T+1, T, T-1, T+1, T, -T, T^{91}, T, -T, -T-1, -T+1, -T, -T^3+T-1, T^6-T^3+T+1, T^4, -T, -T^{15}, T, -T^4, -T^6-T, T^{31}-T^{10}, T^3-1, T^3+1, -T^{20}+T^7, -T, T, T, -T^{13}+T^5-1, -T, -T^9+T^4-T, T^6-T^3+T+1, -T^3+T-1, -T, -T+1, -T-1, -T, T]$

$\mathbf{s_{14}} = [0, T, -T, -T-1, -T+1, -T, -T^3+T-1, T^6-T^3+T+1, -T^9+T^4-T, -T, -T^{13}+T^5-1, T, T, -T, -T^{20}+T^7, T^3+1, T^3-1, T^{31}-T^{10}, -T^6-T, -T^4, T, T^{49}-T^{15}, -T, T^9+T, T, -T^5, -T, -T, T, T^{78}-T^{23}, -T, T, T, T^{13}+1, -T, -T, T, -T^7, -T, T, T-1, T+1, T, -T, -T^{125}+T^{36}, T, -T, -T-1, -T+$

$1, -T, T, T^{20}, -T^3-1, -T^3+1, T^{10}, T^3-1, T^3-T+1, T, T-1, T+1, T, -T, -T^{201}+T^{57}, T, -T, -T-1, -T+1, -T, -T^3+T-1, -T^3+1, -T^{31}, T^6+T, T^4, -T, T^{15}, T, -T^4, -T^6+T^3-T-1, T^3-T+1, T, T-1, T+1, T, -T, -T^{324}+T^{91}, T, -T, -T-1, -T+1, -T, -T^3+T-1, T^6-T^3+T+1, T^4, -T, -T^{49}, T, -T^9-T, -T, T^5, T, T, -T, T^{23}, T, -T, -T, -T^5, T, T^9-T^4+T, -T^6+T^3-T-1, T^3-T+1, T, T-1, T+1, T, -T, T^{146}, T, -T, -T-1, -T+1, -T, -T^3+T-1, T^6-T^3+T+1, -T^9+T^4-T, -T, T^5, T, T, -T, -T^{23}, T, -T, -T, -T^5, T, T^9+T, -T, T^{49}-T^{15}, T, -T^4, -T^6-T, T^{31}-T^{10}, T^3-1, T^3+1, -T^{20}+T^7, -T, T, T, -T^{13}+T^5-1, -T, -T^9+T^4-T, T^6-T^3+T+1, -T^3+T-1, -T, -T+1, -T-1, -T, T]$.
Then

$$
\begin{aligned}
s_8 &= s_7, -s_3, -T^{20}, \overleftarrow{s_3}, -s_4, -T^{10}, \overleftarrow{s_4} \\
s_9 &= s_8, -s_4, T^{31}, \overleftarrow{s_4}, -s_5, -T^{15}, \overleftarrow{s_5} \\
s_{10} &= s_9, -s_5, T^{49}, \overleftarrow{s_5}, -s_6, -T^{23}, \overleftarrow{s_6} \\
s_{11} &= s_{10}, -s_6, T^{78}, \overleftarrow{s_6}, -s_7, T^{36}, \overleftarrow{s_7} \\
s_{12} &= s_{11}, -s_7, -T^{125}, \overleftarrow{s_7}, -s_8, T^{57}, \overleftarrow{s_8} \\
s_{13} &= s_{12}, -s_8, -T^{201}, \overleftarrow{s_8}, -s_9, T^{91}, \overleftarrow{s_9} \\
s_{14} &= s_{13}, -s_9, -T^{324}, \overleftarrow{s_9}, -s_{10}, T^{146}, \overleftarrow{s_{10}}
\end{aligned}
$$

We can illustrate the following conclusion thanks to these initial findings:

Theorem 4.1 *Set for $h \geq 1$*

$$s_h = T^{-1} + T^{-3} + T^{-6} + T^{-11} + \cdots + T^{-\mathcal{L}_{2,n}} = [0, f_h].$$

Then the words f_h, $1 \leq h \leq 11$, are given by $s_h = [0, f_h]$ as listed above. Let for $h \geq 12$

$$f_h = f_{h-1}, 0, f_{h-5}, -T^{\mathcal{L}_{2,h-3}+\mathcal{L}_{2,h-5}+6}, -\overleftarrow{f_{h-5}}, 0, f_{h-4}, T^{\mathcal{L}_{2,h-4}+4}, -\overleftarrow{f_{h-4}}$$

Then

$$s_\infty = \sum_{n=1}^{\infty} T^{-\mathcal{L}_{2,n}} = \lim_{h\to\infty} [0, f_h] \tag{4.2}$$

Proof: Induction on h is used as the evidence. The base case is $h = 12$. The length of s_h is assumed to be even as indeed it appears to be for $h \geq 7$ (this will also be proven during the proof). We have that $s_h = [0, f_h] = p/q$ and the facts point to the polynomial q being of type $T^{-\mathcal{L}_{2,h}}$. Noting that $\mathcal{L}_{2,h+1} - 2\mathcal{L}_{2,h} = -\mathcal{L}_{2,h-2}$, so

$$s_{h+1} = s_h + T^{-\mathcal{L}_{2,h+1}} = \frac{p}{q} + \frac{1}{T^{-\mathcal{L}_{2,h-2}} q^2}$$

and the Folding lemma yields

$$s_{h+1} = [0, f_h, T^{-\mathcal{L}_{2,h-2}}, -\overleftarrow{f_h}]$$

which is an illegal expansion. We still believe that the Folding lemma will lead to the outcome we want. Obviously,

$$s_{h+1} = s_h + T^{-\mathcal{L}_{2,h+1}} = \frac{p}{q} - \frac{1}{-T^{\mathcal{L}_{2,h-2}} q^2} = [0, f_h, T^{\mathcal{L}_{2,h-2}} - q'/q], \tag{4.3}$$

where q' is the denominator of the next-to-last convergent to s_h, and we can take $q = T^{-\mathcal{L}_{2,h}}$ as previously mentioned. As p'/q' is the penultimate convergent of the last convergent, p/q, we know that $pq' - p'q = 1$, which results in

$$s_h(q'/q) = -1/T^{2\mathcal{L}_{2,h}} + (p'/q) \tag{4.4}$$

In light of the presence of $T^{-2\mathcal{L}_{2,h}}$ and $T^{-\mathcal{L}_{2,h+1}}$ to the right of (4.4), the most reasonable supposition for our calculation is that

$$s_h(s_h - \frac{1}{T^{\mathcal{L}_{2,h-3}}} - \frac{2}{T^{\mathcal{L}_{2,h}}})$$

is of the shape (4.4). Thus

$$q'/q = s_h - \frac{1}{T^{\mathcal{L}_{2,h-3}}} - \frac{2}{T^{\mathcal{L}_{2,h}}}. \tag{4.5}$$

Combining (4.5) and (4.3), we get

$$\begin{aligned} s_{h+1} &= [0, f_h, T^{-\mathcal{L}_{2,h-2}} - q'/q] \\ &= [0, f_h, -(s_h - T^{-\mathcal{L}_{2,h-3}} - 2T^{-\mathcal{L}_{2,h}} - T^{-\mathcal{L}_{2,h-2}})] \\ &= [0, f_h, -(s_{h-4} + T^{-\mathcal{L}_{2,h-1}} - T^{-\mathcal{L}_{2,h}})] \end{aligned}$$

So all we need to do is determine the continued fraction expansion of

$$-(s_{h-4} + T^{-\mathcal{L}_{2,h-1}} - T^{-\mathcal{L}_{2,h}}).$$

If $h \geq 12$, then by induction, $\mathcal{L}_{2,h-1} - 2\mathcal{L}_{2,h-4} = \mathcal{L}_{2,h-3} + \mathcal{L}_{2,h-5} + 6$ and the lemma,

$$s_{h-4} + T^{-\mathcal{L}_{2,h-1}} = [0, f_{h-4}, T^{\mathcal{L}_{2,h-3}+\mathcal{L}_{2,h-5}+6}, -\overleftarrow{f_{h-4}}]. \tag{4.6}$$

We will obtain now the expansion of $s_{h-4} + T^{-\mathcal{L}_{2,h-1}} - T^{-\mathcal{L}_{2,h}}$ by noting that $\mathcal{L}_{2,h} - 2\mathcal{L}_{2,h-1} = -\mathcal{L}_{2,h-3}$, so

$$s_{h-4} + T^{-\mathcal{L}_{2,h-1}} - T^{-\mathcal{L}_{2,h}} = [0, f_{h-4}, -T^{\mathcal{L}_{2,h-3}+\mathcal{L}_{2,h-5}+6}, -\overleftarrow{f_{h-4}}, T^{-\mathcal{L}_{2,h-3}} - q'/q].$$

Here q'/q refers to (4.6) in which the word following the 0-th partial quotient is of odd length, $q = T^{\mathcal{L}_{2,h-1}}$, and we find q'/q in virtue of

$$(s_{h-4} + T^{-\mathcal{L}_{2,h-1}})(-s_{h-4} + T^{-\mathcal{L}_{2,h-1}}) = T^{-2\mathcal{L}_{2,h-1}} + p'/T^{\mathcal{L}_{2,h-1}}$$

We have

$$\begin{aligned} T^{-\mathcal{L}_{2,h-3}} - q'/q &= s_{h-4} - T^{-\mathcal{L}_{2,h-1}} + T^{-\mathcal{L}_{2,h-3}} \\ &= s_{h-3} - T^{-\mathcal{L}_{2,h-1}} \\ &= [0, f_{h-3}, -T^{\mathcal{L}_{2,h-4}+4}, -\overleftarrow{f_{h-3}}] \end{aligned}$$

where we have used $\mathcal{L}_{2,h-1} - 2\mathcal{L}_{2,h-3} = \mathcal{L}_{2,h-4} + 4$ Thus

$$s_{h-4} + T^{-\mathcal{L}_{2,h-1}} - T^{-\mathcal{L}_{2,h}} = [0, f_{h-4}, T^{\mathcal{L}_{2,h-3}+\mathcal{L}_{2,h-5}+6}, -\overleftarrow{f_{h-4}}, 0, f_{h-3}, -T^{\mathcal{L}_{2,h-4}+4}, -\overleftarrow{f_{h-3}}].$$

So our inductive assumptions entail for $h \geq 13$ that

$$s_{h+1} = [0, f_h, 0, -f_{h-4}, -T^{\mathcal{L}_{2,h-3}+\mathcal{L}_{2,h-5}+6}, \overleftarrow{f_{h-4}}, 0, -f_{h-3}, T^{\mathcal{L}_{2,h-4}+4}, \overleftarrow{f_{h-3}}] \tag{4.7}$$

□

Corollary 4.1

$$2^{-1} + 2^{-2} + 2^{-4} + 2^{-7} + 2^{-12} + \cdots + 2^{-\mathcal{L}_{2,h}} + \cdots = [0, 1, 1, 1, 3, 2, 6, 1, 57, 1, 497, 1, 1, 8160, 2, 1, 1, 1, 1048486, 1, 8, 7, 2147482623, 1, 65, \cdots].$$

Proof: We replace T by 2 in (4.7) and we apply the identities (1.6) and (1.7). □

Corollary 4.2 *For all integer $x \geq 2$, $\sum_{n=1}^{+\infty} x^{-\mathcal{L}_{2,n}}$ is a transcendental number.*

Proof: We have that,

$$(T^{-\mathcal{L}_{2,h-2}} + T^{-\mathcal{L}_{2,h-1}} - T^{-\mathcal{L}_{2,h}})s_\infty = \text{terms of degree} \geq -\mathcal{L}_{2,h} - \mathcal{L}_{2,h-4} + T^{-\mathcal{L}_{2,h-1}-\mathcal{L}_{2,h+1}} + \quad (4.8)$$

terms of yet lower degree.
The lack of a word of degree $-2\mathcal{L}_{2,h-1}$ due to the fact that $2\mathcal{L}_{2,h-1} = \mathcal{L}_{2,h} + \mathcal{L}_{2,h-3}$ and $\mathcal{L}_{2,h-1} + \mathcal{L}_{2,h-2} = \mathcal{L}_{2,h} - 2$ explains this. We set $q_h = T^{\mathcal{L}_{2,h}+\mathcal{L}_{2,h-4}}(T^{-\mathcal{L}_{2,h-2}} + T^{-\mathcal{L}_{2,h-1}} - T^{-\mathcal{L}_{2,h}})$ and so we have $\deg q_h = \mathcal{L}_{2,h} + \mathcal{L}_{2,h-4} - \mathcal{L}_{2,h-2} = \mathcal{L}_{2,h-1} + \mathcal{L}_{2,h-4}$. The polynomial p_h in the variable T is revealed by equation (4.8) which results in

$$\begin{aligned} |q_h s_\infty - p_h| &= |T|^{-(\mathcal{L}_{2,h-1}+\mathcal{L}_{2,h+1}-\mathcal{L}_{2,h}-\mathcal{L}_{2,h-4})} \\ &= |q_h|^{-1}|T|^{-(\mathcal{L}_{2,h-1}-2\mathcal{L}_{2,h-4})} = |q_h|^{-1}|T|^{-(\mathcal{L}_{2,h-3}+\mathcal{L}_{2,h-5}+6)}. \end{aligned}$$

When this happens, p_h/q_h is a convergent to s_∞, and the following partial quotient has degree of $\mathcal{L}_{2,h-3} + \mathcal{L}_{2,h-5} + 6$. The fact is that for sufficiently large h, using the identity (4.1) and the fact that $\lim_{n\to+\infty} \frac{F_{n+1}}{F_n} = \phi = \frac{1+\sqrt{5}}{2}$ we have

$$(\mathcal{L}_{2,h-3} + \mathcal{L}_{2,h-5} + 6)/(\mathcal{L}_{2,h-1} + \mathcal{L}_{2,h-4}) \to 1 + \phi^{-1} \approx 1.618.$$

As a result, when we specialize T to x, we have a sufficiently high h

$$|s_\infty(x) - p_h(x)/q_h(x)| < q_h^{-3.61}.$$

As a result of Roth's theorem [11], we achieve our intended outcome.

□

We confidently state this conjecture because a bit more appears obvious.

Conjecture 3 *Let $p \geq 1$ be an integer. Let $\mathcal{L}_{p,n}$ be the modified Leonardo $p-$number defined by*

$$\mathcal{L}_{p,n} = \mathcal{L}_{p,n-1} + \mathcal{L}_{p,n-2} + p,$$

with $\mathcal{L}_{p,0} = 0$ if and $\mathcal{L}_{p,1} = 1$. Set

$$s_n = \sum_{h=1}^{n} T^{-\mathcal{L}_{p,h}} = [0, f_n].$$

Then

$$s_\infty = \sum_{n=1}^{\infty} T^{-\mathcal{L}_{p,n}} = \lim_{h\to\infty} [0, f_h] \quad (4.9)$$

where the words f_h is given by

$$f_h = f_{h-1}, 0, f_{h-5}, -T^{\mathcal{L}_{p,h-3}+\mathcal{L}_{p,h-5}+3p}, -\overleftarrow{f_{h-5}}, 0, f_{h-4}, T^{\mathcal{L}_{p,h-4}+2p}, -\overleftarrow{f_{h-4}}$$

for $h \geq h_0$, which is to say that s_∞ has a specializable continued fraction expansion.

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