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On the Diophantine Equation $(p^n)^x + (3^m p + 2)^y = z^2$, where p and $3^m p + 2$ are Prime Numbers

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ABSTRACT: The main objective of this paper aims to give methods for find the Diophantine equation $(p^n)^x + (3^m p + 2)^y = z^2$ in $\mathbb{N}$, where the parameters $p \geq 3$ and $3^m p + 2$ are prime integers. Concretely, we employ a congruence method, and we investigate that the nonexistence solutions of this equation for a prime $p > 3$. Subsequently, we will establish that this equation has no solutions for the prime $p = 3$ and any $m > 1$. In the sequel, for $m = 1$, an analysis via the elliptic curves reveals that if $n = 1$, this equation has a unique solution, given by $(x, y, z) = (5, 4, 122)$.

Key Words: Diophantine equations, elliptic curve, factor and congruent method.

Contents

1. Introduction

The subject of Diophantine equations is very important in number theory. For instance, the study of exponential Diophantine equations, a category that includes famous problems those appearing in the Fermat-Catalan and Beal's conjectures, has a long history, which this equation is typically of the type $x^m + y^n = z^k$ where the aim is to find integer solution under certain conditions on the exponents. When more variables are included to a Diophantine equation as exponents, this equation is formed. Despite attempts to address particular cases, such as Catalan's conjecture, there isn't yet a complete theory for solving these equations. Specifically, according to Catalan [3], Catalan's conjecture states that the only solution to the equation $x^p - y^q = 1$ where $\min\{p, q, x, y\} > 1$, is the single exceptional $3^2 - 2^3 = 1$. For the following examples: when $a^2 + b^2 = c^2$, where a is an even number and $\gcd(a, b, c) = 1$, Cohn [4] proved that the Diophantine equation

$$x^2 + q^m = y^n \tag{1.1}$$

has exactly three families of the solutions (x, m, n) where $q = 2$ and m is an odd integer. Following that, Terai [10] conjectured that the equation (1.1) has only the positive integer solution $(x, m, n) = (a, 2, 2)$ when $y = p$. However, Arif and Abu Muriefah [1] solved that the equation (1.1) has one family of the solution (x, m, n) where $q = 3$ and m is an odd integer. Two years later, Luca [6] showed the existence of exactly one family of the solution (x, m, n) where $q = 3$ and m is an even integer. For following, Arif and Abu Muriefah [2] actually proved that the Diophantine equation (1.1) with $m = 2k + 1$ has exactly two families of the solution (q, n, k, x, y) where q is an odd prime, $q \not\equiv 7 \pmod 8$, n is an odd integer ≥ 5, n is not a multiple of 3 and $(h, n) = 1$ with h is class number of the field $\mathbb{Q}(\sqrt{-q})$. In 2004, Mihăilescu [7] proved this conjecture. In the last few years, Zhu [12] studied that the Diophantine equation $x^2 + q^m = y^3$ where q is a prime and x, y, m are positive integers. Additionally, Terai [11] verified that the Diophantine equation (1.1) with $q = 2c - 1$ and $y = c$ has the only solution $(x, m, n) = (c - 1, 1, 2)$

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where the positive integers (x, m, n) and $2 \leq c \leq 30$ with $c \neq 12, 24$. Moreover, Deng [5] showed that the Diophantine equation $x^2 + q^m = c^{2n}$ has the only solution $(x, m, n) = (c^2 - 1, t, 2)$ where the positive integers (x, m, n), t and $c > 1$. Particullary, he proved that Terai's conjecture is valid if $c = 12$ and 24. Concluded this result when Terai's results is true for $2 \leq c \leq 30$. Recently, Nam [8] solved the Diophantine equation $(p^n)^x + (4^m p + 1)^y = z^2$, where $p \geq 3$ and $4^m p + 1$ are prime integers.

The present paper aims to study the Diophantine equation

$$(p^n)^x + (3^m p + 2)^y = z^2, \tag{1.2}$$

where p, $(3^m p + 2)$ are prime integers and m, n are positive integers.

For a prime number $p > 3$, the analysis of the equation (1.2) establish that no solution exist where either $x = 0$ or $y = 0$, as proven in Lemma 2.2 and 2.3. Moreover, for prime of the form $3N + 2$, the following Theorem 3.1 and 3.2 demonstrate that in any potential solution, at least one of the variables x or y must be even. Above all, Theorem 3.3 confirms that it is impossible for both x and y are odd integers.

For the prime $p = 3$, a detailed analysis of the equation (1.2) as follows:

- If $x = 0$. As shown in Lemma 2.3, the equation (1.2) has no solution.
- If $y = 0$. Lemma 2.4 demonstrate that a solution exist only for $n = 1$, yielding the unique positive integer solution $(x, z) = (1, 2)$.
- When x or y is even and $m > 1$. Theorem 3.4 and 3.5 establish that no solutions exist under these conditions.
- When $m = 1$ and y is even. The equation (1.2) is transformed into an elliptic curve of the form

  $$Y^2 = X^3 - N,$$

  where N is a certain positive integer. This method leads to the next result, present in Theorem 3.6, that the equation (1.2) admit a unique solution $(x, y, z) = (5, 4, 122)$ for $n = 1$.

2. Preliminaries

In this section, we can give some results that will be used in this paper.

Lemma 2.1 *Let any odd positive power to an integer of the form $3a + 2$, $a > 0$ is of the form $3A + 2$ for some positive integer A, i.e., for every odd integer $k \geq 1$, then we have the following statement that $(3a + 2)^k \equiv 2 \pmod 3$.*

Proof: We prove that by induction on k. When $k = 1$, it shows that $(3a + 2)^1 = 3a + 2 \equiv 2 \pmod 3$. Therefore, the assertion is valid for $k = 1$.

Let us assume that the induction is valid for all the odd power integers $k = 2r+1$ with r is nonnegative integer. Then we would get

$$(3a + 2)^{2r+3} = (3a + 2)^2 (3a + 2)^{2r+1}.$$

From inductive assumption, there exist a positive integer b such that $(3a + 2)^{2r+1} = 3b + 2$. Thus, we acquire

$$(3a + 2)^{2r+3} = (3a + 2)^2 (3b + 2) = 3[((3a + 1)^2 + 3)b + 2((a + 1)^2 + 2a^2)] + 2 = 3A + 2,$$

where $A = ((3a + 1)^2 + 3)b + 2((a + 1)^2 + 2a^2) > 0$. Then $(3a + 2)^{2r+3} \equiv 2 \pmod 3$, the claim is proved. □

Lemma 2.2 ([8]) *For a prime number $p > 3$, then we have the Diophantine equation*

$$(p^n)^x + 1 = z^2$$

has no nonnegative integer solutions n.

Lemma 2.3 *For a prime number $p \geq 3$, let m be a positive integer such that $3^m p + 2$ is a prime number. Then we have the Diophantine equation*

$$1 + (3^m p + 2)^y = z^2$$

has no nonnegative solutions.

Proof: Suppose now that the nonnegative integers y and z are such that $1 + (3^m p + 2)^y = z^2$. We separated it into two following facts.

- If $y = 0$, then we get $z^2 = 2$, which is impossible.
- If $y \neq 0$, then we obtain
$$(3^m p + 2)^y = z^2 - 1 = (z + 1)(z - 1).$$

Since the number $3^m p + 2$ is a prime, we assume that there exist integers $\lambda > \theta$ with $\lambda + \theta = y$ such that

$$z + 1 = (3^m p + 2)^\lambda \text{ and } z - 1 = (3^m p + 2)^\theta.$$

Thus, we get

$$(3^m p + 2)^\lambda - (3^m p + 2)^\theta = (3^m p + 2)^\theta[(3^m p + 2)^{\lambda-\theta} - 1] = 2. \quad (2.1)$$

Since $3^m p + 2 > 3$, it implies that $\theta = 0$ and therefore the equation (2.1) reveals $(3^m p + 2)^y = 3$, which is impossible. □

Lemma 2.4 **([8])** *For a positive integer n, then we have the Diophantine equation*

$$(3^n)^x + 1 = z^2$$

has a unique solution is $(x, z) = (1, 2)$ when $n = 1$ and has no solutions when $n > 1$.

3. Main Results

The objective of this section is to establish and discuss key findings regarding the solutions of the Diophantine equation (1.2) when $p > 3$, $3^m p + 2$ are prime numbers. When $x = 0$ or $y = 0$, by using Lemma 2.2 and 2.3, we get the equation (1.2) has no nonnegative integer solution. At first, we always consider x and y are positive integers in the following theorems.

3.1. The case $p > 3$

Now, assume that x is an even integer. Under this assumption, we derive the following profound theorem.

Theorem 3.1 *Assume that $x = 2k$ for a certain positive integer k. Then we have the Diophantine equation (1.2) has no solutions, where $p > 3$, $3^m p + 2$ are prime numbers and n, m are positive integers.*

Proof: From the assumption, we acquire

$$(3^m p + 2)^y = z^2 - (p^n)^{2k} = (z + p^{nk})(z - p^{nk}). \quad (3.1)$$

Since $3^m p + 2$ is a prime and $y \geq 1$, from the equation (3.1), we deduce that there exist integers λ and θ with $\lambda > \theta$, satisfying the given conditions

$$z + p^{nk} = (3^m p + 2)^\lambda \text{ and } z - p^{nk} = (3^m p + 2)^\theta$$

with $\lambda + \theta = y$. So, it comes that

$$(3^m p + 2)^\theta[(3^m p + 2)^{\lambda-\theta} - 1] = 2p^{nk}. \quad (3.2)$$

Since p and $3^m p+2$ are distinct prime numbers, from the equation (3.2) implies that $\theta = 0$. If we combine with condition y is a positive integer, the equation (3.2) reveals

$$2p^{nk} = (3^m p + 2)^y - 1 = (3^m p + 1)[(3^m p + 2)^{y-1} + \cdots + 1]. \tag{3.3}$$

By the equation (3.3), we deduce that $(3^m p + 1) \mid 2p^{nk}$. Since $3^m p + 2$ is a prime number, we obtain $m \geq 1$. Thus, $(3p + 1) \mid 2p^{nk}$, which is impossible. □

Let us now assume that y is an even integer. This assumption leads to the next following theorem.

Theorem 3.2 *Assume that $y = 2\ell$ for a certain positive integer ℓ. Then we have the Diophantine equation (1.2) has no solutions, where $p > 3$, $3^m p+2$ are prime numbers and n, m are positive integers.*

Proof: From the assumption, it follows that

$$(p^n)^x = z^2 - (3^m p + 2)^{2\ell} = [z + (3^m p + 2)^\ell][z - (3^m p + 2)^\ell]. \tag{3.4}$$

Since the number p is a prime and the fact that $x \geq 1$, it follows from the equation (3.4) that there exist integers $\lambda > \theta$, satisfying the below conditions

$$z + (3^m p + 2)^\ell = p^\lambda \quad \text{and} \quad z - (3^m p + 2)^\ell = p^\theta$$

with $\lambda + \theta = nx$. Thus, we obtain

$$p^\theta (p^{\lambda-\theta} - 1) = 2(3^m p + 2)^\ell. \tag{3.5}$$

Since the numbers 2, p and $3^m p + 2$ are distinct prime, from the equation (3.5) allows us to $\theta = 0$. If we combine with condition x is a positive integer, the equation (3.5) appears that

$$2(3^m p + 2)^\ell = p^{nx} - 1 = (p - 1)(p^{nx-1} + \cdots + 1). \tag{3.6}$$

By the equation (3.6), we conclude that $(p - 1)$ is a divisor of $2(3^m p + 2)^\ell$. For a prime integer $p > 3$, then $p - 1 \geq 4$. In the same way, we have that $3^m p + 2$ is also prime and $(p - 1)$ divides $2(3^m p + 2)^\ell$, we acquire

$$p + 2 < (3^m p + 2) \mid (p - 1),$$

which is impossible. Therefore, the equation (3.6) has no solutions. □

Now, let us consider that x and y are odd integers. This condition leads to the next subsequent result.

Theorem 3.3 *Assume that $x = 2k + 1$ and $y = 2\ell + 1$ for a certain positive integers k and ℓ. Then we have the Diophantine equation (1.2) has no solutions, where p is a prime number of the form $3N + 2$ for a certain positive integer N.*

Proof: Let us now that $p = 3N+2$ with N is some positive integer. By using Lemma 2.1, we get that there exists the positive integers (A, B), satisfying the given conditions $(p^n)^x = 3A+2$ and $(3^m p+2)^y = 3B+2$ for all the positive integers (x, y) and $A \neq B$. Hence, we obtain

$$z^2 = 3(A + B) + 4.$$

Since A and B are distinct positive integers, we get $A + B$ is odd number, it follows that z^2 is always odd number. On the other hand, we have 3 and $3^m p + 2$ are odd prime numbers, which means z^2 is even number. Therefore, $z^2 = 3(A + B) + 4 = 2c$ for some integer c. Thus, $A + B$ is even number, it implies that $A = B$, which is not acceptable. □

Example 3.1 *We consider $p = 17$ and m is a positive integer, if it satisfies that the number $q = 17 \cdot 3^m + 2$ is a prime, for the values, $(m, q) \in \{(1, 53), (3, 461), \ldots\}$. As results, then we have the Diophantine equation*

$$(17^n)^x + (17 \cdot 3^m + 2)^y = z^2$$

has no integer solutions.

3.2. The case $p = 3$

In this subsection, we present some results on solutions of the Diophantine equation

$$(3^n)^x + (3^{m+1} + 2)^y = z^2, \tag{3.7}$$

where $m \geq 1$, if it satisfies that the number $q = 3^{m+1} + 2$ is a prime, for the values, $(m, q) \in \{(1, 11), (2, 29), (3, 83), \ldots\}$. According to Lemma 2.3 and 2.4 when $x = 0$ or $y = 0$, we get the equation (3.7) has no solutions. Thus, from now on, we consistently assume that the positive integers x and y. At first, assume that x is an even integer. Under this assumption, we derive the following profound theorem.

Theorem 3.4 *Assume that $m \geq 1$, if it satisfies that the integer $3^{m+1} + 2$ is a prime and $x = 2k$ for a certain positive integer k. Then we have the Diophantine equation (3.7) has no solutions.*

Proof: Since $x = 2k$ with $k \geq 1$, we get

$$(3^{m+1} + 2)^y = z^2 - (3^n)^{2k} = (z + 3^{nk})(z - 3^{nk}). \tag{3.8}$$

Since $3^{m+1} + 2$ is a prime and $y \geq 1$, from equation (3.8) tells us that there exist integers λ and θ with $\lambda > \theta$, satisfying the given conditions

$$z + 3^{nk} = (3^{m+1} + 2)^\lambda \text{ and } z - 3^{nk} = (3^{m+1} + 2)^\theta$$

with $\lambda + \theta = y$. Hence, we get

$$(3^{m+1} + 2)^\theta[(3^{m+1} + 2)^{\lambda - \theta} - 1] = 2 \cdot 3^{nk}. \tag{3.9}$$

Since the primes 2, 3 and $3^{m+1} + 2$ are distinct, from the above equation, leading to $\theta = 0$. If we combine with condition y is a positive integer, the equation (3.9) gives

$$2 \cdot 3^{nk} = (3^{m+1} + 2)^y - 1 = (3^{m+1} + 1)[(3^{m+1} + 2)^{y-1} + \cdots + 1]. \tag{3.10}$$

By the equation (3.10), we get $(3^{m+1} + 1) \mid 2 \cdot 3^{nk}$. Since $3^{m+1} + 2$ is a prime, we obtain $m \geq 1$. Therefore, $10 \mid 2 \cdot 3^{nk}$, which is impossible. □

Let us now assume that y is an even integer. Under this assumption, we derive the next following profound theorem.

Theorem 3.5 *Assume that the integer $m > 1$, if it satisfies that the $3^{m+1} + 2$ is a prime and $y = 2\ell$ for a certain positive integer ℓ. Then we have the Diophantine equation (3.7) has no solutions.*

Proof: Since $y = 2\ell$ with $\ell \geq 1$, we get

$$3^{nx} = z^2 - (3^{m+1} + 2)^{2\ell} = [z + (3^{m+1} + 2)^\ell][z - (3^{m+1} + 2)^\ell]. \tag{3.11}$$

Since 3 is a prime and x is a positive integer, from the equation (3.11), it comes that there exist integers $\lambda > \theta$, satisfying the well-known conditions

$$z + (3^{m+1} + 2)^\ell = 3^\lambda \quad \text{and} \quad z - (3^{m+1} + 2)^\ell = 3^\theta$$

with $\lambda + \theta = nx$. Hence, we acquire

$$3^\theta(3^{\lambda - \theta} - 1) = 2(3^{m+1} + 2)^\ell. \tag{3.12}$$

Since the primes 2, 3 and $3^{m+1} + 2 > 3$ are distinct, from the equation (3.12), it implies that $\theta = 0$, and therefore the equation (3.12) tells us that

$$2(3^{m+1} + 2)^\ell = 3^{nx} - 1.$$

We put $nx = t$. Since $3^{m+1} + 2$ is a prime number, we get

$$3^{3^{m+1}+1} \equiv 1 \pmod{(3^{m+1}+2)}.$$

Let us consider that i and j be integers, satisfying the next conditions $0 \leq i, j < 3^{m+1} + 1$ with $i \neq j$. Then we get

$$3^{i+(3^{m+1}+1)k} \not\equiv 3^{j+(3^{m+1}+1)k} \pmod{(3^{m+1}+2)}$$

for all nonnegative integers k. Dividing it into two following parts.

- Let $t = i + (3^{m+1}+1)k$ be a positive integer, when $0 < i < 3^{m+1}+1$ and k is a nonnegative integer. Then

  $$2(3^{m+1}+2)^{\ell} = 3^t - 1 \not\equiv 0 \pmod{(3^{m+1}+2)},$$

  which is impossible.

- Let $t = \left(3^{m+1}+1\right) k$ be a positive integer, when k is a positive integer. Since $m > 1$, then we have $3^{m+1}+2$ is a prime such that $3^{m+1}+2 > 3^{sk}+1$ for all positive integers s and k. Hence, we acquire

  $$2(3^{m+1}+2)^{l} = 3^{(3^{m+1}+1)k} - 1.$$

  It's clear that 2 is a divisor of $3^{m+1}+1$ for all $m \geq 1$. Therefore, there exists $s \geq 1$ such that $3^{m+1}+1 = 2s$. Thus,

  $$2(3^{m+1}+2)^{\ell} = 3^{2sk} - 1 = 2\left(3^{sk}+1\right)\left(3^{sk-1}+\cdots+1\right),$$

  which is impossible, because $\left(3^{sk}+1\right) \nmid 2\left(3^{m+1}+2\right)^{\ell}$ for all positive integers m, ℓ, s, and k with $sk \geq 3$. So, in this part, the equation (3.7) has no solutions.

□

When $m = 1$, we conclude the given following profound theorem.

Theorem 3.6 *Assume that $y = 2\ell$ for a certain positive integer ℓ. Then we have the Diophantine equation*

$$(3^n)^x + 11^y = z^2$$

has a unique solution, given by $(x, y, z) = (5, 4, 122)$ for $n = 1$.

Proof: According to proof of Theorem 3.5, we get the following equation

$$2 \cdot 11^{\ell} = 3^{nx} - 1. \tag{3.13}$$

We put $nx = t$. Then rewrite the equation (3.13) becomes

$$2 \cdot 11^{\ell} = 3^t - 1. \tag{3.14}$$

Dividing through by two following cases.

- In the case $\ell = 2r$ is even number. Rewriting the equation (3.14), leading to

  $$2 \cdot (11^r)^2 = 3^t - 1. \tag{3.15}$$

  - If $t = 3u$, then this equation (3.15) becomes

    $$(4 \cdot 11^r)^2 = (2 \cdot 3^u)^3 - 2^3. \tag{3.16}$$

    Setting $Y = 4 \cdot 11^r$ and $X = 2 \cdot 3^u$, the equation (3.16) reveals

    $$Y^2 = X^3 - 2^3,$$

    which is defines an elliptic curve. Using SageMath, we establish all the integral points for the last elliptic curve. Specifically, we solve $(X, Y) = (2, 0)$. Therefore, $(2 \cdot 3^u, 4 \cdot 11^r) = (2, 0)$, which is impossible. Thus, the equation (3.16) has no solutions.

– If $t = 3u + 1$, the equation (3.15) leading to

$$2^3 \cdot 3^2 \cdot 2(11^r)^2 = 2^3 \cdot 3^{3u+3} - 2^3 \cdot 3^2. \tag{3.17}$$

Putting $Y = 12 \cdot 11^r$ and $X = 2 \cdot 3^{u+1}$, the equation (3.17) gives

$$Y^2 = X^3 - 72,$$

which is defines an elliptic curve. Using SageMath, we find all the integral points for the above elliptic curve. Specifically, we solve $(X, Y) = (6, 12)$. Thus, $(2 \cdot 3^{u+1}, 12 \cdot 11^r) = (6, 12)$, that means $u = 0$ and $r = 0$, which is impossible since $l = 2r$ is even number. Hence, in this case, the equation (3.17) has no solutions.

– If $t = 3u + 2$, the equation (3.15) indicates that

$$2^3 \cdot 3^4 \cdot 2(11^r)^2 = 2^3 \cdot 3^{3u+6} - 2^3 \cdot 3^4. \tag{3.18}$$

Setting $Y = 36 \cdot 11^r$ and $X = 2 \cdot 3^{u+2}$, the equation (3.18) yielding

$$Y^2 = X^3 - 648,$$

which is defines an elliptic curve. Using SageMath, we compute all the integral points for this elliptic curve. Specifically, we solve

$$(X, Y) \in \{(9, 9), (18, 72), (22, 100), (54, 396), (97, 955), (1809, 76941)\}.$$

Thus, $(2 \cdot 3^{u+2}, 36 \cdot 11^r) = (54, 396)$, which means $u = r = 1$. Therefore, in this case, the Diophantine equation

$$(3^n)^x + 11^y = z^2$$

has a unique solution that for $n = 1$, given by $(x, y, z) = (5, 4, 122)$.

- In the case $\ell = 2r + 1$ is odd number. Rewriting the equation (3.14) appears

$$(22 \cdot 11^r)^2 = 22 \cdot 3^t - 22. \tag{3.19}$$

– If $t = 3u$, the equation (3.19) tells us that

$$22(22 \cdot 11^r)^2 = 22^3 \cdot 3^{3u} - 22^3. \tag{3.20}$$

Putting $Y = 22^2 \cdot 11^r$ and $X = 22 \cdot 3^u$, then the equation (3.20) reveals

$$Y^2 = X^3 - 10648,$$

which is defines an elliptic curve. Again, we use SageMath, we find all the integral points for this elliptic curve. Specifically, we solve $(X, Y) = (22, 0)$. Thus, $(22 \cdot 3^u, 22^2 \cdot 11^r) = (22, 0)$, which is impossible. Therefore, in this case, the equation (3.20) has no solutions.

– If $t = 3u + 1$, the equation (3.19) appears that

$$3^2 \cdot 22^2(22 \cdot 11^r)^2 = 22^3 \cdot 3^{3u+3} - 3^2 \cdot 22^3. \tag{3.21}$$

Setting $Y = 1452 \cdot 11^r$ and $X = 22 \cdot 3^{u+1}$, the equation (3.21) reveals

$$Y^2 = X^3 - 95832.$$

This it defines an elliptic curve. Our analysis using SageMath, we show that there are no integer solutions for this given elliptic curve, we conclude that the elliptic curve described above has no integral points.

– If $t = 3u + 2$, the equation (3.19) reveals

$$3^4 \cdot 22^2 (22 \cdot 11^r)^2 = 22^3 \cdot 3^{3u+6} - 3^4 \cdot 22^3. \quad (3.22)$$

Putting $Y = 4356 \cdot 13^r$ and $X = 22 \cdot 3^{u+2}$, the equation (3.22) becomes

$$Y^2 = X^3 - 862488.$$

This it defines an elliptic curve. In the sequel, from using SageMath, we find all the integral points for the last elliptic curve. Also, the computation reveals that no such integer solutions exist, meaning this elliptic curve has no integral points. So, the analysis of these cases and therefore the proof of Theorem 3.6 is finishes.

□

4. Conclusions

To summarise, we give methods to prove that the Diophantine equation (1.2) has the only solution, given by $(p, n, m, x, y, z) = (3, 1, 1, 5, 4, 122)$, where the parameters that $p \geq 3$, $3^m p + 2$ and n, m are respectively prime and positive integers.

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Conflict of interest

The authors declare that there is no conflict of interest.

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