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Groupoids and their Topological *-Algebras

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ABSTRACT: This study introduces the concept of a topological groupoid and some topological *-algebras are investigated, like the convolution topological *-algebras associated with locally compact groupoids, and in particular, étale groupoids.

Key Words: Groupoids, group algebras, groupoid algebras, Banach algebra, C*-algebra.

Contents

1. Introduction

Groupoids provide a general framework that captures notions of symmetry and dynamics beyond the setting of groups. The notion of groupoids originated with Brandt in 1927. It is most elegantly defined as a small category with inverses.Algebraically, a groupoid can be regarded as a set with a partially defined multiplication that exhibits group-like properties whenever applicable. Although every group is a groupoid, there is a wide variety of groupoids that are not groups.

These notes aim to provide a brief overview of some key topics in the area of topological *-algebras associated with groupoids. The first section begins with a quick overview of groupoids in the algebraic sense, offering illustrative examples, and introducing topological groupoids and locally compact groupoids.

In order to investigate the topological *-algebras derived from groupoids, one usually requires a *-algebra structure on $C_c(G)$, the space of continuous complex-valued functions with compact support. This involves defining the convolution product, which combines functions through integration with respect to a collection of measures known as the Haar system, denoted by $\{\lambda_u,\ u \in G^{(0)}\}$, where $G^{(0)}$ is the unit space of the groupoid G.

Unlike the group case, the existence of a Haar system in groupoids is not guaranteed, and even when it exists, it need not to be unique. In fact, Seda In [5] shows that if the range map is not open, then a groupoid cannot possess a Haar system.

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When G is an étale groupoid, the Haar system is simply a collection of counting measures. Consequently, the construction of the topological *-algebra of an étale groupoid is almost analogous to the discrete group case, Further discussions on these insights will be provided in the subsequent sections of this paper.

2. Preliminaries

2.1. Groupoids (Algebraically)

A groupoid is a mathematical structure that generalizes the concept of a group. There are many definitions of groupoids in mathematics since they are a very flexible and powerful mathematical tool with many applications, but in this section, we will focus on the definition of groupoid as given by Renault in his book *"A Groupoid Approach to C*-Algebras"* [1].
A groupoid is a set G endowed with a partial operation

$$\begin{aligned} G^{(2)} &\to G \\ (g,h) &\mapsto gh \end{aligned} \tag{2.1}$$

where $G^{(2)}$ is a subset of $G \times G$ called the set of composable pairs. (The key point is that the product of an arbitrary pair of elements may not be defined, and the product gh is only defined for pairs $(g,h) \in G^2$), and equipped with an inverse map

$$\begin{aligned} G &\to G \\ g &\mapsto g^{-1} \end{aligned} \tag{2.2}$$

such that the following conditions hold for any $g,h,k \in G$:

(i) If $(g,h) \in G^{(2)}$ and $(h,k) \in G^{(2)}$, then $(gh,k) \in G^{(2)}$ and $(g,hk) \in G^{(2)}$.
Furthermore, $(gh)k = g(hk)$. (written as ghk).

(ii) $(g^{-1})^{-1} = g$ for all $g \in G$.

(iii) For all $g \in G$, $(g,g^{-1}) \in G^{(2)}$, and if $(k,g) \in G^{(2)}$, then $(kg)g^{-1} = k$.

(iv) For all $g \in G$, $(g^{-1},g) \in G^{(2)}$, and if $(g,h) \in G^{(2)}$, then $g^{-1}(gh) = h$.

From (iii) and (iv), we conclude that a unit in a groupoid G is any element that can be written both as gg^{-1} and $g^{-1}g$, for some $g \in G$. The set of all units is called the unit space and denoted by

$$G^{(0)} = \{g^{-1}g \,:\, G \in G\} = \{gg^{-1} \,:\, g \in G\} \tag{2.3}$$
$$= \{g \in G \,:\, g = g^{-1} = gg\}. \tag{2.4}$$

For $g \in G$, the source and range maps are respectively defined as

$$s(g) = g^{-1}g, \qquad r(g) = gg^{-1}.$$

Remark 2.1 *Since in the groupoid not all pairs $(g,h) \in G \times G$ are composable, a pair (g,h) belongs to $G^{(2)}$ if and only if $s(g) = r(h)$. Thus, for a groupoid G, the set of composable pairs is given by*

$$G^{(2)} = \{(g,h) \in G \times G \mid s(g) = r(h)\}. \tag{2.5}$$

(Readers seeking alternative treatments and complete proofs for certain statements in this section are encouraged to consult references such as [2,4], [12], Remark 8.1.5.)

Example 2.1 *(Groups.) Let G be a group with identity element e. Then G is a groupoid with $G^{(2)} = G \times G$ and $G^{(0)} = \{e\}$. In fact, a groupoid is a group if and only if its unit space is a singleton.*

Example 2.2 *Let X be a set and $G = X \times X$. G is a groupoid with*

$$G^{(2)} = \{((x, y)(y, z)) \mid x, y, z \in X\}$$

and the operations defined by

$$(x, y)(y, z) = (x, z) \quad \text{and} \quad (x, y)^{-1} = (y, x).$$

Moreover, $r(x, y) = (x, x)$ *and* $s(x, y) = (y, y)$.

Example 2.3 *(Equivalence Relations.) Let X be a set and $\mathcal{R} \subseteq X \times X$ an equivalence relation on X. Define*

$$\mathcal{R}^{(2)} = \{((x, y), (y, z)) : (x, y), (y, z) \in \mathcal{R}\},$$

which means that $(x, y), (y', z)$ are composable if and only if $y = y'$. Then, for all $(x, y), (y, z) \in \mathcal{R}$ we define the product as

$$(x, y)(y, z) = (x, z)$$

and the inverse as

$$(x, y)^{-1} = (y, x)$$

Moreover, for all $(x, y), (y, z) \in \mathcal{R}$, we have

$$\begin{aligned} r(x, y) &= (x, y)(x, y)^{-1} = (x, y)(y, x) = (x, x) \\ s(x, y) &= (y, y) \\ \mathcal{R}^{(0)} &= \{(x, x) \mid x \in X\} \end{aligned}$$

Example 2.4 *(Transformation groupoids.) Let Γ be a group acting (on the right) on a set X by bijection. Consider the set $G = X \times \Gamma$ and define*

$$G^{(2)} = \{((x, g), (y, h)) \mid g, h \in \Gamma, \ \ x \in X, \ \ \text{and} \ \ y = xg\}.$$

Then the product and the inverse are given by

$$(x, g)(xg, h) = (x, gh), \qquad (x, g)^{-1} = (xg, g^{-1}),$$

for $x \in X$ and $g, h \in G$. We note

$$\begin{aligned} r(x, g) =& (x, g)(x, g)^{-1} = (x, g)(xg, g^{-1}) = (x, e), \\ s(x, g) =& (xg, e), \end{aligned}$$

for x in X, g in G and $G^{(0)} = X \times \{e\} \cong X$.

2.2. Topological groupoids

A topological groupoid consists of a groupoid G and a topology compatible with the groupoid structure. That is, the multiplication and the inverse maps defined in (2.1) and (2.2) are both continuous. (Here, $G^{(2)}$ carries the topology induced from $G \times G$.)

Remark 2.2 Let G be a topological groupoids, we have:

- The topology of $G^{(0)}$ is induced by the open sets of G that contain $G^{(0)}$.
- $G^{(0)}$ is closed if and only if G is Hausdorff.

Now, let's revisit our earlier examples and equip them with a topology, turning them into topological groupoids.

Example 2.5

- *Groups: If G is a topological group, it is a topological groupoid.*
- *Discrete groupoids: Every groupoid is a topological groupoid with the discrete topology.*
- *Equivalence relations: If X is a Hausdorff space and $\mathcal{R}$ is an equivalence relation on X, then $\mathcal{R}$ is a topological groupoid with the subspace topology from $X \times X$.*
- *Transformation groupoids: Let Γ be a Hausdorff group acting continuously on a Hausdorff space X. Then $\Gamma \ltimes X$ with the product topology is a topological groupoid.*

3. Locally Compact Groupoids and theirs Topological *- Algebras

We only consider topological groupoids whose topology is locally compact and Hausdorff. We denote by $C_c(G)$ the algebra of continuous complex valued functions with compact support on G.
For developing an algebraic theory of functions on locally compact groupoids, one needs an analogue of Haar measure on locally compact groups. We adopt the definition given by Renault in [1].
We denote by G^u for $u \in G^{(0)}$ the set $G^u = r^{-1}(\{u\}) = \{g \in G : r(g) = u\}$ and G_u the set $G_u = s^{-1}(\{u\}) = \{g \in G : s(g) = u\}$.

Definition 3.1 *A (left) Haar system on a locally compact Hausdorff groupoid G is a family of positive Radon measures, $\lambda = \{\lambda^u, u \in G^{(0)}\}$, such that:*

(i) For all $u \in G^{(0)}$, $\mathrm{supp}(\lambda^u) = G^u$

(ii) For all $f \in C_c(G)$,

$$G^{(0)} \longrightarrow \mathbb{C},$$
$$u \longmapsto \lambda(f)(u) = \int_G f(x)\, d\lambda^u(x)$$

is continuous.

(iii) For all $f \in C_c(G)$ and all $x \in G$,

$$\int_G f(y)\, d\lambda^{r(x)}(y) = \int_G f(xy)\, d\lambda^{s(x)}(y).$$

These measures are not Haar measures in the strict sense of the term, but they capture similar properties and provide a measure-theoretic framework for the groupoid. It follows from (ii) that $\lambda(f)$ also belongs to $C_c(G^{(0)})$. And we deduce form (iii) of Definition (3.1) that

$$\int_{G^{r(x)}} f(x^{-1}z)\, d\lambda^{r(x)}(z) = \int_{G^{s(x)}} f(y)\, d\lambda^{s(x)}(y). \tag{3.1}$$

Remark 3.1 *Let G be locally compact Hausdorff groupoid with Haar system $\lambda = \{\lambda^u\}_{u \in G^{(0)}}$. Then the map*

$$\lambda : C_c(G) \longrightarrow C_c(G^{(0)}),$$
$$f \longmapsto \lambda(f)$$

is continuous.

Remark 3.2 *Let $\{\lambda^u\}_{u \in G^{(0)}}$ is a left Haar system on locally compact Hausdorff groupoid G. Since $(G^u)^{-1} = G_u$, then, for each $u \in G^{(0)}$, we can associate to λ^u the measure $\lambda_u = (\lambda^u)^{-1}$, with*

$$\int f(x)\, d\lambda_u(x) = \int f(x^{-1})\, d\lambda^u(x).$$

We will call $\{\lambda_u\}_{u \in G^{(0)}}$ a right Haar system on G.

We shall work only with left Haar system.

Example 3.1 *If Γ is a locally compact Hausdorff group acting continuously on a locally compact Hausdorff space X, then $G = X \times \Gamma$ admits a distinguished (left) Haar system $\{\varepsilon_x \times \lambda : x \in X\}$, where λ is a Haar measure on Γ and ε_x is the Dirac measure at x. For $f \in C_c(X \times \Gamma)$. Moreover,*

$$\lambda(f)(u) = \int_G f(x, g)\, d\lambda(g) \quad \text{for all } f \in C_c(X \times \Gamma),$$

with $(x, e) = u \in G^{(0)}$.

Example 3.2 *Let X be a locally compact and Hausdorff space. Consider the groupoid in Example 2.2. Let μ be a positive Radon measure on X with full support (i.e., $\mathrm{supp}(\mu) = X$). Then $\{\varepsilon_x \times \mu \mid x \in X\}$ is a Haar system on $X \times X$ (as a trivial groupoid), where ε_x is the unit point mass at x. Moreover,*

$$\lambda(f)(u) = \int_X f(x,y)\, d\lambda(y) \quad \textit{for all } f \in C_c(X \times X),$$

with $(x,x) = u \in G^{(0)}$.

Unlike the case of locally compact group, Haar system on groupoid need not exist (due to Anton Deitmar [3], who shows that a locally compact groupoid does not necessarily have Haar system). On the other hand, a locally compact groupoid can have a several Haar systems.
One known criterion is that a Haar system can only exist if the range map is open. [Corollary to Lemma 2 in [6], see also [7]].

Remark 3.3 *It may be confusing not to define a measure on all of G. However, if μ is a measure on $G^{(0)}$ then we will obtain a measure ν on G, induced by μ, given by $\nu = \mu \circ \lambda$, and we have*

$$\nu(f) = \int_{G^{(0)}} \int_G f(\gamma)\, d\lambda^u(\gamma)\, d\mu(u) \quad \textit{for } f \in C_c(G).$$

3.1. The convolution topological *-algebra $C_c(G)$

In the remaining sections of this paper, we will assume that G is a groupoid equipped with a Haar system $\lambda = \{\lambda^u\ , u \in G^{(0)}\}$.
For $f, g \in C_c(G)$, the convolution is defined by

$$(f * g)(x) = \int_{G^{r(x)}} f(y)g(y^{-1}x)\, d\lambda^{r(x)}(y) = \int_{G^{s(x)}} f(xy)g(y^{-1})\, d\lambda^{s(x)}(y) \tag{3.2}$$

and the involution by :

$$f^*(x) = \overline{f(x^{-1})}. \tag{3.3}$$

Proposition 3.1 *Let G be a locally compact groupoid. Then $C_c(G)$ is a topological $*$-algebra under the convolution multiplication defined in (3.2) and the involution given in (3.3).*

Proof: Let $f, g \in C_c(G)$. We prove that $f * g \in C_c(G)$. Indeed, if $(f * g)(x) \neq 0$, then there exists y_0 such that $f(xy_0) \neq 0$ and $g(x) \neq 0$. This implies that $\mathrm{supp}(f * g)$ is a subset of $(\mathrm{supp}(f))(\mathrm{supp}(g))$.
Now, we prove that $f * g$ is continuous. Thanks to Tietze extension theorem, we extend the function $(x,y) \mapsto F(x,y) = f(xy)g(y^{-1})$ on $G^{(2)}$ to a bounded continuous function k on $G \times G$. Let $h \in C_c(G)$ such that $h(y) = 1$, if $k(x,y) \neq 0$. Then, we have

$$k(x,y)h(y) = F(x,y) \quad \text{for all}(x,y) \in G^{(2)}.$$

Define a complex-valued function H by

$$\begin{aligned} H : G \times G^0 &\longrightarrow \mathbb{C} \\ (x,u) &\longmapsto H(x,u) = \int_G k(x,y)\, d\lambda^u(y). \end{aligned}$$

We have $f * g = H|_{(G,s(x))}$. Hence, it suffices to show that H is continuous. Let $x_0 \in G$, we shall show that H is continuous at x_0. Let $K = C \times s(C,)$ where C is a compact neighborhood of x_0. Then, for $(x,u) \in K$, we have

$$\begin{aligned} \big|H(x,u) - H(x_0,u_0)\big| &= \Big|\int k(x,y)\, d\lambda^u(y) \ - \int k(x_0,y)\, d\lambda^{u_0}(y)\Big| \\ &\le \int \big|k(x,y) - k(x_0,y)\big|\, |h(y)|\, d\lambda^u(y) + \Big|\int k(x_0,y)\, d\lambda^u(y) - \int k(x_0,y)\, d\lambda^{u_0}(y)\Big| \\ &\le \sup_y \big|k(x,y) - k(x_0,y)\big| \int |h(y)|\, d\lambda^u(y) + \Big|\int k(x_0,y)\, d\lambda^u(y) - \int k(x_0,y)\, d\lambda^{u_0}(y)\Big|. \end{aligned}$$

By uniform continuity of k and the definition of the Haar system in (ii), we have the continuity of H on K. It Follows that the map $x \mapsto H(x, s(x))$ is continuous on C. Since

$$H(x, s(x)) = \int k(x, y)\, d\lambda^{s(x)}(y) = \int f(xy)g(y^{-1})\, d\lambda^{s(x)}(y),$$

we obtain that F is continuous at x_0.
For the associativity, let $f, g, h \in C_c(G)$, $x \in G$. Then, by using (3.2) we have

$$\begin{aligned}
f * (g * h)(x) &= \int_{G^{s(x)}} f(xy)(g * h)(y^{-1})\, d\lambda^{s(x)}(y) \\
&= \int_{G^{s(x)}} f(xy)\, d\lambda^{s(x)}(y) \int_{G^{s(y^{-1})}} g(y^{-1}z)h(z^{-1})\, d\lambda^{s(y^{-1})}(z) \\
&= \int_{G^{s(x)}} f(xy)\, d\lambda^{s(x)}(y) \int_{G^{s(x)}} g(y^{-1}z)h(z^{-1})\, d\lambda^{s(x)}(z), \qquad (s(y^{-1}) = r(y) = s(x)) \\
&= \int_{G^{s(x)}} h(z^{-1})\, d\lambda^{s(x)}(z) \int_{G^{s(x)}} f(xy)g(y^{-1}z)\, d\lambda^{s(x)}(y), \quad \text{(Fubini's Theorem)} \\
&= \int_{G^{s(x)}} h(z^{-1})\, d\lambda^{s(x)}(z) \int_{G^{s(z)}} f(x(zy))g((zy)^{-1}z)\, d\lambda^{s(z)}(y) \\
&= \int_{G^{s(x)}} h(z^{-1})\, d\lambda^{s(x)}(z) \int_{G^{s(xz)}} f((xz)y)g(y^{-1})\, d\lambda^{s(xz)}(y), \; ((zy)^{-1}z = y^{-1} \; ((z, y) \in G^{(2)}) \\
&= \int_{G^{s(x)}} f * g(xz)h(z^{-1})\, d\lambda^{s(x)}(z) \\
&= (f * g) * h(x).
\end{aligned}$$

Notice that f^* is also continuous with compact support $\operatorname{supp}(f^*) = (\operatorname{supp}(f))^{-1}$. Hence, the algebra is stable under the involution.
We prove that for all $f, g \in C_c(G)$, we have $g^* * f^* = (f * g)^*$. Using (3.2), (3.1) and the fact that $s(x) = r(x^{-1})$ for all $x \in G$, we have

$$\begin{aligned}
g^* * f^*(x) &= \int_{G^{r(x)}} g^*(y)f^*(y^{-1}x)\, d\lambda^{r(x)}(y) \\
&= \int_{G^{r(x)}} g^*(x(x^{-1}y))\overline{f(x^{-1}y)}\, d\lambda^{r(x)}(y) \\
&= \int_{G^{r(x)}} g^*(xy)\overline{f(y)}\, d\lambda^{s(x)}(y) \\
&= \overline{\int_{G^{r(x^{-1})}} f(y)g(y^{-1}x^{-1})\, d\lambda^{r(x^{-1})}(y)} \\
&= (f * g)^*(x).
\end{aligned}$$

Also,

$$f^{**} = \overline{f^*(x^{-1})} = f((x^{-1})^{-1}) = f(x).$$

So the map $f \mapsto f^*$ is an involution on $C_c(G)$.
Now we claim that the convolution product $*$ is continuous. Define the function

$$\begin{aligned}
H \, : C_c(G) \times C_c(G) &\longrightarrow C_c(G), \\
(f, g) &\longmapsto f * g,
\end{aligned}$$

with $C_c(G)$ equipped with the inductive limit topology. Suppose that $f_n \to f$ and $g_n \to g$, hence, there exist compact sets K and K' such that, eventually, $\operatorname{supp}(f_n) \subset K$ and $\operatorname{supp}(g_n) \subset K'$. Then, there exist

N such that for $n \geq N$ we have $\mathrm{supp}(f_n * g_n) \subset KK'$ (compact). Also,

$$\begin{aligned}|f*g(x) - f_n * g_n(x)| &\leq \int_{KK'} \left|f(xy)g(y^{-1}) - f_n(xy)g_n(y^{-1})\right| \, d\lambda^{s(x)}(y) \\ &\leq \int_{KK'} |f(xy) - f_n(xy)||g(y^{-1})| \, d\lambda^{s(x)}(y) + \int |f_n(xy)||g(y^{-1}) - g_n(y^{-1})| \, d\lambda^{s(x)}(y).\end{aligned}$$

Therefore, $f_n * g_n$ converges uniformly to $f * g$ on KK'. □

Example 3.3 *If Γ is a locally compact Hausdorff group acting continuously on a locally compact Hausdorff space X, and let $\{\varepsilon_x \times \lambda, \, x \in X\}$ be Haar system on $X \times \Gamma$ (as mentioned in example 3.1). let The convolution be given by $f, g \in C_c(X \times \Gamma)$*

$$\begin{aligned}f*g(x) &= \int f((x,\gamma)(y,\gamma'))\, g((y,\gamma')^{-1})\, d(\varepsilon_{x\gamma} \times \lambda)(y,\gamma') \\ &= \int f(x,\gamma')\, g(x\gamma\gamma', \gamma'^{-1})\, d\lambda(\gamma') \\ &= \int f(x,\gamma')\, g(x\gamma', \gamma'^{-1}\gamma)\, d\lambda(\gamma'),\end{aligned}$$

and the involution by : $f^*(x,\gamma) = \overline{f(x\gamma.\gamma^{-1})}$.

Example 3.4 *Recall from Example 3.2 that the convolution in $G = X \times X$ is giving by*

$$\begin{aligned}f*g(x) &= \int f((x,y)(y,z))\, g((y,z)^{-1})\, d(\varepsilon_x \times \lambda)(y,z) \\ &= \int f(x,z)\, g(z,y)\, d\lambda(z),\end{aligned}$$

and the involution by $f^*(x,y) = \overline{f(y,x)}$.

3.2. The normed *-algebra $(C_c(G), *, \|.\|_I)$

We seek to enrich the previously discussed convolution algebra by introducing a norm on $C_c(G)$, closely related, to the L^1-norm in the locally compact group case. The algebra $C_c(G)$ is equipped with the following norms

$$\|f\|_{I,r} = \sup_{u \in G^{(0)}} \int_{G^u} |f(\gamma)| \, d\lambda^u(\gamma), \qquad \|f\|_{I,s} = \sup_{u \in G^{(0)}} \int_{G^u} |f(\gamma^{-1})| \, d\lambda^u(\gamma), \tag{3.4}$$

and

$$\|f\|_I = \max\{\|f\|_{I,r}, \|f\|_{I,s}\}.$$

Considering the maximum of the norms (I,r) and (I,d) norms, ensures that the involution is isometric on $C_c(G)$.

Remark 3.4 *Before proceeding further in the discussion, it is necessary to introduce the inductive limit topology on $C_c(G)$. Let G be a locally compact and Hausdorff groupoid. Consider the set $\mathcal{K}$, consisting of all compact subsets of G. For any $K \in \mathcal{K}$, let $C_K(G)$ to be a subset of $C_c(G)$ consisting of function with compact support contained in K. It is a normed algebra with the supremum norm. The collection $(C_c(G), C_K(G) : K \in \mathcal{K})$, with the order on $\mathcal{K}$, defined by inclusion, is an inductive system (in the sense of Definition 5.1 in Chapter IV of [8]).*

Proposition 3.2 *Let G be a locally compact groupoid. Then,*

*(i) The $(C_c(G), \|f\|_{I,r})$ is a normed algebra, whereas $(C_c(G), \|f\|_I)$ is a normed *-algebra.*

(ii) The I-norm on $C_c(G)$ defines a topology coarser than the inductive limit topology.

(iii) The involution is isometric with respect to the I-norm.

Before proceeding with the proof of the Proposition 3.2, it is necessary to employ the following Lemma.

Lemma 3.1 *Let $\{\lambda_u\}_{u\in G^{(0)}}$ be a Haar system on a locally compact Hausdorff groupoid G. If K is a compact subset of G, then there is an $M>0$ such that*

$$\lambda^u(K) < M \quad \text{for all } u \in G^{(0)}.$$

Proof: Let $f \in C_c(G)$ such that $\operatorname{supp}(f) = K$ and U be an open subset of G such that $K \subset U \subset G$. By using Urysohn's Lemma for locally compact Hausdorff spaces [see [15], 2.12]. Then there exists $h \in C_0^+(G)$ with $h \equiv 1$ on K and vanishes outside U. We have

$$\begin{aligned}\lambda(h)(u) &= \int h(\gamma)\, d\lambda^u(\gamma)\\ &= \int_K \lambda^u(\gamma) + \int_{U/K} h(\gamma) d\lambda^u(\gamma)\\ &\geq \int_K \lambda^u(\gamma) = \lambda^u(K).\end{aligned}$$

Hence, we have $\lambda^u(K) \leq \lambda(h)(u)$, for all $u \in G^{(0)}$, which implies (by using the continuity in Remark 3.3) that $\sup_{u\in G^{(0)}} \lambda^u(K) \leq \sup_{u\in G^{(0)}} |\lambda(h)(u)| = \|\lambda(h)\|_\infty = M$. □

Proof: of Proposition 3.2.

(i) To establish the norms $\|f\|_{I,r}$ and $\|f\|_{I,s}$ as actual norms, we need to verify certain properties. Let us focus on the (I,r)-norm.
Given $f \in C_c(G)$, let K be a compact set such that $\operatorname{supp}(f) \subset K$. By using the previous Lemma, there exist $M > 0$m such that $\lambda^u(K) \leq M$ for all $u \in G^{(0)}$.Then

$$\|f\|_{I,r} = \sup_{u\in G^{(0)}} \int_{G^u} |f(\gamma)|\, d\lambda^u(\gamma) \leq \|f\|_\infty M.$$

(Similarly, for $\|f\|_{I,s} \leq \|f\|_\infty$, by choosing K symmetric, i.e., $K = K^{-1}$). This implies that $\|f\|_I \leq \infty$. If $f \neq 0$, there exists $u \in G^{(0)}$ such that the restriction of $|f|$ to G^u is non-zero.Therefore $\|f\|_{I,r} > 0$ for $f \neq 0$. Additional properties required for $\|f\|_{I,r}$ as a norm are satisfied trivially.

Next, in order to show that $C_c(G)$ is a normed algebra, we need to prove that for all $f, g \in C_c(G)$,

$$\|f * g\|_I \leq \|f\|_I \|g\|_I.$$

Let $f, g \in C_c(G)$. Then, we have

$$\begin{aligned}\int |f*g(x)|\, d\lambda^u(x) &\leq \int_{G^u}\int_{G^{r(x)}} |f(y)|\, |g(y^{-1}x)|\, d\lambda^{r(x)}(y)\, d\lambda^u(x)\\ &= \int_{G^{r(x)}} |f(y)| \int_{G^u} |g(y^{-1}x)|\, d\lambda^u(x)\, d\lambda^{r(x)}(y)\\ &= \int_{G^{r(x)}} |f(y)| \int_{G^u} |g(z)|\, d\lambda^u(z)\, d\lambda^{r(x)}(y)\\ &\leq \sup_{u\in G^0} \int_{G^u} |g(x)|\, d\lambda^u(x) \int_{G^{r(x)}} |f(y)|\, d\lambda^{r(x)}(y)\\ &\leq \sup_{u\in G^0} \int_{G^u} |g(x)|\, d\lambda^u(x) \sup_{v\in G^0} \int_{G^v} |f(y)|\, d\lambda^v(y)\\ &\leq \|f\|_{I,r}\, \|g\|_{I,r}.\end{aligned}$$

(ii) Now, let us prove that the I-norm on $C_c(G)$ defines a topology coarser than the inductive limit topology. By Proposition 5.7 in Chapter IV in [8], f is continuous in the inductive limit topology if and only if its restriction to $C_K(G)$ is continuous, for an arbitrary $K \in \mathcal{K}$. We have a diagram

$$\begin{array}{ccc} (C_K(G), \|\cdot\|_K) & \xrightarrow{\ j_k\ } & (C_c(G), \tau_{ind}) \\ & {}_{Id_2}\searrow \quad \swarrow_{Id_1} & \\ & (C_c(G), \|\cdot\|_I) & \end{array}$$

Then, it suffices to show that Id_2 is continuous (i.e., there exists $C > 0$ such that $\|f\|_I \leq C\|f\|_K$ for $f \in C_K(G)$ and K is an arbitrary compact).
Suppose that $\{f_n\}$ is a sequence in $C_c(G)$ such that f_n converges to 0 in $C_c(G)$. By the continuity of the map $f \longmapsto \lambda(f)$ [see Remark 3.1], we obtain $\lambda(|f_n|) \to 0$ in $C_c(G^{(0)})$. On other hand, we have

$$\|f_n\|_{I,r} = \sup_{u \in G^{(0)}} \int_{G^u} |f_n| \, d\lambda_u = \sup_{u \in G^{(0)}} \lambda(|f_n|) \to 0$$

□

3.3. The C*algebra $C^*(G)$

We now outline the construction of the full and the reduced C*-algebras associated with a locally compact groupoid. To this end, we first rcall the necessary definitions.

Definition 3.2 *Let $L : C_c(G) \to B(\mathcal{H})$ be a *-homomorphism.*

(i) L is said to be non-degenerate if the linear span of

$$\{L(f)\xi : \ f \in C_c(G), \ \xi \in \mathcal{H}\}$$

is dense in $\mathcal{H}$.

(ii) We say that L is continuous in the inductive limit topology, if, whenever $f_i \to f$ in the inductive limit topology on $C_c(G)$ and $\xi, \eta \in \mathcal{H}$, we have

$$\langle L(f_i)\xi, \eta\rangle \to \langle L(f)\xi, \eta\rangle.$$

(iii) L is called bounded, if

$$\|L(f)\| \leq \|f\|_I, \quad \text{for all } f \in C_c(G).$$

A *-representation $L : C_c(G) \to B(\mathcal{H})$ of $C_c(G)$ is a *-homomorphism from the topological *-algebra $C_c(G)$ into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, that is countinuous with respect to the inductive limit topology.

Example 3.5 *Consider a special class of representations of $C_c(G)$ that play a role analogous to the regular representation of a group. Let μ be any Radon measure on $G^{(0)}$. Define $\nu = \mu \circ \lambda$. Then, we have*

$$\nu(f) = \int_{G^{(0)}} \int_G f(\gamma) \, d\lambda_u(\gamma) \, d\mu(u) \quad \text{for } f \in C_c(G).$$

Set $\mathcal{H} = L^2(G, \nu^{-1})$ such that for $f \in C_c(G)$ and $h \in \mathcal{H}$ we have

$$\nu^{-1}(f) = \int_{G^{(0)}} \int_G f(\gamma^{-1}) \, d\lambda_u(\gamma) \, d\mu(u) = \int_{G^{(0)}} \int_G f(\gamma) \, d\lambda_u(\gamma) \, d\mu(u)$$

and define $Ind\mu : C_c(G) \to B(\mathcal{H})$ *for* $f \in C_c(G)$ *and* $h \in \mathcal{H}$ *by*

$$Ind\mu(f)(h)(\gamma) = \int_G f(\eta)h(\eta^{-1}\gamma)\, d\lambda^{r(\gamma)}(\gamma) = f * h(\gamma). \tag{3.5}$$

Then, $Ind\mu$ *is bounded. It suffice to show that : (see [11],p.17 for the proof)*

$$|\langle Ind\mu(f)\xi, \eta\rangle| \leq \|f\|_I \|\xi\|_2 \|\eta\|_2.$$

Hence, $Ind\mu$ *is a non-degenerate representation of* $C_c(G)$*. Particularly, if* $\mu = \delta_x$ *the point mass measure at* $x \in G^{(0)}$*, let us denote* $Ind\delta_x = \pi_x$*, such that*

$$\begin{aligned} \pi_x\ : C_c(G) &\longrightarrow \mathcal{B}(L^2(G_x)), \\ f &\longmapsto \pi_x(f), \end{aligned}$$

where $(\pi_x(f)\xi)(\gamma) = \int_{G_x} f(\eta)\xi(\eta^{-1}\gamma)\, d\lambda_{r(x)}(\eta) = f * \xi(x).$

Proposition 3.3 *Let* $L : C_c(G) \to B(\mathcal{H})$ *be a representation of* $C_c(G)$*. Then,* L *is continuous in the inductive limit topology.*

Proof: Suppose that $f_i \longrightarrow f$ in the inductive limit topology. By using ((ii)) in Proposition 3.2, we have $\|f_i - f\|_I \longrightarrow 0$. And as L is representation which means that L is I-bounded. Hence,

$$|\langle L(f_i)\xi, \eta\rangle - \langle L(f)\xi, \eta\rangle| \leq \|L(f_i - f)\|\|\xi\|\|\eta\| \leq \|f_i - f\|_I \|\xi\|\|\eta\| \quad \text{for all } \xi, \eta \in \mathcal{H}.$$

□

Remark 3.5 *The representations of* $C_c(G)$ *are precisely the *-homomorphisms that exhibit continuity in the inductive limit topology. This observation strongly supports the claim that I-bounded representations are the most appropriate ones to consider. [See Disintegration Theorem [11] in Section 8.1.]*

Definition 3.3 *Consider a Haar system* $\lambda = \{\lambda^u : u \in G^{(0)}\}$ *on a locally compact Hausdorff groupoid* G*. For every* $f \in C_c(G)$*, define the full norm of* f *as*

$$\|f\| = \sup\{\|L(f)\| : L \textit{ is a *-representation of } C_c(G)\}.$$

The full C*-algebra $C^*(G)$ of G is the completion of $C_c(G)$ with respect to the full norm

$$\|f\|_{\text{full}} = \sup_L \|L(f)\| \tag{3.6}$$

The reduced C*-algebra $C_r^*(G)$ is the completion of $C_c(G)$ with respect to the norm

$$\|f\|_r = \sup_{u \in G^{(0)}} \|\pi_u(f)\| \tag{3.7}$$

4. Étale Groupoids and theirs Topological *-Algebras

4.1. Étale groupoids

In this section, we study the topological *-algebra of étale groupoids. Étale groupoids are the analogs of discrete groups in the groupoid setting. The Haar system on an étale groupoid G is given by the counting measure, which simplifies the general formulas for multiplication and involution.

Definition 4.1 *A topological groupoid* G *is étale, if the associated source and range maps* $s, r : G \to G^{(0)}$ *are local homeomorphisms, i.e., for every point* $g \in G$*, there exists an open neighborhood* $U \subset G$ *of* g*, such that* $r(U)$ *and* $s(U)$ *are open in* $G^{(0)}$ *and*

$$r|_U : U \to r(U) \qquad s|_U : U \to s(U)$$

are homeomorphism.

Example 4.1 *Let us get back to our previous examples.*

- *(Étale Groups). A topological group G is étale if and only if it is discrete.*
- *(Étale Equivalence Relations). Recall Example (2.3). If $\mathcal{R} = \{(x,x) : x \in X\}$, then it is étale as for any $(x,x) \in \mathcal{R}$.*
- *(transformation groupoid) Recall Example (2.4). Then, $G \times X$ is étale if and only if the acting group G is discrete.*

Definition 4.2 *A subset U of an étale groupoid G is called a* bisection, *if the source and range map are one-to-one, when restricted to U.*

The topology of an étale groupoid has a basis consisting of open bisections. If U is an open bisection in G, then we have $r : U \to r(U)$ and $s : U \to s(U)$ are both homeomorphisms onto open subsets of $G^{(0)}$.

Proposition 4.1 *If U and V are bisections, then*

- *$U^{-1} = \{u^{-1} \ : \ u \in U\}$ is a bisection.*
- *$UV = \{uv \ : \ y \in U, \ v \in V, \ (u,v) \in G^{(2)}\}$ is a bisection.*

Example 4.2 *(Bisection.) In example (2.3), we have $X = G^{(0)}$. Then*

$$B = \{(x,x) \mid x \in G^{(0)}\}$$

is a bisection.

Lemma 4.1 *[12, Section 8.4] Let G be an étale groupoid. Then we have*

(i) $G^{(0)}$ is an open subset of G.

(ii) The fibers $G^u = r^{-1}(\{u\})$ and $G_u = s^{-1}(\{u\})$ are discrete in the relative topology.

(iii) If a Haar system exists, it is essentially the counting measure system.

Proof:

(i) Suppose that G is an étale groupoid. Let $u \in G^{(0)}$ and let V be an open neighborhood (a bisection) of u in G. Then $r|_V$ is injective. Now, $U =: V \cup r(V)$ is an open neighborhood of u in $G^{(0)}$ and $U = r^{-1}(U)$ is open in G. Hence, $G^{(0)}$ is open in G.

(ii) If $g \in G^u$, then there exists an open bisection U such that $g \in U$. Since r is one-to-one on U, the singleton set $\{x\} = G \cap U$ is open in G^u.

(iii) Let $\{\lambda_u\}_{u \in G^{(0)}}$ be a Haar system for G. Since the fiber G^u is the support of the measure λ^u and is discrete by part ((i)), every element u in $G^{(0)}$ has positive measure λ^u.
Let $g(x) := \lambda(\chi_{G^{(0)}})(x)$, where $\chi_{G^{(0)}}$ denotes the characteristic function of $G^{(0)}$. By the continuity condition of the Haar system, g is continuous and positive. Again, since the measure λ^u is supported by the fiber G^u for each unit u, we have for $u \in G^{(0)}$,

$$g(u) = \int_G \chi_{G^{(0)}} \, d\lambda^u = \int_{G^{(0)} \cap supp(\lambda^u)} \chi_{G^{(0)}} \, d\lambda^u = \int_{G^u} \chi_{G^{(0)}} \lambda^u = \lambda^u(G_u).$$

Replacing λ_u by $\dfrac{\lambda^u}{g(u)}$, we can assume that $\lambda^u(u) = 1$ for all $u \in G^{(0)}$. Then, by invariance, $\lambda^v(x) = 1$ for any $x \in G_u^v$.

□

We conclude from (iii) that étale groupoids have properties analogous to those of discrete groups.

Similar to the preceding section, we adopt straightforward procedures guided by the contributions of Sims [12] and Putnam [13].

4.2. The convolution topological *-algebra $C_c(G)$

In the remaining sections of this paper, we assume that G be a locally compact Hausdorff étale groupoid. We denote by $C_c(G)$ the set of compactly supported continuous complex-valued functions on G.

Lemma 4.2 *Let G be a locally compact and Hausdorff étale groupoid. Then we have*

$$C_C(G) := span\{f \in C_c(G) \ \ \mathrm{supp}(f) \subseteq U,\ U \text{ is a bisection } \}$$

Proof: Let $f \in C_c(G)$. Then there exist open bisections $\{U_i\}_{i\in I}$, such that $\mathrm{supp}(f) \subseteq \bigcup_{i\in I} U_i$. By compactness of $\mathrm{supp}(f)$, there is a finite subcover $\{U_1, \cdots, U_n\}$ such that $\mathrm{supp}(f) \subseteq \bigcup_{i=1}^n U_i$ for suitable $U_i \subseteq G$. As G is locally compact Hausdorff, there exists a continuous partition of unity $\{h_i \mid i \in \{1,\dots,n\}\}$ on $\bigcup_{i=1}^n U_i$ subordinate to the U_i, i.e., for every $i \in \{1,\dots,n\}$, h_i is a continuous function on G with values in $[0,1]$, and $\mathrm{supp}(h_i) \subseteq U_i$ such that $\sum_{i=1}^n h_i(\gamma) = 1$ for all $\gamma \in \mathrm{supp}(f)$ (see [14], Theorem 2.13). Then, the point-wise product $f_i := f \cdot h_i$ is continuous with compact support, because $\mathrm{supp}(f_i) \subseteq \mathrm{supp}(f)$. It follows that $f = \sum_{i=1}^n f_i$ with $\mathrm{supp}(f_i) \subseteq U_i$, and $\mathrm{supp}(f_i)$ is a bisection. □

Another lemma, that will be used in the upcoming, is the following.

Lemma 4.3 *[13, Lemma 3.3.1] Let G be an étale groupoid. If U and V are open bisections, then the restriction of the product map P to $U \times V \cap G^{(2)}$ is a homeomorphism to its image.*

For $f, g \in C_c(G)$ and for $x \in G$, the convolution is defined by

$$(f * g)(x) = \sum_{\alpha\beta=x} f(\alpha)g(\beta) = \sum_{\alpha\in G^{r(x)}} f(\alpha)g(\alpha^{-1}x) \tag{4.1}$$

and the involution by

$$f^*(x) = \overline{f(x^{-1})}. \tag{4.2}$$

Proposition 4.2 *Let G be a locally compact Hausdorff and étale groupoid. Then $C_c(G)$ is a topological * -algebra under the convolution multiplication (4.1), and the involution (4.2).*

Proof: We first show that these operations are well defined and belong to $C_c(G)$. For a fixed $x \in G$, consider

$$\{(\alpha, \beta) \in G^{(2)} \mid \alpha\beta = x \text{ and } f(\alpha)g(\beta) \neq 0\}.$$

If $\alpha\beta = x$, then $\alpha \in G^{r(x)}$ and $\beta \in G_{s(x)}$. Since these sets are discrete (Lemma (ii)), their intersections with $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ are finite. It follows that the sum defining $(f * g)(x)$ is finite.

We now prove that $f * g \in C_c(G)$. By Lemma (4.2), it suffices to check the case when supp(f and $\mathrm{supp}(g)$ are contained in open bisections U and V. To see that $f * g$ has compact support, note that

$$\mathrm{supp}(f * g) \subseteq \mathrm{supp}(f)\,\mathrm{supp}(g) \subseteq UV,$$

and the product UV is compact whenever U and V are compact.

To establish continuity, let $x = \alpha\beta \in UV$. Then, for every $\gamma \in U, \eta \in V$ such that $x = \gamma\eta$ it follows that

$$\begin{cases} r(\gamma) = r(x) = r(\alpha) \\ s(\eta) = s(x) = s(\beta) \end{cases} \implies \begin{cases} \gamma = \alpha \\ \eta = \beta \end{cases}$$

As an immediate consequence, we obtain

$$(f * g)(x) = \sum_{\gamma\eta=x} f(\gamma)g(\eta) = f(\alpha)\, g(\beta), \quad \gamma \in U,\ \eta \in V.$$

Define

$$F : G^{(2)} \to \mathbb{C}, \qquad F(\gamma,\eta) = f(\gamma)g(\eta).$$

Clearly, F is continuous and supported in $G^{(2)} \cap (U \times V)$. Moreover,

$$(f * g) \circ P = F \quad \Longrightarrow \quad f * g = F \circ P^{-1}$$

where $p : G^{(2)} \to G$ is the multiplication map. By Lemma (4.3), the restriction of P $U \times V$ is a homeomorphism. Therefore,$f * g$ is continuous.

For the associativity, let $f, g, h \in C_c(G)$ and $x \in G$. Then

$$\begin{aligned}
((f * g) * h)(x) &= \sum_{\alpha\beta = x} (f * g)(\alpha)\, h(\beta) \\
&= \sum_{\alpha\beta=x} \left(\sum_{\gamma\delta=\alpha} f(\gamma)g(\delta) \right) h(\beta) \\
&= \sum_{\alpha\beta=x} \sum_{\gamma\delta=\alpha} f(\gamma)g(\delta)h(\beta) \\
&= \sum_{\gamma\delta\beta=x} f(\gamma)g(\delta)h(\beta) \\
&= (f * (g * h))(x).
\end{aligned}$$

Now we show that $g^* * f^* = (f * g)^*$. For $x \in G$,

$$\begin{aligned}
(g^* * f^*)(x) &= \sum_{\alpha\beta=x} g^*(\alpha) f^*(\beta) \\
&= \sum_{\alpha\beta=x} \overline{g(\alpha^{-1})}\, \overline{f(\beta^{-1})} \\
&= \sum_{\alpha\beta=x} \overline{f(\beta^{-1}) g(\alpha^{-1})} \\
&= \sum_{\beta^{-1}\alpha^{-1}=x^{-1}} \overline{f(\beta^{-1}) g(\alpha^{-1})} \\
&= \sum_{\gamma\eta=x^{-1}} \overline{f(\gamma)g(\eta)} \quad (\beta^{-1} = \gamma,\ \alpha^{-1} = \eta) \\
&= \overline{(f * g)(x^{-1})} \\
&= (f * g)^*(x).
\end{aligned}$$

Finally, we claim that the convolution product $*$ is continuous. Define the function

$$\begin{aligned}
H : (C_c(G) \times C_c(G), \Pi(\tau_{\text{ind}} \times \tau_{\text{ind}})) &\longrightarrow (C_c(G), \tau_{\text{ind}}), \\
(f, g) &\longmapsto f * g,
\end{aligned}$$

with $C_c(G)$ equipped with the inductive limit topology. Let $\Pi(\tau_{\text{ind}} \times \tau_{\text{ind}})$ be the topology generated by the product topology $\tau_{\text{ind}} \times \tau_{\text{ind}}$. Suppose that $f_n \to f$ and $g_n \to g$. Hence, there exist open bisections K and K' such that, eventually, $\operatorname{supp}(f_n) \subset K$ and $\operatorname{supp}(n) \subset K'$. Then, $\operatorname{supp} f_n * g_n \subset KK'$. Also, for $x = \alpha\beta \in KK'$, we have

$$\begin{aligned}
|f_n * g_n(x) - f * g(x)| &= \left| \sum_{\gamma\eta=x} f_n(\gamma) g_n(\eta) - \sum_{\gamma\eta=x} f(\gamma)g(\eta) \right| \\
&= |f_n(\alpha)g_n(\beta) - f(\alpha)g(\beta)| \\
&\leq |f_n(\alpha)g_n(\beta) - f_n(\alpha)g(\beta)| + |f_n(\alpha)g(\beta) - f(\alpha)g(\beta)| \\
&\leq |f_n(\alpha)|\, |g_n(\beta) - g(\beta)| + |f_n(\alpha) - f(\alpha)|\, |g(\beta)| \\
&\leq \|f_n\|_\infty\, |g_n(\beta) - g(\beta)| + |f_n(\alpha) - f(\alpha)|\, \|g\|_\infty .
\end{aligned}$$

Since $\|g_n - g\|_\infty, \|f_n - f\|_\infty$ converge uniformly to zero on compact sets, the above expression tends to zero uniformly. Therefore $f_n * g_n$ converges uniformly to $f * g$ on KK'. $\square$

Example 4.3 *If G is discrete group, then $C_c(G) = \mathbb{C}G$, the product is $a * b(g) = \sum_{h\in G} a(h)b(h^{-1}g)$, and the involution $a^*(g) = a(g^{-1})$.*

Example 4.4 *(equivalence relation): Recall Example (2.3) with étale topology. For $f, g \in C_c(G)$ and for all $(x, y) \in \mathcal{R}$ the convolution product is given by (using (4.1))*

$$(f * g)(x, y) = \sum_{z\in[x]} f(x, z)\, g(z, y)$$

and the involution is defined by

$$f^*(x, y) = \overline{f(y, x)}.$$

Lemma 4.4 *Let G be a locally compact, Hausdorff and étale groupoid. Let $f \in C_c(G)$ and U be a bisection such that $\operatorname{supp}(f) \subseteq U$. Then*

(i) $C_c(G^{(0)}) \subseteq C_c(G)$ is a commutative *-subalgebra.

(ii) $\begin{cases} f^* * f \in C_c(G^{(0)}) \\ f * f^* \in C_c(G^{(0)}) \end{cases}$ *and we have* $\|f^* * f\|_\infty = \|f * f^*\|_\infty = \|f\|_\infty$

Proof:

(i) By part ((i)) of Lemma 4.1 and Remark 2.2, $G^{(0)}$ is open and closed in G. Hence, every function $g \in C_c(G^{(0)})$ can be identified with the function $\tilde{f}$ on G as follows

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in G^{(0)} \\ 0, & \text{otherwise} \end{cases}$$

To show commutativity, let $g, g' \in C_c(G^{(0)})$. Then

$$\operatorname{supp}(g * g') \subseteq \operatorname{supp}(g) \cdot \operatorname{supp}(g') \subseteq (G^{(0)}.$$

This is a bisection because r and s are homeomorphisms on $G^{(0)}$. So, we see that $g * g'(x) = g(x) \cdot g'(x)$ for $x = xx \in G^{(0)}$, and the commutativity follows.

(ii) We have $\operatorname{supp}(f^* * f) \subseteq \operatorname{supp}(f^*) \cdot \operatorname{supp}(f) = \operatorname{supp}(f)^{-1} \cdot \operatorname{supp}(f) = s(\operatorname{supp}(f))$ and

$$\begin{aligned} \|f^* * f\|_\infty = \sup_{x\in G^{(0)}} |f^* * f(x)| &= \sup_{x\in G^{(0)}} ||f^* * f(xx)| \\ &= \sup_{x\in G^{(0)}} |f^*(x)\, f(x)| \\ &= \sup_{x\in G^{(0)}} \overline{f(x^{-1})}\, f(x) \\ &= \sup_{x\in G^{(0)}} |f(x)|^2 = \|f\|_\infty^2 \end{aligned}$$

$\square$

4.3. The normed *-algebras $(C_c(G), *, \|.\|_I)$

The "I-norm" on $C_c(G)$, when G is étale, is given by

$$\|f\|_I = \max\{\|f\|_{I,r}, \|f\|_{I,s}\},$$

where

$$\|f\|_{I,r} = \sup_{u\in G^{(0)}} \sum_{\gamma\in G^u} |f(\gamma)|, \qquad \|f\|_{I,s} = \sup_{u\in G^{(0)}} \sum_{G^u} |f(\gamma^{-1})|. \tag{4.3}$$

Following the same path as earlier, we obtain that $C_c(G)$ is a normed *-algebra with respect to the I-norm.

Proposition 4.3 *Let G be a locally compact groupoid then, The I-norm on $(C_c(G,*))$ is a norm satisfying, for every $f, g \in C_c(G)$,*

(i) $\|f * g\|_I \leq \|f\|_I \, \|g\|_I$,

(ii) $\|f^*\|_I = \|f\|_I$.

Proof: It is clear that $\|\cdot\|_I$ is homogeneous and satisfies the triangle inequality . To show that it is finite, we have for $f \in C_c(G)$ that $f|_{C_K}(G)$ is continuous for an arbitrary K. Then there is a finite collection of open bisection $\{U_i\}_{i=1}^N$ that cover K. Let $\{h_i\}_i$ partition of unity for K subordinate to the U_i. Then we have

$$\|f\|_I = \|\sum_{i=1}^{N} \|h_i.f\|_I \leq \sum_{i=1}^{i=N} \|h_i.f\|_I \leq \sum_{i=1}^{N} \|h_i.f\|_\infty \leq N\|f\|_\infty < \infty. \tag{4.4}$$

For $f, g \in C_c(G)$, we have

$$\begin{aligned}
\sum_{\gamma\in G^u} |f * g(\gamma)| &= \sum_{\gamma\in G^u} \left| \sum_{\alpha\in G^{r(\gamma)}} f(\alpha)\, g(\alpha^{-1}\gamma) \right| \\
&\leq \sum_{\gamma\in G^u} \sum_{\alpha\in G^{r(\gamma)}} |f(\alpha)|\, |g(\alpha^{-1}\gamma)| \\
&= \sum_{\alpha\in G^u} \sum_{\gamma\in G^{r(\alpha)}} |f(\alpha)|\, |g(\alpha^{-1}\gamma)| \\
&\leq \sum_{\alpha\in G^u} |f(\alpha)| \left(\sum_{\gamma\in G^{r(\alpha)}} |g(\alpha^{-1}\gamma)| \right) \\
&\leq \sum_{\alpha\in G^u} |f(\alpha)| \left(\sum_{\eta\in G^{s(\alpha)}} |g(\eta)| \right) \\
&\leq \sum_{\alpha\in G^u} |f(\alpha)|\, \|g\|_I \\
&\leq \|f\|_I \|g\|_I .
\end{aligned}$$

□

4.4. The C*algebra $C^*(G)$

The full C*-algebra of a discrete group can be seen either as the universal C*-algebra generated by a unitary representation, or as the universal C*-algebra generated by a *-representation of $C_c(G)$. We will use the latter as it provides a more general and applicable perspective.

In the following, we want to introduce a C*-norm on $C_c(G)$. For this, we need to discuss representations of $C_c(G)$. Let $\mathcal{H}$ be a Hilbert space and

$$\pi : C_c(G) \to B(\mathcal{H}), \quad \text{for all } f \in C_c(G)$$

be a *-representation. As consequence of Proposition ((i)) obviously $\pi|_{C_c(G^{(0)})}$ is a *-representation, and we have the following result.

Proposition 4.4 *Let G be a locally compact Hausdorff étale groupoid, $\mathcal{H}$ be a Hilbert space and $\pi : C_c(G) \to B(H)$ be a *-representation of the latter. Then there exists a constant $K_f > 0$ such that*

$$\|\pi(f)\| \leq K_f.$$

And if $\mathrm{supp}(f) \subseteq U$, *is an open bisection we may take $K_f = \|f\|_\infty$.*

Proof: By lemma 4.2 every $f \in C_c(G)$ can be written as $\sum_{i=1}^n f_i$, where $f_i \in C_c(G)$ such that, for each i, $\mathrm{supp}(f_i) \subset U_i$ and $\{U_i\}_i$ is a collection of bisection. By using the proof of Proposition 4.3, we get

$$\|\pi(f)\| = \|\pi\left(\sum_{i=1}^n f_i\right)\| \leq \sum_{i=1}^n \|\pi(f_i)\|$$

And

$$\|\pi(f_i)\|^2 = \|\pi(f_i)\pi(f_i)^*\| = \|\pi(f_i * f_i^*)\|. \tag{4.5}$$

By Lemma 4.4, $f_i * f_i^* \in C_c(G^{(0)})$ and the restriction of π to the commutative *-algebra $(C_c(G^{(0)}))$ becomes a *-homomorphism. We claim that and $\pi|_{G^{(0)}}$ is a C*- homemorphism. Then it is norm decreasing. Hence, $\|\pi(h)\| \leq \|h\|_\infty$ for all $h \in C_c(G^{(0)})$ and we have

$$\|\pi(f_i)\|^2 = \|\pi(f_i)\pi(f_i)^*\| = \|\pi(f_i * f_i^*)\| \leq \|f_i * f_i^*\|_\infty. \tag{4.6}$$

For the last inequality, take $h = f_i * f_i^*$. For $x \in G^{(0)}$ it is clear that

$$f_i * f_i^*(x) = f_i * f_i^*(r(x)) = f_i * f_i^*(xx^{-1}) = f_i(x) * f_i^*(x^{-1}) = |f_i(x)|^2.$$

Getting back to equation (4.6), we get

$$\|\pi(f_i)\|^2 \leq \|f_i * f_i^*\|_\infty = \|f_i\|_\infty^2 \implies \|\pi(f)\| \leq n\|f\|_\infty = K_f.$$

If f is supported on a bisection, then there is just one term in the sum then $K_f = \|f\|_\infty$. □

Lemma 4.5 *Let G be a locally compact Hausdorff étale groupoid. Then any *-representation π of $C_c(G)$ is continuous in the inductive limit topology and satisfies*

$$\|\pi(f)\| \leq \|f\|_I. \tag{4.7}$$

Proof: By the previous Proposition, it is clear that π is continuous in the inductive limit topology. To show that it is I-norm bounded, observe that, for $f \in C_c(G)$, holds $\|f\|_\infty \leq \|f\|_I$. Since continuity is equivalent to boundedness for linear maps on normed spaces, we deduce that π is I-norm bounded.
The completion of $C_c(G)$ in the I-norm yields a Banach *-algebra, hence, the extension of π to this completion is a *-homomorphism from the Banach *-algebra $\overline{C_c(G)}^I$ into $(B(H))$. Applying spectral theory, write $\rho_A : A \to [0, \infty)$ for the spectral-radius function on a Banach algebra A. For each $f \in C_c(G)$, we have

$$\|\pi(f)\|^2 = \|\pi(f^*f)\| = \rho_{B(H)}(\pi(f^*f)) \leq \rho_{\overline{Cc(G)}^I}(f^*f) \leq \|f^*f\|_I \leq \|f\|_I^2.$$

□

Example 4.5 *Let $x \in G^{(0)}$. For each $f \in C_c(G)$ we define*

$$\pi_x : C_c(G) \longrightarrow \mathcal{B}(\ell^2(G_x)), \qquad (\pi_x(f)\xi)(\gamma) = \sum_{\alpha \in G^{r(\gamma)}} f(\alpha)\,\xi(\alpha^{-1}\gamma),$$

for $\xi \in \ell^2(G_x)$ and $\gamma \in G_x$.

Well-definedness. *If $\alpha \in G^{r(\gamma)}$, then*

$$s(\alpha^{-1}) = r(\alpha) = r(\gamma),$$

hence $(\alpha^{-1}, \gamma) \in G^{(2)}$, so that $\alpha^{-1}\gamma$ is defined. Moreover, since $\gamma \in G_x$, we have

$$s(\gamma) = x \quad \Longrightarrow \quad s(\alpha^{-1}\gamma) = x,$$

so the terms of the sum indeed belong to G_x. As, f is compactly supported and $G^{r(\gamma)}$ is discrete, the sum is finite. Therefore $(\pi_x(f)\xi)(\gamma)$ is well-defined for all $\gamma \in G_x$.

Boundedness. *Assume that f is supported on a bisection. Then for each $\gamma \in G_x$, the set $G_x \cap \operatorname{supp}(f)$ contains at most one point, which we denote by η_γ. Thus*

$$(\pi_x(f)\xi)(\gamma) = \begin{cases} f(\eta_\gamma)\,\xi(\eta_\gamma^{-1}\gamma), & \text{if } \eta_\gamma \text{ exists,} \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned}
\|\pi_x(f)\xi\|_2^2 &= \sum_{\gamma \in G_x} |(\pi_x(f)\xi)(\gamma)|^2 \\
&= \sum_{\gamma \in G_x} \sum_{\alpha \in G^{r(\gamma)}} |f(\alpha)\,\xi(\alpha^{-1}\gamma)|^2 \\
&= \sum_{\gamma \in G_x} |f(\eta_\gamma)|^2\,|\xi(\eta_\gamma^{-1}\gamma)|^2 \\
&\le \|f\|_\infty^2 \sum_{\gamma \in G_x} |\xi(\eta_\gamma^{-1}\gamma)|^2 \\
&\le \|f\|_\infty^2 \sum_{\gamma \in G_x} |\xi(\gamma)|^2 \\
&\le \|f\|_\infty^2\,\|\xi\|_2^2.
\end{aligned}$$

Hence $\pi_x(f)$ is bounded with $\|\pi_x(f)\| \le \|f\|_\infty$.

Multiplicativity. *To show that $\pi_x(f)\pi_x(g) = \pi_x(f * g)$, it suffices to check on the basis $\{\delta_\gamma : \gamma \in G_x\}$ of $\ell^2(G_x)$. For $\gamma_0 \in G_x$, compute*

$$(\pi_x(f)\delta_{\gamma_0})(\gamma) = \sum_{\alpha \in G^{r(\gamma)}} f(\alpha)\,\delta_{\gamma_0}(\alpha^{-1}\gamma) = f(\gamma\gamma_0^{-1}),$$

hence

$$\pi_x(f)\delta_{\gamma_0} = \sum_{u \in G_x} f(u\gamma_0^{-1})\,\delta_u. \tag{4.8}$$

Therefore,

$$\begin{aligned}\pi_x(f)\,\pi_x(g)\delta_\gamma &= \pi_x(f)\left(\sum_{\beta\in G_x} g(\beta\gamma^{-1})\delta_\beta\right)\\ &= \sum_{\beta\in G_x} g(\beta\gamma^{-1})\pi_x(f)\delta_\beta\\ &= \sum_{\beta\in G_x} g(\beta\gamma^{-1})\sum_{\alpha\in G_x} f(\alpha\beta^{-1})\delta_\alpha\\ &= \sum_{\alpha\in G_x}\left(\sum_{\beta\in G_x} f(\alpha\beta^{-1})g(\beta\gamma^{-1})\right)\delta_\alpha,\end{aligned}$$

while

$$\begin{aligned}\pi_x(f*g)\delta_\gamma &= \sum_{\alpha\in G_x}(f*g)(\alpha\gamma^{-1})\delta_\alpha\\ &= \sum_{\alpha\in G_x}\left(\sum_{\delta\in G^{r(\alpha)}} f(\delta)g(\delta^{-1}\alpha\gamma^{-1})\right)\delta_\alpha.\end{aligned}$$

For fixed $\alpha\in G_x$, the two expressions in the brackets are the same. This follows from the fact that the maps

$$\begin{aligned}\phi : G_{s(\alpha)} &\longrightarrow G^{r(\alpha)} \\ \beta &\longmapsto \alpha\beta^{-1}\end{aligned}\qquad\qquad \begin{aligned}\psi : G^{r(\alpha)} &\longrightarrow G_{s(\alpha)} \\ \delta &\longmapsto \delta^{-1}\alpha\end{aligned}$$

are inverse bijections, since we have

$$\phi(\psi(\delta)) = \phi(\delta^{-1}\alpha) = \alpha(\delta^{-1}\alpha)^{-1} = \alpha\alpha^{-1}\delta = \delta.$$

$*$-preserving property. *For $\gamma,\eta\in G_x$,*

$$\begin{aligned}\langle\pi_x(f^*)\delta_\gamma,\delta_\eta\rangle &= \langle\sum_{\alpha\in G^x} f^*(\alpha\gamma^{-1})\delta_\alpha,\delta_\eta\rangle\\ &= \sum_{\alpha\in G^x} f(\gamma\alpha^{-1})\langle\delta_\alpha,\delta_\eta\rangle\\ &= f(\gamma\eta^{-1})\\ &= \sum_{\alpha\in G^x} f(\alpha\eta^{-1})\langle\delta_\gamma,\delta_\alpha\rangle\\ &= \langle\delta_\gamma,\sum_{\alpha\in G^x} f(\alpha\eta^{-1})\delta_\alpha\rangle\\ &= \langle\delta_\gamma,\pi_x(f)\delta_\eta\rangle,\end{aligned}$$

which shows that $\pi_x(f^*) = \pi_x(f)^*$. Therefore π_x is a $*$-representation of $C_c(G)$ on $\ell^2(G_x)$.

Remark 4.1 *Let G be a locally compact Hausdorff étale groupoid. The left regular representation of G π^r is the direct sum of the *-representation (4.8)*

$$\pi^r = \bigoplus_{x\in G^{(0)}}\pi_x$$

Definition 4.3 *Let G be a locally compact Hausdorff étale groupoid. For every $f \in C_c(G)$, define the full norm of f as*

$$\|f\| = \sup\{\|\pi(f)\| : \pi \text{ is a *-representation of } C_c(G)\}.$$

This defines a C-norm on $C_c(G)$.*

The full C*-algebra $C^*(G)$ of G is the completion of $C_c(G)$ with respect to the norm in definition (3.3).

$$\|f\|_{\text{full}} = \sup_{\pi} \|\pi(f)\| \tag{4.9}$$

The reduced C*-algebra $C^*_r(G)$ is the completion of $C_c(G)$ with respect to the norm

$$\|f\|_r = \sup_{u \in G^{(0)}} \|\pi_u(f)\| \tag{4.10}$$

Remark 4.2 *For all $f \in C_c(G)$, we have*

$$\|f\|_\infty \leq \|f\|_r \leq \|f\|_{full}.$$

If f is supported on a bisection, then we have equality throughout.

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