Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 3** : 1–9.
ISSN-0037-8712
doi:10.5269/bspm.79130

Asymptotic Profiles of Global Solutions for a PDE Including a Singular Nonlinearity

Zahia Daoui, Arij Bouzelmate and Said El Aboudi

ABSTRACT: We examine solutions of global nature to the following elliptic equation with singular nonlinearity

$$\Delta_p U + \lambda_1 U + \lambda_2 |x|^\gamma |U|^{q-1} U = 0, \quad \text{in } \mathbb{R}^N,$$

such that $p > 2$, $q \geq 1$, $N \geq 1$, $\lambda_1 > 0$, $\lambda_2 > 0$, and $\gamma < 0$.
We prove the global existence of radial solutions and examine conditions ensuring their strict positivity. Furthermore, we describe the asymptotic profile of positive solutions in the limit near infinity.

Key Words: Global solutions, singular nonlinearity, asymptotic profiles.

Contents

1. Introduction

This work addresses the elliptic equation

$$\Delta_p U + \lambda_1 U + \lambda_2 |x|^\gamma |U|^{q-1} U = 0, \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

such that $p > 2$, $q \geq 1$, $N \geq 1$, $\lambda_1 > 0$, $\lambda_2 > 0$ and $\gamma < 0$.

Throughout the paper, the analysis is confined to radial solutions satisfying $\upsilon(x) = \upsilon(|x|)$, which reduce the original PDE to an ordinary differential equation where the study is restricted to continuous functions at the origin.

It is worth noting that several works in the literature have investigated various cases related to equation (1.1) while considering the previously stated restriction. For instance, when $p > 2$, prior work in [3] investigated singularities occurring at isolated points and their asymptotics near infinity , to the equation

$$\Delta_p \upsilon + |\upsilon|^{q-1}\upsilon = 0. \tag{1.2}$$

The authors also provided a classification of solutions, with no restriction on their sign, for all values of q. A related situation is studied in [5], where the parameters λ_1, β, and γ are negative and the equation is

$$\Delta_p \upsilon + \lambda_1 \upsilon + \beta r \upsilon' + r^\gamma |\upsilon|^{q-1}\upsilon = 0. \tag{1.3}$$

The case $p = 2$ is studied in [7], where the focus is on establishing the nonexistence of radial solutions of positive sign related to the next equation

$$\Delta \upsilon - \frac{A}{r^\gamma}\upsilon + \upsilon^{q-1} = 0, \quad A, \gamma > 0 \text{ and } q > 2. \tag{1.4}$$

2020 *Mathematics Subject Classification:* 35A01, 35A02, 35B08, 35B09, 35B40, 35J60.
Submitted September 23,2025. Published February 22, 2026

Moreover, [10] focuses on the study of solutions that decrease rapidly with a predetermined number of zeros by establishing their existence and unicity, and also how they vary with respect to parameters concerning the equation

$$\Delta \upsilon + \frac{r}{2}\upsilon' + \frac{1}{p-1}\upsilon + |\upsilon|^{p-1}\upsilon = 0. \tag{1.5}$$

Various other cases of (1.1) were previously considered in [1,2,4,6,8,9,11,12,13]. In this paper, the problem considered takes the form

$$(Q)\left\{ \begin{array}{l} \left(|\upsilon'|^{p-2}\,\upsilon'\right)' + \dfrac{N-1}{r}\,|\upsilon'|^{p-2}\,\upsilon' + \lambda_1\upsilon + \lambda_2 r^{\gamma}|\upsilon|^{q-1}\upsilon = 0, \quad r > 0, \\ \upsilon(0) = \zeta, \quad \upsilon'(0) = 0 \end{array} \right. \tag{1.6}$$

where $p > 2$, $q \geq 1$, $N \geq 1$, $-p < \gamma < 0$, $-N < \gamma < 0$, $\lambda_1 > 0$, $\lambda_2 > 0$ and $\zeta \in \mathbb{R}^*$.
Owing to the symmetry $\upsilon(.,\zeta,\lambda_1,\lambda_2) = -\upsilon(.,-\zeta,\lambda_1,\lambda_2)$, it is sufficient to consider the case $\zeta > 0$.

We aim to study the global solutions of problem (Q) defined on $[0,+\infty)$, called entire solutions and satisfying $\upsilon \in C^0([0,+\infty[) \cap C^1([0,+\infty[)$, $|\upsilon'|^{p-2}\upsilon' \in C^1(]0,+\infty[)$.

Thus, Section 2 investigates the existence of global solutions. Section 3 proves that, all solutions remain strictly positive for a convenient initial data. We also analyse the long time profiles of the global solutions.

2. Global Solutions

This part focuses on establishing the existence results for global and local solutions to problem (Q). We state two main theorems: the first ensures local existence via a fixed-point argument, while the second establishes global existence.

Theorem 2.1 *The problem (Q) admits a unique solution υ over the maximal interval $[0, r_{\max})$.*

Proof: Denote by υ a solution to problem (Q). Then, for all $r \in [0, r_{max}[$ it holds that

$$\left(r^{N-1}\,|\upsilon'|^{p-2}\,\upsilon'\right)'(r) = -\lambda_1 r^{N-1}\upsilon(r) - \lambda_2 r^{\gamma+N-1}|\upsilon|^{q-1}\upsilon(r). \tag{2.1}$$

An integration over the interval $(0,r)$ of equation (2.1) leads to

$$\upsilon(r) = \zeta - \int_0^r \Lambda(\Psi[\upsilon](s))ds, \tag{2.2}$$

where

$$\Lambda(s) = |s|^{\frac{2-p}{p-1}}s, \quad s \in \mathbb{R}, \tag{2.3}$$

and Ψ is a function defined by

$$\Psi[\upsilon](s) = s^{1-N}\int_0^s \sigma^{N-1}\left\{\lambda_1 + \lambda_2\sigma^{\gamma}|\upsilon|^{q-1}\right\}\upsilon(\sigma)d\sigma. \tag{2.4}$$

Take $\rho > 0$ and $\zeta > \lambda > 0$. Our setting involves the complete metric space

$$E_{\zeta,\lambda,\rho} = \{w \in C([0,\rho]) : \quad \|w - \zeta\| \leqslant \lambda\}. \tag{2.5}$$

We denote by $C([0,\rho])$ the complete space consisting of continuous real functions on $[0,\rho]$, equipped with the uniform norm $\|\cdot\|_0$.
Let us consider the operator Π on $E_{\zeta,\lambda,\rho}$, defined as

$$\Pi[w](r) = \zeta - \int_0^r \Lambda\left(\Psi[w](s)\right)ds.$$

We show first that Π is a self-mapping on $E_{\zeta,\lambda,\rho}$ for small ρ.
Let $w \in E_{\zeta,\lambda,\rho}$. Thus,

$$\Psi[w](s) \leqslant \left(\frac{\lambda_1(\zeta+\lambda)}{N}s^{-\gamma} + \frac{\lambda_2(\zeta+\lambda)^q}{N+\gamma}\right)s^{\gamma+1}. \tag{2.6}$$

We choose $\rho \leqslant \left(\frac{\lambda_1(\gamma+N)}{\lambda_2 N(\zeta+\lambda)^{q-1}}\right)^{\frac{1}{\gamma}}$, then for each $s \in]0,\rho]$, the next inequality is satisfied

$$\Psi[w](s) \leqslant \frac{2\lambda_2(\zeta+\lambda)^q}{N+\gamma}s^{\gamma+1}. \tag{2.7}$$

It results that

$$|\Pi[w](r) - \zeta| \leqslant \frac{p-1}{p+\gamma}\left(\frac{2\lambda_2(\zeta+\lambda)^q}{N+\gamma}\right)^{\frac{1}{p-1}} r^{\frac{\gamma+p}{p-1}}.$$

Hence, for ρ small enough, we get

$$|\Pi[w](r) - \zeta| \leqslant \lambda \quad \text{for } w \in E_{\zeta,\lambda,\rho}.$$

Next, we establish that Π is a contraction mapping from $E_{\zeta,\lambda,\rho}$ into $E_{\zeta,\lambda,\rho}$.
Given $r \in [0,\rho]$ and $w_1, w_2 \in E_{\zeta,\lambda,\rho}$, this yields

$$|\Pi[w_1](r) - \Pi[w_2](r)| \leqslant \int_0^r |\Lambda(\Psi[w_1](s)) - \Lambda(\Psi[w_2](s))|ds, \tag{2.8}$$

such that Λ and Ψ are respectively given by (2.3) and (2.4) .
Note $\mu(s) = \min(\Psi[w_1](s), \Psi[w_2](s))$. We use (2.4) to get

$$\Psi[w](s) \geqslant \left(\frac{\lambda_1(\zeta-\lambda)}{N}s^{-\gamma} + \frac{\lambda_2(\zeta-\lambda)^q}{N+\gamma}\right)s^{\gamma+1}, \tag{2.9}$$

then,

$$\Psi[w](s) \geqslant \frac{\lambda_2(\zeta-\lambda)^q}{N+\gamma}s^{\gamma+1}, \tag{2.10}$$

and

$$|\Pi[w_1](r) - \Pi[w_2](r)| \leqslant \int_0^r (\mu(s))^{\frac{2-p}{p-1}}|\Psi[w_1](s) - \Psi[w_2](s)|ds. \tag{2.11}$$

By using (2.4) and (2.5) , we derive

$$|\Psi(w_1)(s) - \Psi[w_2](s)| \leqslant \left(\frac{\lambda_1}{N}s + \frac{\lambda_2 q(\zeta+\lambda)^{q-1}}{N+\gamma}s^{\gamma+1}\right)\|w_1 - w_2\|_0. \tag{2.12}$$

Due to (2.10), we have

$$\mu(s) \geqslant \frac{\lambda_2(\zeta-\lambda)^q}{N+\gamma}s^{\gamma+1}. \tag{2.13}$$

Therefore,

$$|\Pi[w_1](r) - \Pi[w_2](r)| \leqslant \left(\frac{\lambda_2(\zeta-\lambda)^q}{N+\gamma}\right)^{\frac{2-p}{p-1}} \int_0^r s^{\frac{(\gamma+1)(2-p)}{p-1}}|\Psi[w_1](s) - \Psi[w_2](s)|\,ds. \tag{2.14}$$

Then by (2.12), we get

$$|\Pi[w_1](r) - \Pi[w_2](r)| \leq \left[C_1 r^{\frac{\gamma(2-p)+p}{p-1}} + C_2 r^{\frac{\gamma+p}{p-1}}\right]\|w_1 - w_2\|_0, \tag{2.15}$$

where

$$C_1 = \frac{\lambda_1(p-1)}{N(\gamma(2-p)+p)}\left(\frac{\lambda_2(\zeta-\lambda)^q}{N+\gamma}\right)^{\frac{2-p}{p-1}},$$

and

$$C_2 = \frac{\lambda_2 q(p-1)(\zeta+\lambda)^{q-1}}{(\gamma+p)(\gamma+N)}\left(\frac{\lambda_2(\zeta-\lambda)^q}{N+\gamma}\right)^{\frac{2-p}{p-1}}.$$

Consequently, Π defines a contraction on $E_{\zeta,\lambda,\rho}$.
Accordingly, for $r > 0$ sufficiently close to 0, the existence of a unique solution υ to problem (Q) follows from Banach's fixed-point. □

Theorem 2.2 *The solution to Problem* (Q) *can be extended on* $[0, +\infty[$.

Proof: Consider υ a solution established on $[0, r_{max}[$ to problem (Q). Assuming for contradiction, that $r_{max} < +\infty$, thus

$$\lim_{r\to r_{max}} |\upsilon(r)| = \lim_{r\to r_{max}} |\upsilon'(r)| = +\infty. \tag{2.16}$$

For each $r \in [0, r_{\max}[$, introduce the function E such that

$$E(r) = \frac{p-1}{p}|\upsilon'(r)|^p + \frac{\lambda_1}{2}\upsilon^2(r) + \frac{\lambda_2}{q+1}r^\gamma|\upsilon(r)|^{q+1}.$$

Given that $p > 2$, $q \geq 1$, $\gamma < 0$, and $\lambda_1 > 0$ and $\lambda_2 > 0$, we deduce that

$$\lim_{r\to r_{\max}} E(r) = +\infty.$$

Considering (1.6), it can be seen that

$$E'(r) = -\frac{N-1}{r}|\upsilon'(r)|^p + \frac{\lambda_2\gamma}{q+1}r^{\gamma-1}|\upsilon(r)|^{q+1}. \tag{2.17}$$

In addition, combining (2.16) with the conditions $N \geq 1$, $p > 2$, $\lambda_2 > 0$, and $\gamma < 0$, it yields that

$$\lim_{r\to r_{\max}} E'(r) = -\infty,$$

which is impossible. □

3. Main Results

This section is concerned with establishing conditions that guarantee the strict positivity of solutions to problem (Q). We also analyze the limit of υ and $r^c\upsilon$ near infinity with $c > 0$.

Theorem 3.1 *Assume that* $q > 1$ *and* $\gamma > \dfrac{-p(N-1)}{p-1}$. *For suitably small initial data, the solution of Problem* (Q) *stays strictly positive throughout* $(0, +\infty)$.

To prove the preceding theorem, we begin with a preliminary proposition.

Proposition 3.1 *Let* υ *be a solution of problem* (Q). *Then*

i) If $r_0 > 0$ *represents the first zero of* υ, *then* $\upsilon'(r_0) < 0$.

ii) $\upsilon'(r) < 0$ *for all* $r \in S_\upsilon = \{r > 0 : \upsilon(r) > 0\}$.

Proof: i) Consider $r_0 > 0$ such that $\upsilon(r_0) = 0$ and $\upsilon(r) > 0$ on $(0, r_0)$. Thus, $\upsilon'(r_0) \leq 0$ and by continuity of υ, the interval $(r_0 - \epsilon, r_0)$ forms a left neighborhood where $\upsilon > 0$ and $\upsilon' < 0$, for some $\epsilon > 0$.
Assume for contradiction that the derivative $\upsilon'(r_0) = 0$. Introduce now the function V such that

$$V(r) = r^{N-1}|\upsilon'|^{p-2}\upsilon'(r). \tag{3.1}$$

From equation (1.6), we derive

$$V'(r) = -\lambda_1 r^{N-1}\upsilon(r) - \lambda_2 r^{\gamma+N-1}|\upsilon(r)|^{q-1}. \tag{3.2}$$

Since $\lambda_1 > 0$, $\lambda_2 > 0$, and $\upsilon > 0$ on $(0, r_0)$, it yields that $V'(r) < 0$ on $(0, r_0)$. Hence, for all $r \in (r_0-\epsilon, r_0)$, one obtains

$$0 = V(r_0) < V(r),$$

which yields $\upsilon'(r) > 0$ for every $r \in (r_0 - \epsilon, r_0)$. However, this yields a contradiction with $\upsilon'(r) \le 0$ on $(r_0 - \epsilon, r_0)$. Thus, we must have $\upsilon'(r_0) < 0$.
ii) Applying (1.6), it results that

$$\left(|\upsilon'|^{p-2}\,\upsilon'\right)'(r) \underset{0}{\sim} \frac{-\lambda_2\zeta^q}{N} r^{\gamma},$$

then, $\upsilon'(r) < 0$ for r close to 0.
Suppose υ' has a first zero at r_1. It follows that $(|\upsilon'|^{p-2}\upsilon')'(r_1) \ge 0$. However, from equation (1.6), we have

$$(|\upsilon'|^{p-2}\upsilon')'(r_1) = -\lambda_1\upsilon(r_1) - \lambda_2 r^{\gamma}|\upsilon|^{q-1}\upsilon(r_1) < 0,$$

yielding an absurdity. Thus, $\upsilon'(r) < 0$ for all $r \in S_\upsilon$. □

Theorem 3.1 will now be proved.

Proof: For a contradiction argument, suppose that υ fails to remain strictly positive. Let $r_0 > 0$ such that $\upsilon(r_0) = 0$ and $\upsilon(r) > 0$ on $(0, r_0)$. Define G as

$$\begin{aligned} G(r) &= \frac{p-1}{p}\,|\upsilon'(r)|^p + \frac{\lambda_1}{2}\upsilon^2(r) - \frac{\lambda_2}{q+1} r^{\gamma}|\upsilon(r)|^{q+1} \\ &= r^{\gamma}|\upsilon(r)|^{q+1}\left[\frac{-\lambda_2}{q+1} + \frac{\lambda_1}{2} r^{-\gamma}|\upsilon(r)|^{1-q} + \frac{p-1}{p} r^{-\gamma}|\upsilon(r)|^{-q-1}\,|\upsilon'(r)|^p\right]. \end{aligned} \tag{3.3}$$

Then, by using (1.6), we have

$$G'(r) = -\frac{N-1}{r}\,|\upsilon'(r)|^p - 2\lambda_2 r^{\gamma}|\upsilon(r)|^{q-1}\upsilon(r)\upsilon'(r) - \frac{\lambda_2\gamma}{q+1} r^{\gamma-1}|\upsilon(r)|^{q+1}. \tag{3.4}$$

Observing that $\lim_{r\to 0} r^{-\gamma}G(r) = \frac{-\lambda_2\zeta^{q+1}}{q+1}$, we deduce $G(r) < 0$ when r is small enough.
As $\upsilon(r_0) = 0$, it yields from Proposition 3.1 that $\upsilon'(r_0) < 0$. Then $G(r_0) > 0$.
We infer that there exists $\theta \in (0, r_0)$ for which $G(\theta) = 0$, denoting the first zero of G such that $G'(\theta) \ge 0$.
Considering equation (3.4) together with the condition $G(\theta) = 0$, we deduce that

$$G'(\theta) = \theta^{\gamma-1}\upsilon^{q+1}(\theta)\left[\frac{-\lambda_2}{q+1}\left(\frac{p(N-1)}{p-1} + \gamma\right) + \frac{\lambda_1 p(N-1)}{2(p-1)}\theta^{-\gamma}\upsilon^{1-q}(\theta) + 2\lambda_2\theta\frac{|\upsilon'(\theta)|}{\upsilon(\theta)}\right]. \tag{3.5}$$

We show that $G'(\theta) < 0$ when the parameter ζ is sufficiently small.
In fact, observing that $G(\theta) = 0, |\upsilon'(\theta)|^p > 0$ and due to (3.3), we deduce

$$-\frac{\lambda_1}{2}\upsilon^2(\theta) + \frac{\lambda_2}{q+1}\theta^{\gamma}|\upsilon|^{q+1}(\theta) > 0.$$

Then,

$$\theta(\zeta) < \left(\frac{2\lambda_2}{\lambda_1(q+1)}\right)^{\frac{-1}{\gamma}} \zeta^{\frac{1-q}{\gamma}}.$$

Hence, $\lim_{\zeta\to 0}\theta(\zeta)=0$. Together with $\lim_{r\to 0} r^{-\gamma}\upsilon^{1-q}(r)=0$ and $\lim_{r\to 0} r\upsilon'(r)=0$, this yields

$$\lim_{\zeta\to 0}\theta^{-\gamma}\upsilon^{1-q}(\theta)=0,$$
$$\lim_{\zeta\to 0}\frac{\theta\,|\upsilon'(\theta)|}{\upsilon(\theta)}=0.$$

Then, by combining these results with (3.5) and the positivity of $\left(\frac{p(N-1)}{p-1}+\gamma\right)$ (by hypothesis), we conclude that for sufficiently small ζ, we have $G'(\theta)<0$, which contradicts the inequality $G'(\theta)\geq 0$. □

We now give asymptotic properties of strictly positive solutions υ to problem (Q).

Theorem 3.2 *The following holds*

$$\lim_{r\to+\infty}\upsilon(r)=0 \text{ and } \lim_{r\to+\infty}\upsilon'(r)=0. \tag{3.6}$$

Proof: As $\upsilon(r)>0$ and $\upsilon'(r)<0$ for all $r>0$, it follows that $\lim_{r\to+\infty}\upsilon(r)=\ell\in[0,+\infty[$, and it must also hold that $\lim_{r\to+\infty}\upsilon'(r)=0$. If $\ell>0$, then from (1.6)

$$\lim_{r\to+\infty}\left(|\upsilon'|^{p-2}\upsilon'\right)'(r)=-\lambda_1\ell<0,$$

it results in $\lim_{r\to+\infty}\upsilon'(r)=-\infty$, standing in contradiction with $\lim_{r\to+\infty}\upsilon'(r)=0$. Then, $\lim_{r\to+\infty}\upsilon(r)=0$. □

Theorem 3.3 *Suppose that one of the following conditions holds*

i) $N\leq p$ *and* $q>p-1$.

ii) $p-1<q\leq\dfrac{(\gamma+N)(p-1)}{N-p}$.

Then,

$$\lim_{r\to+\infty} r^{\frac{\gamma+p}{q+1-p}}\upsilon(r)=K, \quad \text{where } K>0. \tag{3.7}$$

Proof: Step 1: $r^{\frac{\gamma+p}{q+1-p}}\upsilon(r)$ is bounded.
By equation (2.1), the following inequality holds for any $r>0$.

$$(r^{N-1}|\upsilon'|^{p-2}\upsilon')'(r)<-\lambda_2 r^{\gamma+N-1}\upsilon^q(r). \tag{3.8}$$

An integration of (3.8) over $(0,r)$ for any $r>0$, ensures

$$r^{N-1}|\upsilon'|^{p-2}\upsilon'(r)<-\lambda_2\int_0^r s^{\gamma+N-1}\upsilon^q(s)\,ds. \tag{3.9}$$

Given that υ is strictly decreasing due to Proposition 3.1, we deduce that

$$r^{N-1}|\upsilon'|^{p-2}\upsilon'(r)<-\frac{\lambda_2}{N+\gamma}r^{N+\gamma}\upsilon^q(r). \tag{3.10}$$

Hence, for every $r>0$,

$$\upsilon'(r)<-\left(\frac{\lambda_2}{N+\gamma}\right)^{\frac{1}{p-1}} r^{\frac{\gamma+1}{p-1}}\upsilon^{\frac{q}{p-1}}(r). \tag{3.11}$$

Taking into account that $q > p-1$ and υ is strictly positive, it results that

$$\frac{p-1}{p-q-1}\left(\upsilon^{\frac{p-q-1}{p-1}}\right)'(r) < -\left(\frac{\lambda_2}{N+\gamma}\right)^{\frac{1}{p-1}} r^{\frac{\gamma+1}{p-1}}. \tag{3.12}$$

Thus,

$$\left(\upsilon^{\frac{p-q-1}{p-1}}\right)'(r) > \frac{q+1-p}{p-1}\left(\frac{\lambda_2}{N+\gamma}\right)^{\frac{1}{p-1}} r^{\frac{\gamma+1}{p-1}}. \tag{3.13}$$

Integrating inequality (3.13) over $(0, r)$ for $r > 0$ yields

$$\upsilon^{\frac{p-q-1}{p-1}}(r) > \frac{q+1-p}{\gamma+p}\left(\frac{\lambda_2}{N+\gamma}\right)^{\frac{1}{p-1}} r^{\frac{p+\gamma}{p-1}}. \tag{3.14}$$

Hence,

$$\upsilon(r) < \left(\frac{q+1-p}{\gamma+p}\left(\frac{\lambda_2}{N+\gamma}\right)^{\frac{1}{p-1}}\right)^{-\frac{p-1}{q+1-p}} r^{-\frac{\gamma+p}{q+1-p}}. \tag{3.15}$$

Consequently, $r^{\frac{\gamma+p}{q+1-p}}\upsilon(r)$ remains bounded.

Step 2: The function $r^{\frac{\gamma+p}{q+1-p}}\upsilon(r)$ is strictly increasing for every $r > 0$.
Introduce the function g defined as

$$g(r) = \frac{p+\gamma}{q+1-p}\upsilon(r) + r\,\upsilon'(r), \quad r > 0. \tag{3.16}$$

Clearly,

$$\left(r^{\frac{\gamma+p}{q+1-p}}\upsilon(r)\right)' = r^{\frac{p+\gamma}{q+1-p}-1}g(r), \quad r > 0. \tag{3.17}$$

Whether $r^{\frac{\gamma+p}{q+1-p}}\upsilon(r)$ is monotone is governed by the sign of $g(r)$. According to (1.6), we derive

$$\begin{aligned}(p-1)|\upsilon'(r)|^{p-2}g'(r) = (p-1)&\left(\frac{\gamma+p}{q+1-p} - \frac{N-p}{p-1}\right)|\upsilon'(r)|^{p-2}\upsilon'(r) \\ &- \lambda_1 r\upsilon(r) - \lambda_2 r^{\gamma+1}|\upsilon(r)|^{q-1}\upsilon(r),\end{aligned} \tag{3.18}$$

for any $r > 0$ satisfying $\upsilon'(r) \neq 0$.
Proposition 3.1 allows that $\upsilon'(r) < 0$ for every $r > 0$, and consequently relation (3.18) gives

$$(p-1)\frac{g'(r)}{\upsilon'(r)} = \left(p - N + \frac{(p-1)(p+\gamma)}{q+1-p}\right) + \frac{\lambda_1 r\,\upsilon(r)}{|\upsilon'(r)|^{p-1}} + \frac{\lambda_2 r^{\gamma+1}\upsilon^q(r)}{|\upsilon'(r)|^{p-1}}. \tag{3.19}$$

According to $i)$ and $ii)$, it follows that $g'(r) < 0$ for all $r > 0$.
Moreover, note that $g(0) = \dfrac{p+\gamma}{q+1-p}\upsilon(0) > 0$. For a contradiction argument, assume that r_0 is the first zero of g. Thus, $g(r) < g(r_0) = 0$ for every $r > r_0$. Consequently, by (3.17), $r^{\frac{\gamma+p}{q+1-p}}\upsilon(r)$ converges as r tends to $+\infty$, while $\lim\limits_{r\to+\infty} g(r) \in [-\infty, 0[$.
Combining $\lim\limits_{r\to+\infty}\upsilon(r) = 0$ (by Theorem 3.2) with expression (3.16), it yields $\lim\limits_{r\to+\infty} r\upsilon'(r) \in [-\infty, 0[$, which contradicts the positivity of υ. Then, $g(r) > 0$ for every $r > 0$ and $r^{\frac{\gamma+p}{q+1-p}}\upsilon(r)$ is strictly increasing.

Step 3: $\lim\limits_{r\to+\infty} r^{\frac{\gamma+p}{q+1-p}}\upsilon(r) = K > 0.$

Step 1 and Step 2 ensure the boundedness and the increasing nature of $r^{\frac{\gamma+p}{q+1-p}}\upsilon(r)$. Thus, a positive constant $K > 0$ exists for which $\lim\limits_{r\to+\infty} r^{\frac{\gamma+p}{q+1-p}}\upsilon(r) = K > 0$.

□

4. Conclusion

In this work, we have investigated the qualitative properties of solutions to problem (Q). Beyond establishing both local and global existence and uniqueness results, particular attention was given to the positivity of solutions. In particular, we showed that when the initial datum ζ is sufficiently small, the corresponding solution remains strictly positive. This analysis provides a first step toward a deeper understanding of how the initial conditions shape the long-term dynamics of the system.

A second aspect of our study focused on the asymptotic behavior of the solution υ and its derivative υ', we also identified an interesting asymptotic equivalence of υ near infinity of the form $K\, r^{-\frac{\gamma+p}{q+1-p}}$ with $K > 0$ and $\frac{\gamma+p}{q+1-p} > 0$. Our results describe the manner in which the solution decays as $r \to +\infty$, providing a crucial information to understand qualitative profiles of the solutions.

Finally, taken together, these results contribute to a more comprehensive understanding of the problem and open the way for further investigations. In particular, the determination of the eventual asymptotic equivalent of υ when $q < p - 1$ or $q > \frac{(\gamma+N)(p-1)}{N-p}$ remains an open question to be addressed in future work.

Acknowledgment

We wish to convey our sincere thanks to the editor and reviewers for the insightful feedback and recommendations provided, which helped improve the clarity and quality of this work.

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Zahia Daoui,
Department of Mathematics,
Lar2a Laboratory, Faculty of Sciences, Abdelmalek Essaadi University,
Tetouan, Morocco.

E-mail address: `zahia.daoui@etu.uae.ac.ma`

and

Arij Bouzelmate,
Department of Mathematics,
Lar2a Laboratory, Faculty of Sciences, Abdelmalek Essaadi University,
Tetouan, Morocco.
E-mail address: `abouzelmate@uae.ac.ma`,

and

Said El Aboudi,
Department of Mathematics,
Lar2a Laboratory, Faculty of Sciences, Abdelmalek Essaadi University,
Tetouan, Morocco.
E-mail address: `said.elaboudi1@etu.uae.ac.ma`