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Analytical and Geometrical Properties of New Class of Univalent Functions Associated with the Fractional Derivative Operator

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ABSTRACT: In this work, we introduce and analyze a new subclass $\mathcal{F}^{\varrho}_{0,z}(\varepsilon, \lambda, \eta, \mu)$ of analytic univalent functions related to the fractional derivative operator within the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. We investigate coefficient estimates, distortion bounds and growth theorems, convex set, radius of convexity, radius of starlikeness, arithmetic mean, and weighted mean, and also we establish some basic results like extreme points, Hadamard product, and the closure theorem for the functions in the class.

Keywords: Univalent function, analytical properties, geometrical properties, fractional derivative operator.

Contents

1. Introduction and Preliminaries

The study of univalent functions occupies a central position in geometric function theory (GFT), supported by analytic function theory, conformal mappings, and their strong interconnections with applied fields such as engineering, fluid dynamics and mathematical physics. Let $\mathcal{A}$ be the class of analytic functions $\xi(z)$ defined in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$, of the form

$$\xi(z) = z + \sum_{s=2}^{\infty} a_s z^s. \tag{1.1}$$

Within this family, the subclass $\mathcal{S} \subseteq \mathcal{A}$ consists of functions that are univalent in $\mathbb{U}$ and normalized, such that $\xi(0) = 0,\ \xi'(0) - 1 = 0$.

The function $\xi(z) \in \mathcal{S}$ is called starlike of order $\beta, (0 < \beta \le 1)$, if it satisfies

$$\Re\left(\frac{z\xi'(z)}{\xi(z)}\right) > \beta, \quad z \in \mathbb{U}.$$

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This is denoted by $\mathcal{S}^*(\beta)$, with the inclusions $\mathcal{S}^*(\beta) \subseteq \mathcal{S}^*(0) = \mathcal{S}^*$, where $\mathcal{S}^*$ represents the family of functions starlike with respect to the origin.

Similarly, a function $\xi(z) \in \mathcal{S}$ is said to be convex of order $\beta, (0 < \beta \leq 1)$, if

$$\Re\left(1 + \frac{z\xi''(z)}{\xi'(z)}\right) > \beta, \quad z \in \mathbb{U}.$$

This subclass denoted by $\mathcal{C}(\beta)$ with the relation $\mathcal{C}(\beta) \subseteq \mathcal{C}(0) = \mathcal{C}$, where $\mathcal{C}$ denote standard convex class. It is a well-known fact that $z\xi'(z) \in \mathcal{S}^*(\beta)$ if and only if $\xi(z) \in \mathcal{C}(\beta)$. Robertson [24] first introduced the classes $\mathcal{S}^*(\beta)$ and $\mathcal{C}(\beta)$, which later studied by Schild [26] , Pinchuk [23], Owa and Srivastava [19,20], among others (see also, Duren [10] and Goodman [12]).

Additionally, a function $\xi(z) \in \mathcal{S}$ is called close-to-convex of order$\beta, (0 < \beta \leq 1)$, if it satisfies

$$\Re\big(z\xi'(z)\big) > \beta, \quad z \in \mathbb{U}.$$

We also consider the subclass $\mathcal{N} \subset \mathcal{S}$ consisting of analytic univalent functions in $\mathbb{U}$ with negative coefficients, represented by

$$\xi(z) = z - \sum_{s=2}^{\infty} a_s z^s \tag{1.2}$$

The subclasses $\mathcal{S}^*_{\mathcal{N}}(\beta)$ and $\mathcal{C}_{\mathcal{N}}(\beta)$ denote, respectively, the starlike and convex of order β within $\mathcal{N}$. These are defined as

$$\mathcal{S}^*_{\mathcal{N}}(\beta) = \mathcal{S}^*(\beta) \cap \mathcal{N}, \quad 0 < \beta \leq 1, \quad z \in \mathbb{U},$$

and

$$\mathcal{C}_{\mathcal{N}}(\beta) = \mathcal{C}(\beta) \cap \mathcal{N}, \quad 0 < \beta \leq 1, \quad z \in \mathbb{U}.$$

These classes were introduced and studied by Silverman [27]. Moreover, it is easy to verify that $z\xi'(z) \in \mathcal{S}^*_{\mathcal{N}}(\beta)$ if and only if $\xi(z) \in \mathcal{C}_{\mathcal{N}}(\beta)$.
The convolution product for functions $\xi_1(z), \xi_2(z) \in \mathcal{N}$ is defined as

$$\xi_1(z) * \xi_2(z) = z - \sum_{s=2}^{\infty} c_s d_s z^s \tag{1.3}$$

where, $\xi_1(z) = z - \sum_{s=2}^{\infty} c_s z^s$ and $\xi_2(z) = z - \sum_{s=2}^{\infty} d_s z^s$.

In geometric function theory, operator theory plays a crucial role. Numerous researchers have proposed and examined subclasses of univalent functions with negative coefficients using different operators. Fractional calculus, which extends classical differentiation and integration to non-integer or complex orders, has been applied in GFT to define specialized differential and integral operators that often preserve univalence. Hohlov [15,16] developed operators involving the Hadamard product with Gauss hypergeometric functions and established conditions for univalence preservation. Later, Kiryakova and Saigo [18] extended these ideas to the generalized fractional calculus (GFC) using special functions and integral transforms. These approaches provide criteria that ensure the stability of classes of univalent or convex functions under fractional operators. Notable examples include the Dziok-Srivastava [11] and Srivastava-Wright operators [30], both of which are widely applied to GFT and GFC to derive univalence conditions, covering many classical cases. A further generalization of the fractional calculus was provided by Srivastava and Owa [29], who introduced the following:

Definition 1.1 *Let $\xi(z)$ be an analytic function of the form (1.1) defined in a simply connected domain of the $z-$ plane containing the origin. The fractional integral of $\xi(z)$ for order $\varrho > 0$ is given by*

$$\mathcal{D}^{-\varrho}_{0,z}\xi(z) = \frac{1}{\Gamma(\varrho)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\varrho}} d\zeta, \tag{1.4}$$

while the fractional derivative of $\xi(z)$ for order $\varrho, (0 \leq \varrho < 1)$ is defined as

$$\mathcal{D}^{\varrho}_{0,z}\xi(z) = \frac{1}{\Gamma(1-\varrho)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\varrho}} d\zeta, \quad (z-\zeta) > 0 \tag{1.5}$$

where the multi-valued terms $(z-\zeta)^{1-\varrho}$ *and* $(z-\zeta)^{\varrho}$ *are made single-valued by requiring* $\log(z-\zeta)$ *to be real whenever* $(z-\zeta)>0$ *(See [30]).*

Using the formation, the Srivastava-Owa fractional derivative of order $m+\varrho$ is expressed as

$$\mathcal{D}_{0,z}^{m+\varrho}\xi(z)=\frac{d^m}{dz^m}\mathcal{D}_{0,z}^{\varrho}\xi(z),\quad m\in\mathbb{N}_0,\quad \text{and}\quad 0\le\varrho<1. \tag{1.6}$$

Based on this, Srivastava and Owa (1983) [28] introduced the operator $\varOmega_{0,z}^{\varrho}:\mathcal{A}_0\longrightarrow\mathcal{A}_0$, defined by

$$\begin{aligned}\varOmega_{0,z}^{\varrho}\xi(z)&=\Gamma(2-\varrho)z^{\varrho}\mathcal{D}_{0,z}^{\varrho}\xi(z)\\&=z+\sum_{s=2}^{\infty}\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}a_s z^s,\end{aligned} \tag{1.7}$$

where $\mathcal{D}_{0,z}^{\varrho}\xi(z)$ is given in (1.5). For specific values of ϱ, the operator reduces to the well-known cases:

$$\begin{aligned}\varOmega_{0,z}^{0}\xi(z)&=\xi(z)=z+\sum_{s=2}^{\infty}a_s z^s\\ \varOmega_{0,z}^{1}\xi(z)&=z\xi'(z)=z+\sum_{s=2}^{\infty}s a_s z^s,\\ \varOmega_{0,z}^{j}\xi(z)&=\varOmega_{0,z}(\varOmega_{0,z}^{j-1}\xi(z))=z+\sum_{s=2}^{\infty}s^j a_s z^s.\end{aligned} \tag{1.8}$$

Equation (1.8) is known as the Sălăgean Operator [25]. By taking $j=-k$ and $k\in\mathbb{N}$, it also includes the Libera Bernardi integral operator [4]. For, more applications and details of the fractional derivative operator in analytic function theory, applied mathematics and nonlinear studies see [1,5,6,7,8,9,21,22]. While several studies have examined the subclasses of univalent functions involving fractional operators, the effect of the operator $\varOmega_{0,z}^{\varrho}$ on functions with negative coefficients has not been extensively explored. The fractional derivative provides an intermediate operator that bridges between a function and its higher-order derivatives. This helps in refining coefficient estimates, distortion theorems, growth results, and other analytic characteristics more precisely. We use fractional derivative operators in univalent function theory to extend classical results, develop new subclasses, obtain sharper bounds, and connect geometric function theory with fractional calculus-based real world applications. This motivates us to define and study the class $\mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$.

This paper focuses on the study of univalent functions associated with the fractional derivative operator $\varOmega_{0,z}^{\varrho}$. In this context, we introduce the class $\mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$, consisting of functions of the form (1.2) that satisfy the analytic condition

$$\left|\frac{\eta z^2(\varOmega_{0,z}^{\varrho}\xi(z))''+\lambda\left\{z(\varOmega_{0,z}^{\varrho}\xi(z))'-\varOmega_{0,z}^{\varrho}\xi(z)\right\}}{\mu z(\varOmega_{0,z}^{\varrho}\xi(z))'+(1-\lambda)\varOmega_{0,z}^{\varrho}\xi(z)}\right|<\varepsilon. \tag{1.9}$$

Where $\varOmega_{0,z}^{\varrho}\xi(z)$ is defined by equation (1.7), $z\in\mathbb{U},\ 0<\varepsilon<1,\ 0\le\lambda<1,\ 0\le\eta\le1,$ and $0\le\mu<1$. The parameters $\varepsilon,\lambda,\eta,$ and μ play a crucial role in regulating the geometric characteristics of the class $\mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$. The bound ε restricts the analytic expression in (1.9), thereby controlling variation in behavior and preserving univalence. The parameters λ and μ regulate the influence of the function and its first derivative, affecting starlikeness, growth, and distortion features. The parameter η modifies the effect of the second derivative, enabling more precise control over convexity and angular curvature. Thus, the combined effect of these parameters allows a versatile framework for shaping and studying the geometric nature of the class.

Remark 1.1 *If* $\eta=0$ *and* $\varrho=0$*, then class* $\mathcal{F}_{0,z}^{0}(\varepsilon,\lambda,0,\mu)\equiv\mathcal{S}(\lambda,\mu,\varepsilon)$ *is the class define in [3] (also see [13]) that consists the function* $\xi(z)\in\mathcal{N}$ *satisfying*

$$\left|\frac{\lambda\left\{\xi'(z)-\frac{\xi(z)}{z}\right\}}{\mu\xi'(z)+(1-\lambda)\frac{\xi(z)}{z}}\right|<\varepsilon.$$

In recent years, numerous subclasses of analytic univalent functions have been introduced and studied from various perspectives. Significant contributions in this direction include the work of Amourah and Darus [2], Hern *et al.* [14], Janteng and Hern [17] and Hasoon and Al-Ziadi [13] among others.

Motivated by these developments, we investigate precise results for the class $\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$ including coefficient bounds, distortion and growth theorems, radii of starlikeness and convexity, mean values results (arithmetic and weighted) as well as Hadamard product, and closure properties.

2. Coefficient Bound

In this section, we establish the necessary and sufficient conditions for functions $\xi(z)$ belonging to the class $\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$.

Theorem 2.1 *Let the function $\xi(z)$ be defined by (1.2). Then, $\xi(z)$ belongs to the class $\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$ if and only if the following inequality is satisfied:*

$$\sum_{s=2}^{\infty}\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}a_s\le\varepsilon(\mu+1-\lambda) \tag{2.1}$$

where the parameters are given by

$$0<\varepsilon<1,\ 0\le\eta\le1,\ 0\le\lambda<1,\ 0\le\mu<1 \text{ and } z\in\mathbb{U}.$$

Furthermore, the condition (2.1) is sharp, and the extremal case is attained for the function

$$\xi(z)=z-\frac{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}{\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]}z^s \tag{2.2}$$

Proof: Assume that inequality (2.1) holds and let $|z|=1$. Then we obtain

$$\begin{aligned}
&\left|\eta z^2(\Omega^{\varrho}_{0,z}\xi(z))''+\lambda\left\{z(\Omega^{\varrho}_{0,z}\xi(z))'-\Omega^{\varrho}_{0,z}\xi(z)\right\}\right|-\varepsilon\left|\mu z(\Omega^{\varrho}_{0,z}\xi(z))'+(1-\varrho)\Omega^{\varrho}_{0,z}\xi(z)\right|\\
&=\left|-\sum_{s=2}^{\infty}\left[(s-1)(\eta s+\lambda)\right]\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}a_sz^s\right|\\
&\qquad-\varepsilon\left|(\mu+1-\lambda)z-\sum_{s=2}^{\infty}(s\mu+1-\lambda)\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}a_sz^s\right|\\
&\le\sum_{s=2}^{\infty}\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}a_s-\varepsilon(\mu+1-\lambda)\\
&\le 0.
\end{aligned}$$

Hence, by maximum modulus principle, $\xi(z)\in\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$.
Conversely, we consider that $\xi(z)\in\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$, so we get

$$\left|\frac{\eta z^2(\Omega^{\varrho}_{0,z}\xi(z))''+\lambda\left\{z(\Omega^{\varrho}_{0,z}\xi(z))'-\Omega^{\varrho}_{0,z}\xi(z)\right\}}{\mu z(\Omega^{\varrho}_{0,z}\xi(z))'+(1-\lambda)\Omega^{\varrho}_{0,z}\xi(z)}\right|<\varepsilon,\quad z\in\mathbb{U}$$

This implies that

$$\left|\eta z^2(\Omega^{\varrho}_{0,z}\xi(z))''+\lambda\left\{z(\Omega^{\varrho}_{0,z}\xi(z))'-\Omega^{\varrho}_{0,z}\xi(z)\right\}\right|<\varepsilon\left|\mu z(\Omega^{\varrho}_{0,z}\xi(z))'+(1-\lambda)\Omega^{\varrho}_{0,z}\xi(z)\right|.$$

Therefore, we get

$$\begin{aligned}
&\left|-\sum_{s=2}^{\infty}\left[(s-1)(\eta s+\lambda)\right]\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}a_sz^s\right|\\
&\qquad<\varepsilon\left|(\mu+1-\lambda)z-\sum_{s=2}^{\infty}(s\mu+1-\lambda)\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}a_sz^s\right|
\end{aligned}$$

Thus, it follows that

$$\sum_{s=2}^{\infty}\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}a_s<\varepsilon(\mu+1-\lambda).$$

This completes the proof. □

Corollary 2.1 *If $\xi(z)\in\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$, then coefficients of $\xi(z)$ satisfy the bound*

$$a_s\leq\frac{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}{\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]},\quad \textit{for}\quad s\geqslant 2. \tag{2.3}$$

3. Distortion and Growth Bounds

In this section, we derive the distortion and growth bounds for the functions belonging to class $\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$.

Theorem 3.1 *Let function $\xi(z)$ be defined by (1.2). If $\xi(z)\in\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$, then for $0<|z|=r<1$, the following inequality holds:*

$$\begin{aligned} r-\frac{\varepsilon(\mu+1-\lambda)\Gamma(3-\varrho)r^2}{2\Gamma(2-\varrho)\left[(2\eta+\lambda)+\varepsilon(2\mu+1-\lambda)\right]}&\leq|\xi(z)|\\ &\leq r+\frac{\varepsilon(\mu+1-\lambda)\Gamma(3-\varrho)r^2}{2\Gamma(2-\varrho)\left[(2\eta+\lambda)+\varepsilon(2\mu+1-\lambda)\right]}.\end{aligned}$$

Both bounds are sharp for the extremal function

$$\xi(z)=z-\frac{\varepsilon(\mu+1-\lambda)\Gamma(3-\varrho)}{2\Gamma(2-\varrho)\left[2\eta+\lambda+\varepsilon(2\mu+1-\lambda)\right]}z^2. \tag{3.1}$$

Proof: From the Theorem 2.1, we have

$$\sum_{s=2}^{\infty}\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}a_s\leq\varepsilon(\mu+1-\lambda).$$

It follows that

$$\begin{aligned}\left[2\eta+\lambda+\varepsilon(2\mu+1-\lambda)\right]&\frac{\Gamma(2-\varrho)\Gamma(3)}{\Gamma(3-\varrho)}\sum_{s=2}^{\infty}a_s\\ &\leq\sum_{s=2}^{\infty}\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]\frac{\Gamma(2-\varrho)\Gamma(s+1)}{\Gamma(s+1-\varrho)}a_s\\ &\leq\varepsilon(\mu+1-\lambda).\end{aligned}$$

Thus

$$\sum_{s=2}^{\infty}a_s\leq\frac{\varepsilon(\mu+1-\lambda)\Gamma(3-\varrho)}{2\Gamma(2-\varrho)\left[2\eta+\lambda+\varepsilon(2\mu+1-\lambda)\right]}.$$

For $\xi(z)\in\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$, we obtain

$$\begin{aligned}|\xi(z)|&=\left|z-\sum_{s=2}^{\infty}a_sz^s\right|\\ &\leq|z|+|z|^2\sum_{s=2}^{\infty}a_s\\ &\leq r+\frac{\varepsilon(\mu+1-\lambda)\Gamma(3-\varrho)}{2\Gamma(2-\varrho)\left[2\eta+\lambda+\varepsilon(2\mu+1-\lambda)\right]}r^2.\end{aligned}$$

On the other hand

$$
\begin{aligned}
|\xi(z)| &= \left| z - \sum_{s=2}^{\infty} a_s z^s \right| \\
&\geqslant |z| - |z|^2 \sum_{s=2}^{\infty} a_s \\
&\geqslant r - \frac{\varepsilon(\mu+1-\lambda)\Gamma(3-\varrho)}{2\Gamma(2-\varrho)\left[2\eta+\lambda+\varepsilon(2\mu+1-\lambda)\right]} r^2.
\end{aligned}
$$

This completes the proof. □

Theorem 3.2 *Let the analytic function $\xi(z)$ defined by (1.2) belongs to the class $\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$. Then, for $0<|z|=r<1$, the following inequality for the derivative holds:*

$$
\begin{aligned}
1 - \frac{2\varepsilon(\mu+1-\lambda)\Gamma(3-\varrho)r}{2\Gamma(2-\varrho)\left[(2\eta+\lambda)+\varepsilon(2\mu+1-\lambda)\right]} &\leq |\xi'(z)| \\
&\leq 1 + \frac{2\varepsilon(\mu+1-\lambda)\Gamma(3-\varrho)r}{2\Gamma(2-\varrho)\left[(2\eta+\lambda)+\varepsilon(2\mu+1-\lambda)\right]}.
\end{aligned}
$$

These bounds are sharp and are attained by the extremal function defined in (3.1).

Proof: For any function $\xi(z)\in\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$, we have

$$
\begin{aligned}
|\xi'(z)| &= \left| 1 - \sum_{s=2}^{\infty} s a_s z^{s-1} \right| \\
&\leq 1 + |z| \sum_{s=2}^{\infty} s a_s \\
&\leq 1 + \frac{2\varepsilon(\mu+1-\lambda)\Gamma(3-\varrho)}{2\Gamma(2-\varrho)\left[2\eta+\lambda+\varepsilon(2\mu+1-\lambda)\right]} r.
\end{aligned}
$$

Similarly, for the lower estimate,

$$
\begin{aligned}
|\xi'(z)| &= \left| 1 - \sum_{s=2}^{\infty} s a_s z^{s-1} \right| \\
&\geqslant 1 - |z| \sum_{s=2}^{\infty} s a_s \\
&\geqslant 1 - \frac{2\varepsilon(\mu+1-\lambda)\Gamma(3-\varrho)}{2\Gamma(2-\varrho)\left[2\eta+\lambda+\varepsilon(2\mu+1-\lambda)\right]} r.
\end{aligned}
$$

Thus, the required growth estimate for $|\xi'(z)|$ follows, and the extremal function in (3.1) verifies the sharpness of the estimate. This completes the proof. □

4. Radius of Convexity and Starlikeness

In this section, we obtain the radii of convexity and starlikeness for the functions $\xi(z)$ in the class $\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$.

Theorem 4.1 *Let $\xi(z)\in\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$, then $\xi(z)$ is univalently convex of order $\psi,(0\leq\psi<1)$ within the disk $|z|<R_1$, where*

$$
R_1 = \inf_s \left[\frac{(1-\psi)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]\Gamma(2-\varrho)\Gamma(s+1)}{s\varepsilon(\mu+1-\lambda)(s-\psi)} \right]^{\frac{1}{s-1}}, \quad s \geqslant 2.
$$

The bound is sharp for the extremal function defined in (2.2).

Proof: For convexity of order ψ, it is sufficient to verify that

$$\left|\frac{z\xi''(z)}{\xi'(z)}\right| \leq 1-\psi, \quad (0 \leq \psi < 1)$$

inside $|z| < R_1$. Now, using the expression of $\xi(z)$, we have

$$\left|\frac{z\xi''(z)}{\xi'(z)}\right| \leq \frac{\sum_{s=2}^{\infty} s(s-1)a_s |z|^{s-1}}{1-\sum_{s=2}^{\infty} sa_s |z|^{s-1}}.$$

Thus the condition

$$\sum_{s=2}^{\infty} \frac{s(s-\psi)}{1-\psi} a_s |z|^{s-1} \leq 1 \tag{4.1}$$

is required. Applying theorem 2.1, this holds if

$$\frac{s(s-\psi)}{1-\psi} |z|^{s-1} \leq \frac{[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]\,\Gamma(2-\varrho)\Gamma(s+1)}{\Gamma(s+1-\varrho)\varepsilon(\mu+1-\lambda)},$$

Hence,

$$|z| \leq \left[\frac{(1-\psi)\,[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]\,\Gamma(2-\varrho)\Gamma(s+1)}{s(s-\psi)\Gamma(s+1-\varrho)\varepsilon(\mu+1-\lambda)}\right]^{\frac{1}{s-1}}, \quad (s \geqslant 2).$$

Taking infimum over $(s \geq 2)$ gives the radius R_1. We get

$$R_1 = \inf_s \left[\frac{(1-\psi)\,[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]\,\Gamma(2-\varrho)\Gamma(s+1)}{s\varepsilon(\mu+1-\lambda)(s-\psi)}\right]^{\frac{1}{s-1}}$$

Hence proved. □

Theorem 4.2 *Let $\xi(z) \in \mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$, then the function $\xi(z)$ is univalently starlike of order $\psi, (0 \leq \psi < 1)$ within the disk $|z| < R_2$, where*

$$R_2 = \inf_s \left[\frac{(1-\psi)\,[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]\,\Gamma(2-\varrho)\Gamma(s+1)}{\varepsilon(\mu+1-\lambda)(s-\psi)}\right]^{\frac{1}{s-1}}, \quad s \geqslant 2.$$

The bound is sharp for the extremal function given in (2.2).

Proof: For starlike of order ψ, it is sufficient to prove that

$$\left|\frac{z\xi'(z)}{\xi(z)} - 1\right| \leq 1-\psi, \quad (0 \leq \psi < 1)$$

for $|z| < R_2$. From the definition of $\xi(z)$, we obtain Now

$$\left|\frac{z\xi'(z)}{\xi(z)} - 1\right| \leq \frac{\sum_{s=2}^{\infty}(s-1)a_s |z|^{s-1}}{1-\sum_{s=2}^{\infty} a_s |z|^{s-1}}.$$

Thus, the condition reduces to

$$\sum_{s=2}^{\infty} \frac{(s-\psi)}{1-\psi} a_s |z|^{s-1} \leq 1. \tag{4.2}$$

By Theorem 2.1, this holds if

$$\frac{(s-\psi)}{1-\psi} |z|^{s-1} \leq \frac{[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]\,\Gamma(2-\varrho)\Gamma(s+1)}{\Gamma(s+1-\varrho)\varepsilon(\mu+1-\lambda)},$$

Therefore,

$$|z| \leq \left[\frac{(1-\psi)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]\Gamma(2-\varrho)\Gamma(s+1)}{(s-\psi)\Gamma(s+1-\varrho)\varepsilon(\mu+1-\lambda)}\right]^{\frac{1}{s-1}}, \quad (s \geqslant 2).$$

Taking infimum over $(s \geqslant 2)$ yields the radius $|z| = R_2$, we get

$$R_2 = \inf_s \left[\frac{(1-\psi)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]\Gamma(2-\varrho)\Gamma(s+1)}{\varepsilon(\mu+1-\lambda)(s-\psi)}\right]^{\frac{1}{s-1}}$$

Hence proved. □

5. Weighted Mean and Arithmetic Mean

In this sections, we examine the properties of the Weighted Mean and Arithmetic Mean for the subclass $\mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$.

Definition 5.1 *[13] The Weighted Mean $W_\sigma(z)$ for analytic functions $\phi_1(z),\phi_2(z) \in \mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$ is given by*

$$W_\sigma(z) = \frac{1}{2}\left[(1-\sigma)\phi_1(z)+(1+\sigma)\phi_2(z)\right], \quad 0<\sigma<1.$$

Theorem 5.1 *Let $\phi_1(z),\phi_2(z) \in \mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$, then their weighted mean $W_\sigma(z)$ is also in the class $\mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$.*

Proof: From definition (5.1) of weighted mean $W_\sigma(z)$, we have

$$\begin{aligned}
W_\sigma(z) &= \frac{1}{2}\left[(1-\sigma)\phi_1(z)+(1+\sigma)\phi_2(z)\right] \\
&= \frac{1}{2}\left[(1-\sigma)\left(z-\sum_{s=2}^{\infty}a_s z^s\right)+(1+\sigma)\left(z-\sum_{s=2}^{\infty}b_s z^s\right)\right] \\
&= z-\sum_{s=2}^{\infty}\frac{1}{2}\left[(1-\sigma)a_s+(1+\sigma)b_s\right]z^s.
\end{aligned}$$

Since $\phi_1(z),\phi_2(z) \in \mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$, by Theorem 2.1 they satisfy

$$\sum_{s=2}^{\infty}\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}a_s \leq \varepsilon(\mu+1-\lambda)$$

and similarly for b_s

$$\sum_{s=2}^{\infty}\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}b_s \leq \varepsilon(\mu+1-\lambda).$$

Hence,

$$\begin{aligned}
&\sum_{s=2}^{\infty}\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}\frac{1}{2}\left[(1-\sigma)a_s+(1+\sigma)b_s\right] \\
&= \frac{1}{2}(1-\sigma)\sum_{s=2}^{\infty}\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}a_s \\
&+ \frac{1}{2}(1+\sigma)\sum_{s=2}^{\infty}\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}b_s \\
&\leq \frac{1}{2}\varepsilon(1-\sigma)(\mu+1-\lambda)+\frac{1}{2}\varepsilon(1+\sigma)(\mu+1-\lambda) \\
&= \varepsilon(\mu+1-\lambda).
\end{aligned}$$

Therefore, weighted mean $W_\sigma(z) \in \mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$. □

Definition 5.2 *[13] The Arithmetic Mean $m(z)$ for analytic functions $\xi_1(z), \xi_2(z), \xi_3(z), ..., \xi_n(z)$, is defined as*

$$m(z) = \frac{1}{n}\sum_{\iota=1}^{n}\xi_\iota(z), \quad \forall\ n \in \mathbb{N}.$$

Theorem 5.2 *Let $\xi_1(z), \xi_2(z), \xi_3(z), ..., \xi_\rho(z)$ are defined as*

$$\xi_\iota(z) = z - \sum_{s=2}^{\infty} a_{s,\iota} z^s, \quad (a_{s,\iota} \geqslant 0, \iota = 1, 2, 3, ...\rho, \quad s \geqslant 2), \tag{5.1}$$

and belongs to the class $\mathcal{F}^{\varrho}_{0,z}(\varepsilon, \lambda, \eta, \mu)$. The Arithmetic Mean for the $\xi_\iota(z)(\iota = 1, 2, ..., \rho)$ that defined as

$$m(z) = \frac{1}{\rho}\sum_{\iota=1}^{\rho}\xi_\iota(z). \tag{5.2}$$

Then the Arithmetic Mean is also in the class $\mathcal{F}^{\varrho}_{0,z}(\varepsilon, \lambda, \eta, \mu)$.

Proof: From equation (5.1) and (5.2), we have

$$\begin{aligned} m(z) &= \frac{1}{\rho}\sum_{\iota=1}^{\rho}\left(z - \sum_{s=2}^{\infty} a_{s,\iota} z^s\right) \\ &= z - \sum_{s=2}^{\infty}\left(\frac{1}{\rho}\sum_{\iota=1}^{\rho} a_{s,\iota}\right) z^s. \end{aligned}$$

Since $\xi_\iota(z) \in \mathcal{F}^{\varrho}_{0,z}(\varepsilon, \lambda, \eta, \mu)$ for every $(\iota = 1, 2, 3, ...\rho)$, by Theorem 2.1, they satisfy

$$\begin{aligned} &\sum_{s=2}^{\infty}[(s-1)(\eta s + \lambda) + \varepsilon(s\mu + 1 - \lambda)]\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}\left(\frac{1}{\rho}\sum_{\iota=1}^{\rho} a_{s,\iota}\right) \\ &= \frac{1}{\rho}\sum_{\iota=1}^{\rho}\left(\sum_{s=2}^{\infty}[(s-1)(\eta s + \lambda) + \varepsilon(s\mu + 1 - \lambda)]\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)} a_{s,\iota}\right) \\ &\leq \frac{1}{\rho}\sum_{\iota=1}^{\rho}\varepsilon(\mu + 1 - \lambda) \\ &= \varepsilon(\mu + 1 - \lambda) \end{aligned}$$

Thus $m(z) \in \mathcal{F}^{\varrho}_{0,z}(\varepsilon, \lambda, \eta, \mu)$. This completes the proof.

□

6. Convex Set

Theorem 6.1 *The class $\mathcal{F}^{\varrho}_{0,z}(\varepsilon, \lambda, \eta, \mu)$ forms a convex set.*

Proof: Let functions $\xi(z), \varphi(z) \in \mathcal{F}^{\varrho}_{0,z}(\varepsilon, \lambda, \eta, \mu)$. For every real number ν with $0 \leq \nu \leq 1$, we must show that,

$$(1-\nu)\xi(z) + \nu\varphi(z) \in \mathcal{F}^{\varrho}_{0,z}(\varepsilon, \lambda, \eta, \mu). \tag{6.1}$$

We have,

$$(1-\nu)\xi(z) + \nu\varphi(z) = z - \sum_{n=2}^{\infty}[(1-\nu)a_s + \nu b_s] z^n.$$

Using Theorem 2.1, it follows that

$$\begin{aligned}
&\sum_{s=2}^{\infty}[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]\,[(1-\nu)a_s+\nu b_s]\,\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}\\
&=(1-\nu)\sum_{s=2}^{\infty}[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]\,\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}a_s\\
&\qquad+\nu\sum_{s=2}^{\infty}[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]\,\frac{\Gamma(s+1)\Gamma(2-\varrho)}{\Gamma(s+1-\varrho)}b_s\\
&\leq(1-\nu)\varepsilon(\mu+1-\lambda)+\nu\varepsilon(\mu+1-\lambda)\\
&=\varepsilon(\mu+1-\lambda).
\end{aligned}$$

Therefore, the class $\mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$ is convex. □

7. Extreme Point

In this part, we establish the set of extreme points for the class $\mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$.

Theorem 7.1 *Define*

$$\xi_1(z)=z,\quad \xi_s(z)=z-\frac{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}{\Gamma(2-\varrho)\Gamma(s+1)\,[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]}z^s,\ \textit{for } s=2,3,4,\ldots$$

Then a function $\xi(z)$ belongs to $\mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$ if and only if it can be expressed as

$$\xi(z)=\sum_{s=1}^{\infty}\theta_s\xi_s(z)\quad \textit{where}\quad \theta_s\geqslant 0\quad \textit{and}\quad \sum_{s=1}^{\infty}\theta_s=1.$$

Proof: Consider that

$$\xi(z)=\sum_{s=1}^{\infty}\theta_s\xi_s(z)$$

Then we obtain

$$\xi(z)=z-\sum_{s=2}^{\infty}\frac{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}{\Gamma(2-\varrho)\Gamma(s+1)\,[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]}\theta_s z^s.$$

Hence, $\xi(z)$ clearly belongs to $\mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$ because

$$\begin{aligned}
&\sum_{s=2}^{\infty}\frac{\{\Gamma(2-\varrho)\Gamma(s+1)\,[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]\}\,.\,\{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)\}}{\{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)\Gamma(2-\varrho)\Gamma(s+1)\}\,.\,\{[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]\}}\theta_s\\
&=\sum_{s=2}^{\infty}\theta_s\\
&=1-\theta_1\leq 1.
\end{aligned}$$

Conversely, let $\xi(z)\in\mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$. We now prove that

$$\xi(z)=\sum_{s=1}^{\infty}\theta_s\xi_s(z).$$

Now $\xi(z)\in\mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$ from Theorem 2.1, the coefficient a_s satisfy

$$a_s\leq\frac{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}{\Gamma(2-\varrho)\Gamma(s+1)\,[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]},\ s\geqslant 2$$

Define

$$\theta_s = \frac{\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]}{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}a_s, \quad s=2,3,4,...$$

$$\theta_1 = 1-\sum_{s=2}^{\infty}\theta_s$$

Then it follows that

$$\xi(z)=\sum_{s=1}^{\infty}\theta_s\xi_s(z),$$

which completes the proof. □

8. Hadamard Product

In this section, we present the Hadamard product for functions in the class $\mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$.

Theorem 8.1 *Let $h_1(z)$ and $h_2(z)$ belong to the class $\mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$. Then their Hadamard product $h_1 * h_2 \in \mathcal{F}_{0,z}^{\varrho}(l,\lambda,\eta,\mu)$ where*

$$h_1(z)=z-\sum_{s=2}^{\infty}a_sz^s, \quad h_2(z)=z-\sum_{s=2}^{\infty}b_sz^s \quad \textit{and} \quad h_1*h_2(z)=z-\sum_{s=2}^{\infty}a_sb_sz^s$$

and the parameter l satisfies

$$l \geqslant \frac{\varepsilon^2(\mu+1-\lambda)(\eta s+\lambda)\Gamma(s+1-\varrho)(s-1)}{\left[\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]^2 - \varepsilon^2(\mu+1-\lambda)(\mu s+1-\lambda)\Gamma(s+1-\varrho)\right]}.$$

Proof: $h_1 * h_2 \in \mathcal{F}_{0,z}^{\varrho}(\varepsilon,\lambda,\eta,\mu)$ and so

$$\frac{\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]}{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}a_s \leq 1 \tag{8.1}$$

and

$$\frac{\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]}{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}b_s \leq 1. \tag{8.2}$$

For the convolution $h_1 * h_2$, we require the smallest real number l such that

$$\frac{\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+l(s\mu+1-\lambda)\right]}{l(\mu+1-\lambda)\Gamma(s+1-\varrho)}a_sb_s \leq 1 \tag{8.3}$$

Using Cauchy-Schwartz inequality, we obtain

$$\frac{\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]}{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}\sqrt{a_sb_s} \leq 1 \tag{8.4}$$

therefore it is sufficient to show that,

$$\begin{aligned}&\frac{\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+l(s\mu+1-\lambda)\right]}{l(\mu+1-\lambda)\Gamma(s+1-\varrho)}a_sb_s\\ &\leq \frac{\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]}{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}\sqrt{a_sb_s}\end{aligned}$$

This inequality is equivalent to

$$\sqrt{a_s b_s} \leq \frac{[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]\, l}{[(s-1)(\eta s+\lambda)+l(s\mu+1-\lambda)]\,\varepsilon} \tag{8.5}$$

from equation (8.4), we have

$$\sqrt{a_s b_s} \leq \frac{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}{\Gamma(2-\varrho)\Gamma(s+1)\,[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]}$$

so, it is enough to prove that

$$\frac{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}{\Gamma(2-\varrho)\Gamma(s+1)\,[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]} \leq \frac{[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]\, l}{[(s-1)(\eta s+\lambda)+l(s\mu+1-\lambda)]\,\varepsilon} \tag{8.6}$$

Simplifies equation (8.6), we get

$$l \geqslant \frac{\varepsilon^2(\mu+1-\lambda)(\eta s+\lambda)\Gamma(s+1-\varrho)(s-1)}{\Big[\Gamma(2-\varrho)\Gamma(s+1)\,[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]^2 - \varepsilon^2(\mu+1-\lambda)(\mu s+1-\lambda)\Gamma(s+1-\varrho)\Big]}.$$

□

9. Closure Theorem

In this section, we established the closure property for the class $\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$.

Theorem 9.1 *Let $\xi_j \in \mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$, $\quad j=1,2,3,...,n$ then*

$$g(z)=\sum_{j=1}^{n} c_j\xi_j(z) \in \mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu),\ \textit{where } \xi_j(z)=z-\sum_{s=2}^{\infty} a_{s,j}z^s \ \textit{and } \sum_{j=1}^{n} c_j = 1.$$

Proof: We consider,

$$\begin{aligned} g(z) &= \sum_{j=1}^{n} c_j\xi_j(z) \\ &= z-\sum_{s=2}^{\infty}\sum_{j=1}^{n} c_j a_{s,j} z^s \\ &= z-\sum_{s=2}^{\infty} e_s z^s, \end{aligned}$$

where $e_s=\sum_{j=1}^{n} c_j a_{s,j}$. Thus $g(z)\in\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$ if

$$\sum_{s=2}^{\infty} \frac{\Gamma(2-\varrho)\Gamma(s+1)\,[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]}{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)} e_s \leq 1,$$

That is , if

$$\begin{aligned} &\sum_{s=2}^{\infty}\sum_{j=1}^{n} \frac{\Gamma(2-\varrho)\Gamma(s+1)\,[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]}{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)} c_i a_{s,i} \\ &\quad = \sum_{j=1}^{n} c_i \sum_{s=2}^{\infty} \frac{\Gamma(2-\varrho)\Gamma(s+1)\,[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)]}{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)} a_{s,i} \\ &\quad \leq \sum_{j=1}^{n} c_j = 1 \end{aligned}$$

Since,

$$\frac{\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]}{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}a_{s,j}\leq 1 \text{ as } \xi_j\in\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu).$$

□

Theorem 9.2 *Let* $f,g\in\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$*, then*

$$h(z)=z-\sum_{s=2}^{\infty}(a_s^2+b_s^2)z^s$$

$h(z)\in\mathcal{F}^{\varrho}_{0,z}(l,\lambda,\eta,\mu)$ *where*

$$l\geqslant\frac{2\varepsilon^2(\mu+1-\varrho)\Gamma(s+1-\varrho)(s-1)(\eta s+\lambda)}{\Big[\Gamma(s+1)\Gamma(2-\varrho)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]^2-2\varepsilon^2(\mu+1-\varrho)\Gamma(s+1-\varrho)(s\mu+1-\lambda)\Big]}.$$

Proof: Since $f,g\in\mathcal{F}^{\varrho}_{0,z}(\varepsilon,\lambda,\eta,\mu)$ then from Theorem 2.1 we have,

$$\sum_{s=2}^{\infty}\left[\frac{\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]}{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}a_s\right]^2\leq 1 \quad \textit{and}$$

$$\sum_{s=2}^{\infty}\left[\frac{\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]}{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}b_s\right]^2\leq 1.$$

From last inequalities, we get

$$\sum_{s=2}^{\infty}\frac{1}{2}\left[\frac{\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]}{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}\right]^2\left(a_s^2+b_s^2\right)\leq 1. \tag{9.1}$$

but $h(z)\in\mathcal{F}^{\varrho}_{0,z}(l,\lambda,\eta,\mu)$ if and only if

$$\sum_{s=2}^{\infty}\frac{\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+l(s\mu+1-\lambda)\right]}{l(\mu+1-\lambda)\Gamma(s+1-\varrho)}\left(a_s^2+b_s^2\right)\leq 1. \tag{9.2}$$

Where $0<l<1$, from equation (9.1) and (9.2) ,

$$\frac{\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+l(s\mu+1-\lambda)\right]}{l(\mu+1-\lambda)\Gamma(s+1-\varrho)}\leq\frac{1}{2}\left[\frac{\Gamma(2-\varrho)\Gamma(s+1)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]}{\varepsilon(\mu+1-\lambda)\Gamma(s+1-\varrho)}\right]^2$$

simplifying above inequality, we get

$$l\geqslant\frac{2\varepsilon^2(\mu+1-\lambda)\Gamma(s+1-\varrho)(s-1)(\eta s+\lambda)}{\Big[\Gamma(s+1)\Gamma(2-\varrho)\left[(s-1)(\eta s+\lambda)+\varepsilon(s\mu+1-\lambda)\right]^2-2\varepsilon^2(\mu+1-\lambda)\Gamma(s+1-\varrho)(s\mu+1-\lambda)\Big]}.$$

□

10. Conclusions

In the present work, we have investigated a new class of analytic univalent functions associated with the fractional derivative operator and established several significant results. We have derived analytical aspects such as coefficient estimates, growth theorems, distortion bounds, closure theorems, and the Hadamard product. Furthermore, we have examined geometrical properties such as radius of convexity, radius of starlikeness, and extremal points for the newly defined analytic class related to the fractional derivative operator.

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