Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.79110

Effects of Duplication Operations on Signless Laplacian Spectrum and Network Measures*

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ABSTRACT: This article investigates the spectral properties of graphs under vertex and edge duplication, analyzing how these operations affect the eigenvalues of Signless Laplacian operators. We derive the spectra for graphs with duplicated vertices and edges, and evaluate key invariants such as Kirchhoff index, global mean-first passage time, and spanning tree counts. Our results link structural graph transformations to their spectral outcomes, offering insights for graph-based modeling in network science.

Key Words: Graph invariants, laplacian spectrum, kirchhoff index, vertex duplication, edge duplication, spanning trees.

Contents

1. Introduction

Spectral graph theory bridges linear algebra and graph theory by analyzing graph structures through the eigenvalues and eigenvectors of associated matrices. Key matrices include the adjacency matrix A, the Laplacian $L = D - A$ (with D as the degree matrix), and the signless Laplacian $Q = D + A$ [1]. These matrices reveal crucial connectivity properties; for instance, the second smallest eigenvalue of L (the algebraic connectivity) plays a vital role in graph partitioning and robustness analysis [2]. Modern applications leverage spectral properties for graph isomorphism testing, community detection, and modeling dynamical processes on networks [3]. The normalized Laplacian $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$ is particularly useful for studying random walks and Markov chains [4]. Beyond mathematics, spectral methods impact quantum chemistry (Hückel theory), computer vision (graph-based segmentation), and complex network analysis [5]. Current research emphasizes efficient spectral partitioning algorithms [6,7] and connections between spectral gaps and graph expansion [8].

The Laplacian spectrum, comprising the eigenvalues of $L(G)$, provides critical insights into graph structure by revealing connectivity patterns, spanning tree properties, and synchronization phenomena [9,10]. Of particular importance is the smallest positive eigenvalue, known as the algebraic connectivity,

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2010 *Mathematics Subject Classification:* 35B40, 35L70.
Submitted April 25, 2025. Published September 23, 2025

which quantifies the graph's overall connectivity level. The Laplacian spectrum has been extensively studied due to its fundamental role in spectral graph theory, network analysis, and combinatorial optimization [13,14,15]. Recent research has highlighted the importance of the signless Laplacian matrix $Q(G) = D(G) + A(G)$, particularly for its applications in graph coloring and structural analysis [11]. Notably, bipartite graphs exhibit unique spectral properties for $Q(G)$, with zero eigenvalues occurring exclusively in these cases [12]. Researchers utilize the spectral characteristics of $Q(G)$ to study graph bipartitioning and derive properties related to domination numbers and other combinatorial invariants [16]. Current investigations focus on the relationship between Laplacian and signless Laplacian spectra, yielding new results on graph energy and spectral bounds [17].

Vertex duplication constitutes a fundamental operation in graph theory that introduces a new vertex while preserving the adjacency patterns of an existing vertex. In the resulting graph G', the duplicate vertex v' maintains identical neighborhood relations $N_{G'}(v') = N_G(v)$ as the original vertex v in G [18]. This operation plays a crucial role in modeling network evolution, particularly in biological and social systems where entities replicate their connections. The comprehensive duplication framework presented in [19] illustrates how networks evolve through successive vertex duplications and connection replications, ultimately forming complex topologies. Within topological graph theory, vertex duplication techniques have proven valuable for studying graph genus properties [20,21], offering researchers powerful tools for analyzing graph structure and topology.

The vertex duplication operation constructs a new graph G_1 by creating a copy of vertex v_k and introducing an edge $e = v'v''$ between the duplicates, with neighborhood relations $N(v') = \{v_k, v''\}$ and $N(v'') = \{v_k, v'\}$. This operation finds important applications in modeling real-world networks like communication systems and transportation infrastructure where entity replication occurs. However, in streaming bipartite graphs, duplicate edges pose challenges for accurate butterfly counting (2×2 bicliques). To address this, Meng et al. developed DEABC, an algorithm that efficiently handles duplicate edges while counting butterflies, achieving improved accuracy and memory efficiency in streaming contexts [22]. Edge duplication significantly influences graph labeling problems. Jesintha et al. examined edge product cordial labeling in duplicated prism graphs, revealing how edge duplication affects labeling properties [23]. These investigations demonstrate the fundamental role of edge duplication in both theoretical and applied graph theory research. Extending these foundations, our work investigates the spectral properties of vertex and edge duplication operations. We apply the obtained spectra to compute several important graph characteristics, including the Kirchhoff index, global mean-first passage time, average path length, and spanning tree count.

Lemma 1.1 ([24]) *Let A, B, C, D be square matrices of appropriate dimensions where both A and D are invertible. Consider the block matrix*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Then the determinant of T can be computed as:

$$\det(T) = \det(D) \cdot \det\left(A - BD^{-1}C\right)$$

Moreover, if D is invertible, the matrix $A - BD^{-1}C$ is called the Schur complement of D in T.

2. Main Results

In this section, we establish two theoretical results that characterize the Signless Laplacian (SL) spectra under specific graph operations. The first theorem provides a detailed formulation and proof of the SL spectrum for a graph obtained through vertex duplication. This result captures how the spectral structure of the original graph transforms when a vertex and its adjacency pattern are replicated. The second theorem addresses the effect of edge duplication on the SL spectrum, offering a spectral characterization when multiple edges are introduced between pairs of vertices. These theorems form the foundation for the subsequent analysis, where we utilize the derived spectra to explore various spectral-based graph invariants and network descriptors.

Theorem 2.1 *Let $\Gamma = (V, E)$ be a κ-regular graph with m vertices and eigenvalues $\nu_1, \nu_2, \ldots, \nu_m$. Construct the graph Γ^* by performing the vertex duplication operation. Then the signless Laplacian spectrum of Γ^* consists of:*

1. *The eigenvalue 1 with multiplicity m*

2. *For each eigenvalue ν_j of Γ, the pair:*

$$\frac{\kappa + \nu_j + 5 \pm \sqrt{(\kappa + \nu_j - 1)^2 + 8}}{2}$$

   each with multiplicity 1

Proof: Let $w_1, w_2, \ldots, w_m$ be the vertices of graph Γ; then the adjacency matrix $\mathcal{A}(\Gamma)$ is given by:

$$\mathcal{A}(\Gamma) = [\, a_{ij} \,]_{m \times m}, \quad \text{where} \quad a_{ij} = \begin{cases} 0 & \text{if } i = j, \\ b_{ij} & \text{if } i < j, \\ b_{ji} & \text{if } i > j. \end{cases}$$

We construct Γ^* by replacing each vertex w_i with an edge $f_i = \{w_i', w_i''\}$ for $i = 1, 2, \ldots, m$. The signless Laplacian matrix $\mathcal{Q}(\Gamma^*)$ is:

$$\mathcal{Q}(\Gamma^*) = [\, q_{ij} \,], \quad \text{where}$$

$$q_{ij} = \begin{cases} \kappa + 2 & \text{if } i = j \text{ and } 1 \le i \le m, \\ b_{ij} & \text{if } 1 \le i, j \le m \text{ and } i \ne j, \\ 1 & \text{if } \begin{cases} (i \le m \text{ and } j = m + 2i - 1) \text{ or } (i \le m \text{ and } j = m + 2i), \\ (j \le m \text{ and } i = m + 2j - 1) \text{ or } (j \le m \text{ and } i = m + 2j), \end{cases} \\ 2 & \text{if } i = j \text{ and } i > m, \\ 1 & \text{if } |i - j| = 1 \text{ and } i, j > m \text{ and } \left\lfloor \frac{i-m+1}{2} \right\rfloor = \left\lfloor \frac{j-m+1}{2} \right\rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\mathcal{C} = [\, c_{ij} \,]_{k \times n}, \quad \text{where} \quad c_{ij} = \begin{cases} 1 & \text{if } j = 2i - 1 \text{ or } j = 2i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the signless Laplacian matrix can be expressed as:

$$\mathcal{Q}(\Gamma^*) = \begin{bmatrix} \mathcal{A}(\Gamma) + (\kappa + 2)\mathcal{I} & \mathcal{C} \\ \mathcal{C}^T & \mathcal{I}_m \otimes (2\mathcal{I} + \mathcal{A}(P_2)) \end{bmatrix}$$

The characteristic polynomial is computed as:

$$
\begin{aligned}
\chi\left(\mathcal{Q}(\Gamma^*);\xi\right) &= \begin{vmatrix} (\xi-(\kappa+2))\mathcal{I}-\mathcal{A}(\Gamma) & -\mathcal{C} \\ -\mathcal{C}^T & \mathcal{I}_m \otimes ((\xi-2)\mathcal{I}-\mathcal{A}(P_2)) \end{vmatrix} \\
&= \left|\mathcal{I}_m \otimes ((\xi-2)\mathcal{I}-\mathcal{A}(P_2))\right| \cdot \\
&\quad \left|(\xi-(\kappa+2))\mathcal{I}-\mathcal{A}(\Gamma)-\mathcal{C}\left(\mathcal{I}_m \otimes ((\xi-2)\mathcal{I}-\mathcal{A}(P_2))\right)^{-1}\mathcal{C}^T\right| \\
&= \left((\xi-2)^2-1\right)^m \cdot \\
&\quad \left|(\xi-(\kappa+2))\mathcal{I}-\mathcal{A}(\Gamma)-\mathcal{C}\left[\mathcal{I}_m \otimes \left(\frac{1}{(\xi-2)^2-1}\left((\xi-2)\mathcal{I}+\mathcal{A}(P_2)\right)\right)\right]\mathcal{C}^T\right| \\
&= \left(\xi^2-4\xi+3\right)^m \cdot \\
&\quad \left|(\xi-(\kappa+2))\mathcal{I}-\mathcal{A}(\Gamma)-\mathcal{C}\left[\mathcal{I}_m \otimes \left(\frac{1}{(\xi^2-4\xi+3)}\left((\xi-2)\mathcal{I}+\mathcal{A}(P_2)\right)\right)\right]\mathcal{C}^T\right| \\
&= \left(\xi^2-4\xi+3\right)^m \cdot \\
&\quad \left|(\xi-(\kappa+2))\mathcal{I}-\mathcal{A}(\Gamma)-\mathcal{C}\left(\mathcal{I}_m \otimes ((\xi-2)\mathcal{I}+\mathcal{A}(P_2))\right)\mathcal{C}^T\right|.
\end{aligned}
$$

Now,

$$
\mathcal{C}\left(\mathcal{I}_{\boldsymbol{m}} \otimes \left((\xi-2)\mathcal{I}+\mathcal{A}\left(P_2\right)\right)\right)\mathcal{C}^T
$$

$$
\begin{aligned}
\mathcal{M} &= \mathcal{C}\times\mathcal{D}\times\mathcal{C}^T \\
&= \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & \\ 0 & 0 & 1 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}_{k\times n}
\begin{bmatrix} \xi-2 & 1 & & & \\ 1 & \xi-2 & & & \\ & & \ddots & & \\ & & & \xi-2 & 1 \\ & & & 1 & \xi-2 \end{bmatrix}_{n\times n}
\begin{bmatrix} 1 & 1 & & & \\ 0 & 0 & & & \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix}_{n\times m} \\
&= \begin{bmatrix} \xi-1 & \xi-1 & & & \\ & \xi-1 & \xi-1 & & \\ & & \ddots & & \\ & & & \xi-1 & \xi-1 \end{bmatrix}_{k\times n}
\begin{bmatrix} 1 & 1 & & & \\ 0 & 0 & & & \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix}_{n\times m} \\
&= (2\xi-2)\mathcal{I}_m
\end{aligned}
$$

Building upon the previous results, we have:

$$
\begin{aligned}
\chi\left(\mathcal{Q}(\Gamma^*);\xi\right) &= \left|\left(\xi^2-4\xi+3\right)\left((\xi-(\kappa+2))\mathcal{I}-\mathcal{A}(\Gamma)\right)-\mathcal{C}\left(\mathcal{I}_m \otimes ((\xi-2)\mathcal{I}+\mathcal{A}(P_2))\right)\mathcal{C}^T\right| \\
&= \left|\left(\xi^2-4\xi+3\right)\left((\xi-(\kappa+2))\mathcal{I}-\mathcal{A}(\Gamma)\right)-(2\xi-2)\mathcal{I}_m\right|
\end{aligned}
$$

Given that $\nu_1, \nu_2, \ldots, \nu_m$ are the eigenvalues of $\mathcal{A}(\Gamma)$, we obtain:

$$
\begin{aligned}
\chi\left(\mathcal{Q}(\Gamma^*);\xi\right) &= \prod_{j=1}^{m}\left(\xi^2-4\xi+3\right)\left(\xi-\kappa-2-\nu_j\right)-(2\xi-2) \\
&= \prod_{j=1}^{m}(\xi-1)(\xi-3)\left(\xi-\kappa-2-\nu_j\right)-2(\xi-1) \\
&= (\xi-1)^m\prod_{j=1}^{m}\left[\xi^2-\xi\left(\kappa+\nu_j+5\right)+\left(3\kappa+3\nu_j+4\right)\right]
\end{aligned}
$$

The roots of the characteristic polynomial are:

- $\xi = 1$ (with multiplicity m)
- For each $j = 1, 2, \ldots, m$, the roots:

$$\frac{(\kappa+\nu_j+5) \pm \sqrt{(\kappa+\nu_j)^2 - 2(\kappa+\nu_j) + 9}}{2}$$

Therefore, the signless Laplacian spectrum of Γ^* consists of:

$$\begin{cases} 1^m, \\ \left(\frac{(\kappa+\nu_j+5)+\sqrt{\kappa^2+\nu_j^2+2\kappa\nu_j-2\kappa-2\nu_j+9}}{2}\right)^1, \\ \left(\frac{(\kappa+\nu_j+5)-\sqrt{\kappa^2+\nu_j^2+2\kappa\nu_j-2\kappa-2\nu_j+9}}{2}\right)^1 \end{cases}$$

for $j = 1, 2, \ldots, m$.

Theorem 2.2 *Let $\Gamma = (V, E)$ be a κ-regular graph with m vertices, n edges, and eigenvalues $\nu_1, \nu_2, \ldots, \nu_m$. Construct the graph $\Gamma^\diamond$ by performing edge duplication graph operation, then the signless Laplacian spectrum of $\Gamma^\diamond$ consists of:*

1. *The eigenvalue 2 with multiplicity $n - m$*
2. *For each eigenvalue ν_j of Γ, the pair:*

$$\frac{2\kappa+\nu_j+2 \pm \sqrt{(2\kappa+\nu_j)^2 - 4(2\kappa+\nu_j) + 8}}{2}$$

each with multiplicity 1

for $j = 1, 2, \ldots, m$.

Let $\Gamma = (V, E)$ be a κ-regular graph with vertex set $V = \{w_1, w_2, \ldots, w_m\}$ and eigenvalues $\mu_1, \mu_2, \ldots, \mu_m$. The adjacency matrix $\mathcal{A}(\Gamma)$ is given by:

$$\mathcal{A}(\Gamma) = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_m \end{array} \begin{array}{c} \begin{matrix} w_1 & w_2 & w_3 & \cdots & w_m \end{matrix} \\ \begin{bmatrix} 0 & b_{12} & b_{13} & \cdots & b_{1m} \\ b_{21} & 0 & b_{23} & \cdots & b_{2m} \\ b_{31} & b_{32} & 0 & \cdots & b_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & 0 \end{bmatrix} \end{array}$$

Construct the graph Γ^* by vertex duplication:

- For each vertex w_i, add a new edge $f_i = \{w_i', w_i''\}$ connecting two new vertices

The signless Laplacian matrix $\mathcal{Q}(\Gamma^*)$ is:

$$\mathcal{Q}(\Gamma^*) = [q_{ij}]_{2m\times 2m} \quad \text{with} \quad q_{ij} = \begin{cases} \kappa+2 & \text{if } i = j \leq m, \\ b_{ij} & \text{if } i \neq j \leq m, \\ 1 & \text{if } (i \leq m, j > m) \text{ and } (i, j-m) \in E, \\ 1 & \text{if } (j \leq m, i > m) \text{ and } (j, i-m) \in E, \\ 2 & \text{if } i = j > m, \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{C} = \begin{bmatrix} 1 & 0 & \cdots & 1 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 1 \end{bmatrix}$

Then, the signless Laplacian matrix can be expressed as:

$$\mathcal{Q}(\Gamma^*) = \begin{bmatrix} 2\kappa\mathcal{I}_m + \mathcal{A}(\Gamma) & \mathcal{C} \\ \mathcal{C}^T & 2\mathcal{I}_n \end{bmatrix}$$

The characteristic polynomial is computed as:

$$\begin{aligned}
\chi(\mathcal{Q}(\Gamma^*);\xi) &= \begin{vmatrix} (\xi-2\kappa)\mathcal{I}_m - \mathcal{A}(\Gamma) & -\mathcal{C} \\ -\mathcal{C}^T & (\xi-2)\mathcal{I}_n \end{vmatrix} \\
&= |(\xi-2)\mathcal{I}_n| \cdot \left|(\xi-2\kappa)\mathcal{I}_m - \mathcal{A}(\Gamma) - \mathcal{C}\left((\xi-2)\mathcal{I}_n\right)^{-1}\mathcal{C}^T\right| \\
&= (\xi-2)^n \cdot \left|(\xi-2\kappa)\mathcal{I}_m - \mathcal{A}(\Gamma) - \frac{1}{\xi-2}\mathcal{C}\mathcal{C}^T\right| \\
&= (\xi-2)^{n-m} \cdot |(\xi-2)(\xi-2\kappa)\mathcal{I}_m - (\xi-2)\mathcal{A}(\Gamma) - (\mathcal{A}(\Gamma)+\kappa\mathcal{I}_m)|
\end{aligned}$$

Given that $\mu_1, \mu_2, \ldots, \mu_m$ are the eigenvalues of $\mathcal{A}(\Gamma)$, we obtain:

$$\chi(\mathcal{Q}(\Gamma^*);\xi) = (\xi-2)^{n-m}\prod_{j=1}^{m}\left[\xi^2 - \xi(2\kappa+\mu_j+2) + (3\kappa+\mu_j)\right]$$

The roots of the characteristic polynomial are:

- $\xi = 2$ (with multiplicity $n-m$)
- For each $j = 1, 2, \ldots, m$, the roots:

$$\frac{(2\kappa+\mu_j+2) \pm \sqrt{4\kappa^2+\mu_j^2+4\kappa\mu_j-4\kappa+4}}{2}$$

Therefore, the signless Laplacian spectrum of Γ^* consists of:

$$\begin{cases} 2^{n-m}, \\ \left(\frac{(2\kappa+\mu_j+2)+\sqrt{4\kappa^2+\mu_j^2+4\kappa\mu_j-4\kappa+4}}{2}\right)^1, \\ \left(\frac{(2\kappa+\mu_j+2)-\sqrt{4\kappa^2+\mu_j^2+4\kappa\mu_j-4\kappa+4}}{2}\right)^1 \end{cases}$$

for $j = 1, 2, \ldots, m$.

3. Graph Spectral Quantities

Let $\mathcal{M}_\Gamma$ and $\mathcal{R}_\Gamma$ be the product of all nonzero eigenvalues of $\mathcal{L}(\Gamma^{(g)})$ and the sum of the reciprocals of these eigenvalues, respectively. That is:

$$\mathcal{M}_\Gamma = \prod_{k=2}^{|\Gamma^{(g)}|}\eta_k \quad \text{and} \quad \mathcal{R}_\Gamma = \sum_{k=2}^{|\Gamma^{(g)}|}\frac{1}{\eta_k}$$

3.1. Implications for Vertex Duplication Operation

Based on the above defined $\mathcal{M}_\Gamma$ and $\mathcal{R}_\Gamma$, we can derive several important network measures for the vertex-duplicated graphs $\Gamma^{(g)}$:

- Kirchhoff Index: $\mathcal{K}(\Gamma^{(g)}) = |\Gamma^{(g)}| \cdot \mathcal{R}_\Gamma$
- Global Mean-First Passage Time: $\mathcal{T}(\Gamma^{(g)}) = \frac{2|\mathcal{E}^{(g)}|}{|\Gamma^{(g)}|-1}\mathcal{R}_\Gamma$
- Average Path Length: $\mathcal{D}(\Gamma^{(g)}) = \frac{|\Gamma^{(g)}|}{|\Gamma^{(g)}|-1}\mathcal{R}_\Gamma$
- Number of Spanning Trees: $\mathcal{N}(\Gamma^{(g)}) = \frac{1}{|\Gamma^{(g)}|}\mathcal{M}_\Gamma$

3.1.1. Resistance-Based Graph Index. The electrical network model of graphs was first created by Klein and Randic [25] to represent the graph topology. A graph features unit resistors placed on each edge of its structure. The effective electrical resistance measured between two any vertices u and v in graph Γ becomes the resistance distance notation ρ_{uv} when applying unit current to u and extracting from v. Resistance distance measures the direct node connections and all possible current pathways within a network structure which makes it an effective tool for evaluating complex network connectivity robustness. The Kirchhoff index of Γ is defined as:

$$\mathcal{K}(\Gamma) = \frac{1}{2}\sum_{u\in V}\sum_{v\in V}\rho_{uv}(\Gamma)$$

which sums the resistance distances between all vertex pairs in Γ. This can be expressed equivalently through the graph's Laplacian spectrum as:

$$\mathcal{K}(\Gamma) = |V|\sum_{k=2}^{|V|}\frac{1}{\eta_k}$$

For the vertex-duplicated graph $\Gamma^{(g)}$, we obtain the explicit formula:

$$\mathcal{K}(\Gamma^{(g)}) = \sum_{u<v}\rho_{uv}(\Gamma) = |\Gamma^{(g)}|\sum_{k=2}^{|\Gamma^{(g)}|}\frac{1}{\eta_k} = |\Gamma^{(g)}|\sum_{i=0}^{g-1}\sum_{j=0}^{m-1}\frac{1}{\eta_{ij}}, \quad (i,j)\neq(0,0)$$

For the specific case of the $P(g)$ construction:

$$\mathcal{K}(P(g)) = 3m\sum_{i=0}^{g-1}\sum_{j=0}^{m-1}\frac{m(4\kappa+4\nu_j+9)}{(3\kappa+3\nu_j+4)}, \quad (i,j)\neq(0,0)$$

3.1.2. Global Mean-First Passage Time. To determine network efficiency for random walks the mean-first passage time from node u to node v provides essential information. The measure shows how many steps a random walker needs on average to reach vertex v starting from vertex u for the initial visit. This measurement tells us how instantly information flows through the network from one place to another. The global mean-first passage time (GMFPT) named as $\langle\mathcal{F}_\Gamma\rangle$ represents the overall efficiency index for the entire network. The GMFPT result comes from averaging $\mathcal{F}_{uv}$ measurements between every unique pair of nodes (u,v) except when u=v across all starting points (V). This single number shows how well the network carries information through its entire structure according to Noh and Rieger [26].

$$\langle\mathcal{F}_\Gamma\rangle = \frac{1}{|V|(|V|-1)}\sum_{u\neq v}\mathcal{F}_{uv}(\Gamma)$$

The commuting time $\mathcal{C}_{uv}$ between nodes u and v relates to resistance distance:

$$\mathcal{C}_{uv} = \mathcal{F}_{uv} + \mathcal{F}_{vu} = 2|\mathcal{E}|\rho_{uv}$$

where $|\mathcal{E}|$ denotes the number of edges in Γ. For the vertex-duplicated graph $\Gamma^{(g)}$, the GMFPT becomes:

$$\langle \mathcal{F}_{\Gamma^{(g)}} \rangle = \frac{2|\mathcal{E}^{(g)}|}{|\Gamma^{(g)}|(|\Gamma^{(g)}| - 1)} \sum_{u<v} \rho_{uv} = \frac{2|\mathcal{E}^{(g)}|}{|\Gamma^{(g)}| - 1} \sum_{k=2}^{|\Gamma^{(g)}|} \frac{1}{\eta_k}$$

Expressed in terms of the eigenvalue decomposition:

$$\langle \mathcal{F}_{\Gamma^{(g)}} \rangle = \frac{2|\mathcal{E}^{(g)}|}{|\Gamma^{(g)}| - 1} \sum_{i=0}^{g-1} \sum_{j=0}^{m-1} \frac{1}{\eta_{ij}}, \quad (i, j) \neq (0, 0)$$

For the specific construction $P(g)$:

$$\langle \mathcal{F}_{P(g)} \rangle = \frac{2|\mathcal{E}^{(g)}|}{|\Gamma^{(g)}| - 1} \sum_{i=0}^{g-1} \sum_{j=0}^{m-1} \frac{m(4\kappa + 4\nu_j + 9)}{(3\kappa + 3\nu_j + 4)}, \quad (i, j) \neq (0, 0)$$

Using the network parameters $|\Gamma^{(g)}| = 3m$ and $|\mathcal{E}^{(g)}| = \frac{\kappa m + 6m}{2}$, we can express GMFPT purely in terms of network size:

$$\langle \mathcal{F}_{P(g)} \rangle = \frac{m(\kappa + 6)}{3m - 1} \sum_{i=0}^{g-1} \sum_{j=0}^{m-1} \frac{m(4\kappa + 4\nu_j + 9)}{(3\kappa + 3\nu_j + 4)}, \quad (i, j) \neq (0, 0)$$

3.1.3. Average Path Length. Real-world networks exhibit small-world properties which show that huge complex networks contain remarkably few steps needed to reach from one random node to another. The average path length calculation provides analytical data regarding this concept. The fundamental characteristics of network topology are measured through the clustering coefficient and degree distribution along with the average path length. The average path length acts as an evaluation method for how efficiently information and mass distributions spread throughout networks. The definition describes this measure as an average calculation of all unique shortest path lengths known as δ_{uv} between each possible pair of nodes u and v that exists in the network. The mathematical equation defines the speed and efficiency of network communication between any two connected network sections. The specific class of network structures g in the graph P(g) has its average path length depicted by $\mathcal{D}_g$ through computation of all node pair shortest distances [27].

$$\mathcal{D}_g = \frac{2}{|\Gamma^{(g)}|(|\Gamma^{(g)}| - 1)} \sum_{u<v} \delta_{uv}$$

Assuming the associated electrical network is a complete graph, there exists a fundamental relation between the effective resistance ρ_{uv} and the shortest path length δ_{uv}:

$$\rho_{uv} = \frac{2\delta_{uv}}{|\mathcal{N}|}$$

where $|\mathcal{N}|$ denotes the total number of nodes in the complete graph. Using this relation, we can rewrite the average path length as:

$$\mathcal{D}_g = \frac{2}{|\Gamma^{(g)}|(|\Gamma^{(g)}| - 1)} \cdot \frac{|\Gamma^{(g)}|}{2} \sum_{u<v} \rho_{uv} = \frac{|\Gamma^{(g)}|}{|\Gamma^{(g)}| - 1} \sum_{k=2}^{|\Gamma^{(g)}|} \frac{1}{\eta_k}$$

$$\mathcal{D}_g = \frac{|\Gamma^{(g)}|}{|\Gamma^{(g)}| - 1} \sum_{i=0}^{g-1} \sum_{j=0}^{m-1} \frac{1}{\eta_{ij}}, \quad (i, j) \neq (0, 0)$$

$$\mathcal{D}_g = \frac{3m}{3m-1} \sum_{i=0}^{g-1} \sum_{j=0}^{m-1} \frac{m(4\kappa + 4\nu_j + 9)}{(3\kappa + 3\nu_j + 4)}, \quad (i,j) \neq (0,0)$$

3.1.4. The Number of Spanning Trees. The number of spanning trees is one of the most fundamental and widely studied quantities in the analysis of complex networks. This metric provides valuable insights into various important aspects of network behavior, including reliability, transport efficiency, self-organized criticality, and properties of loop-erased and standard random walks. Additionally, it is deeply connected to the behavior of resistor networks, where the structure of the network can be analyzed through its electrical characteristics. More significantly, the number of spanning trees serves as a powerful indicator of structural robustness. A higher number of such trees generally implies a greater number of alternate paths between nodes, thereby contributing to the resilience and redundancy of the network in the presence of edge or node failures. The exact number of spanning trees, denoted by $\mathcal{N}_{\mathrm{ST}}(P(g))$, for the graph $P(g)$ with generation index $g \geq 1$, can be computed using the well-known Kirchhoff's Matrix-Tree Theorem. According to this theorem, the number of spanning trees can be obtained by taking the product of all nonzero eigenvalues of the signless Laplacian matrix associated with the graph [28].

$$\mathcal{N}_{\mathrm{ST}}(P(g)) = \frac{\prod_{k=2}^{|\Gamma^{(g)}|} \eta_k}{|\Gamma^{(g)}|} = \frac{\mathcal{A}_g}{|\Gamma^{(g)}|} = \frac{\prod_{i=0}^{g-1} \prod_{j=0}^{m-1} \eta_{ij}}{|\Gamma^{(g)}|}, \quad (i,j) \neq (0,0)$$

$$\mathcal{N}_{\mathrm{ST}}(P(g)) = \prod_{i=0}^{g-1} \prod_{j=0}^{m-1} \frac{3\kappa + 3\nu_j + 4}{3}, \quad (i,j) \neq (0,0)$$

3.2. Implications for Edge Duplication Operations

In this section, based on the previously defined quantities $\mathcal{A}_g$ and $\mathcal{B}_g$ in Lemma 1, we use these expressions to compute various structural measures of the network $P(g)$ under edge duplication. These include the Kirchhoff index, global mean-first passage time, average path length, and the number of spanning trees.

3.2.1. Kirchhoff Index. For the edge-duplicated graph, the Kirchhoff index is given by:

$$\mathcal{KF}(P(g)) = \frac{m(2+\kappa)}{2} \sum_{i=0}^{g-1} \sum_{j=0}^{m-1} \frac{2(\mathcal{M} - m)(5\kappa + 2\nu_j + 2)}{(3\kappa + \nu_j)}, \quad (i,j) \neq (0,0)$$

3.2.2. Global Mean-First Passage Time. For the edge-duplicated graph, the global mean-first passage time (GMFPT) can be expressed in terms of the network size $|\Gamma^{(g)}| = \frac{2m+m\kappa}{2}$ and the number of edges $|\mathcal{E}^{(g)}| = \frac{3m\kappa}{2}$ as:

$$\langle \mathcal{F}_{\Gamma^{(g)}} \rangle = \frac{6m\kappa}{2m + m\kappa - 2} \sum_{i=0}^{g-1} \sum_{j=0}^{m-1} \frac{2(\mathcal{M} - m)(5\kappa + 2\nu_j + 2)}{(3\kappa + \nu_j)}, \quad (i,j) \neq (0,0)$$

3.2.3. Average Path Length. Similarly, the average path length for the edge-duplicated graph is:

$$\mathcal{D}_g = \frac{2m + m\kappa}{2m + m\kappa - 2} \sum_{i=0}^{g-1} \sum_{j=0}^{m-1} \frac{2(\mathcal{M} - m)(5\kappa + 2\nu_j + 2)}{(3\kappa + \nu_j)}, \quad (i,j) \neq (0,0)$$

3.2.4. The Number of Spanning Trees. For the edge-duplicated graph, the number of spanning trees is given by:

$$\mathcal{N}_{\mathrm{ST}}(P(g)) = \prod_{i=0}^{g-1} \prod_{j=0}^{m-1} \frac{4(\mathcal{M} - m)(\nu_j + 3\kappa)}{(2m + m\kappa)}, \quad (i,j) \neq (0,0)$$

4. Conclusion

By leveraging two key theoretical theorems, we examined how vertex and edge duplication affect the spectral properties of graphs, focusing on their signless Laplacian spectra. Our findings provide a precise characterization of the shifts in signless Laplacian eigenvalues resulting from these operations. Using the derived spectral information, we calculated critical graph invariants, including the Kirchhoff index, global mean-first passage time, average path length, and spanning tree count. The results reveal that even simple modifications in network structure can significantly influence both spectral dynamics and topological features. This study enhances the understanding of network duplication processes while offering an analytical framework for assessing dynamic network transformations. Future research could extend these principles to weighted, directed, and multilayer networks to uncover novel spectral patterns.

Acknowledgments

This research was conducted independently without any institutional or financial support.

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