Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 2** : 1–9.
ISSN-0037-8712
doi:10.5269/bspm.79133

New Notation Dedicated to Higher Education Linear Algebra Instructors: Product Between a Vector and a Family of Vectors

Rafik Bouifden* and Aziz Haddi

ABSTRACT: This article presents a new mathematical concept called the external scalar product of a matrix by a family of vectors. This new product is important when the property to be demonstrated concerns the family and is no longer the nature of its vector. This is analogous to the integration technique by substitution, where the variable of integration is changed to simplify the calculation. Thanks to this product, we have discovered a deletion method that has very interesting applications in the teaching of linear algebra. The primary objective of this tool is to simplify the teaching of several concepts in linear algebra.

Key Words: External scalar product, the notation "$\dot{\ltimes}$", matrices, family of vectors, linear algebra.

Contents

1. Notations

- Let $n \in \mathbb{N}$. We denote $[1, n]$ as the set $\{1, 2, 3, \ldots, n\}$;
- $\mathbb{K}$ denotes one of the fields $\mathbb{R}$ or $\mathbb{C}$.
- E denotes a finite-dimensional vector space over $\mathbb{K}$.
- E^m is the Cartesian product of E repeated m times.
- The family $\mathcal{B}_n = (e_1, \ldots, e_n)$ always denotes the canonical basis of the Cartesian vector space $(\mathbb{K}^n, +, \cdot)$.
- Let $(i, j) \in [1, m] \times [1, n]$. The elementary matrix E_{ij} is defined by : $E_{ij} = \delta_{i,p}\delta_{q,j}$, where $1 \le p \le m, 1 \le q \le n$, and where the Kronecker delta δ_{ij} is given by $\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$
- $\mathcal{B}(E)$ the set of the bases of the vector space E.
- $\mathcal{BON}(E)$ the set of the orthonormal bases of the Euclidean space $(E, \langle ., . \rangle)$.
- $\mathcal{O}_n(\mathbb{R})$ the set of orthogonal matrices.
- $\mathcal{GL}_n(\mathbb{K})$ the group of all invertible $n \times n$ matrices with entries in $\mathbb{K}$.

* Corresponding author.
2020 *Mathematics Subject Classification:* 97D40, 97U70, 15A72, 97H60.
Submitted September 23, 2025. Published January 21, 2026

2. Introduction

Driven by ongoing innovation and a growing grasp of conceptual knowledge, linear algebra has long been a cornerstone of mathematics. Keyser [7] emphasizes the critical need of human effort to realize ideals—even those that seem unattainable—since he argues that the progress of society is intimately linked to them. of adding fresh mathematical ideas to improve theoretical understanding and practical applications. Regarding this, Strang [10] emphasizes that progress in every part of the subject depends on ongoing development in mathematical ideas.

Particularly in linear algebra and operator theory, the investigation and description of different product operations are still being actively studied in modern mathematics. For example, in their recent study of inequalities for convex functions of tensor and Hadamard products, Dragomir et al. [4] make use of tensor and Hadamard products extensively. selfadjoint operators in Hilbert spaces, so stressing the continuing need to better grasp the sophisticated features of well-known mathematical tools. Similarly, the development of totally fresh mathematical concepts like the dualtype octonions invented by Dağdeviren and Kürüz [3] shows ongoing attempts to close in on gaps in the literature and grow the intellectual field of mathematical structures. These advances show that often leading to new discoveries, the development of fresh mathematical objects and their related activities is a valid and vital component of mathematical growth. novel angles of view and unanticipated uses.

This study suggests a fresh perspective on some fundamental linear algebra ideas based on the spirit of invention and the continuous development of mathematical frameworks. via the construction of an external scalar product. Offering a fresh viewpoint, especially by enabling a worldwide view of matrices [9] and more easy transfer of matrix properties to vectors, this new tool is intended to provide one. This method supports the focus on worldwide studies in linear algebra Hoffman and Kunze suggest [6], as well as the educational advantages of creative approaches Zhang and Wang explored. The notation "$\dot{\ltimes}$" chosen is a conscious choice that recognizes that symbols play a key role in mathematical practice and that they can help explain new ideas ([2]; [5]; [1]).

Furthermore, the deletion approach, which has significant pedagogical and practical benefits in higher education, shows the practical value of this external scalar product. This covers simplifying linear transformation matrix calculation, enabling the quest for orthogonal bases in Euclidean spaces, and setting up a bijection between bases and their duals. Emphasizing abstractions and generalizations [8], these applications respond to calls for linear algebra instruction combining both conceptual and procedural components. By This new invention seeks to improve the learning experience by means of interactive and visual features that encourage greater grasp of abstract ideas. Encourage original thoughts inside this basic area.

3. Presentation of the Product and its Properties

Definition 3.1 *Let* $X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ x_n \end{pmatrix} \in \mathbb{K}^n$ *and* $\mathcal{F} = (u_1, u_2, \dots u_n)$ *a family of vectors of the* $\mathbb{K}$*-vector space* E*. Then the external scalar product, denoted* $X \dot{\ltimes} \mathcal{F}$ *is defined by:*

$$X \dot{\ltimes} \mathcal{F} = \sum_{i=1}^{n} x_i u_i$$

We have two mechanical operations that remind us of the usual scalar product (with the only difference, we replaced the usual product of $\mathbb{K}$ by the external law of space E).

Remark 3.1 If $\mathcal{F} = Y = (y_1, y_2, ..., y_n) \in \mathbb{K}^n$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ x_n \end{pmatrix} \in \mathbb{K}^n$. Then:

$$X \dot{\ltimes} \mathcal{F} = X \dot{\ltimes} Y = \sum_{i=1}^{n} x_i y_i$$

(It is indeed the canonical scalar product of $\mathbb{K}^n$, $< X, Y >$*)*

This seems to justify the scalar terminology attributed to this new product.

Remark 3.2 The external scalar product $X \dot{\ltimes} \mathcal{F}$ generalizes the usual matrix product .

Let $A \in M_n(\mathbb{K})$ that we look at as a family of its columns: $A = (C_1, C_2, ..., C_n)$, where each column plays the role of a vector.

By definition

$$X \dot{\ltimes} A = \sum_{i=1}^{n} x_i C_i$$

Hence the symmetric relation

$$X \dot{\ltimes} A = AX \tag{3.1}$$

where AX is the usual matrix product $A \times X$.

4. Propositions and Benefits of the External Scalar Product

Here is the global formulation of the external law of a $\mathbb{K}$-vector space $(E, +, .)$

Proposition 4.1 *Let* $m \in \mathbb{N}^*$

The application $\begin{array}{l} \dot{\ltimes} : \mathbb{K}^m \times E^m \to E \\ (X, \mathcal{F}) \mapsto X \dot{\ltimes} \mathcal{F} \end{array}$ *is* $\mathbb{K}$*- bilinear.*

Proof: *Linearity in the first argument:* For all $X, Y \in \mathbb{K}^m$, $\mathcal{F} \in E^m$, and $\alpha, \beta \in \mathbb{K}$,

$$(\alpha X + \beta Y) \dot{\ltimes} \mathcal{F} = \sum_{i=1}^{m} (\alpha x_i + \beta y_i).u_i = \alpha \sum_{i=1}^{m} x_i.u_i + \beta \sum_{i=1}^{m} y_i.u_i = \alpha(X \dot{\ltimes} \mathcal{F}) + \beta(Y \dot{\ltimes} \mathcal{F}).$$

Linearity in the second argument: For all $X \in \mathbb{K}^m$, $\mathcal{F}_1 = (u_1, u_2, \dots u_n), \mathcal{F}_2 = (v_1, v_2, \dots v_n) \in E^m$, and $\alpha, \beta \in \mathbb{K}$,

$$X \dot{\ltimes} (\alpha \mathcal{F}_1 + \beta \mathcal{F}_2) = \sum_{i=1}^{m} x_i.(\alpha u_i + \beta v_i) = \alpha \sum_{i=1}^{m} x_i.u_i + \beta \sum_{i=1}^{m} x_i.v_i = \alpha(X \dot{\ltimes} \mathcal{F}_1) + \beta(X \dot{\ltimes} \mathcal{F}_2).$$

Thus, $\dot{\ltimes}$ is $\mathbb{K}$-bilinear.

Remark 4.1 Note that for $m = 1$, we recover the external law "." of the vector space E.

Proposition 4.2 *Let* X *and* Y *be elements of* $\mathbb{K}^n$*, and let* $\mathcal{F}$ *and* $\mathcal{F}'$ *be elements of* E^n*, with* $\alpha \in \mathbb{K}$*.*

1. *Mixed distributivity :* $\begin{cases} (X + Y) \dot{\ltimes} \mathcal{F} = X \dot{\ltimes} \mathcal{F} + Y \dot{\ltimes} \mathcal{F} \\ X \dot{\ltimes} (\mathcal{F} + \mathcal{F}') = X \dot{\ltimes} \mathcal{F} + X \dot{\ltimes} \mathcal{F}' \end{cases}$
2. *Mixed associativity :* $(\alpha.X) \dot{\ltimes} \mathcal{F} = X \dot{\ltimes} (\alpha.\mathcal{F}) = \alpha.(X \dot{\ltimes} \mathcal{F})$
3. *Mixed unitarity :* $(1_{\mathbb{K}}.1_{\mathbb{K}^n}) \dot{\ltimes} \mathcal{F} = 1_{\mathbb{K}^n} \dot{\ltimes} (1_{\mathbb{K}}.\mathcal{F}) = 1_{\mathbb{K}}.(1_{\mathbb{K}^n} \dot{\ltimes} \mathcal{F})$

Note how this new notation allows the simple formulation of several properties, particularly in properties or proofs where the nature of the elements of the family $\mathcal{F}$ is not of interest.

Proposition 4.3 *Let $\mathcal{F}$ be a family of E.*

1. *The family $\mathcal{F}$ is linearly independent if and only if :*

$$(\forall X \in K^n) \quad (X \dot{\ltimes} \mathcal{F} = 0_E \Rightarrow X = 0_{\mathbb{K}^n})$$

2. *The family $\mathcal{F}$ spans E if and only if :*

$$(\forall x \in E) \quad (\exists X \in K^n) : x = X \dot{\ltimes} \mathcal{F}$$

3. *The family $\mathcal{F}$ is a basis of E if and only if :*

$$(\forall x \in E) \quad (\exists! X \in K^n) : x = X \dot{\ltimes} \mathcal{F} \tag{4.1}$$

Note that in the global notation $x = X \dot{\ltimes} \mathcal{F}$, the coordinate matrix of the vector x in the basis $\mathcal{F}$ becomes visible namely it is the vector column X. More generally, the vector space $(\mathbb{K}^n, +, .)$ serves as a standard finite-dimensional model for vector spaces over the field $\mathbb{K}$. This means that for any vector space E of dimension n with basis $\mathcal{B}$, there is an associated coordinate isomorphism $\begin{array}{c} M_{\mathcal{B}} : E \to \mathbb{K}^n \\ x = X \dot{\ltimes} \mathcal{B} \mapsto X \end{array}$ $(*)$ where X is the coordinate column vector of x in $\mathcal{B}$. This correspondence allows one to represent vectors in E by their coordinates in $\mathbb{K}^n$.

Visual coding: In practice, one often omits the basis notation $\mathcal{B}$ in expressions like $X \dot{\ltimes} \mathcal{B}$, focusing simply on the coordinate vector X itself, which encodes the vector with respect to the chosen basis.

4.1. Orthonormed basis of an Euclidean space

We assume that $\mathbb{K} = \mathbb{R}$. Recall that the standard scalar product of $\mathbb{R}^n$ is defined by :

$$\forall (X, Y) \in \mathbb{R}^n \times \mathbb{R}^n \quad \langle X, Y \rangle_c = X.Y = \sum_{i=1}^{n} x_i y_i$$

The Euclidean space $\mathbb{R}^n$ remains a model in Euclidean spaces, i.e if we choose in $(*)$ $\mathcal{B}$ an orthonormed basis in Euclidean space $(E, \langle ., . \rangle)$, then the erase isomorphism $M_{\mathcal{B}}$ becomes an isometry.

$$\begin{array}{c} M_B : (E, \langle ., . \rangle) \to (\mathbb{R}^n, \langle ., . \rangle) \\ X \dot{\ltimes} \mathcal{B} \mapsto X \end{array}$$

Note that the family $\mathcal{B}$ is an ortohonormed base of $(E, \langle ., . \rangle)$, if and only if, we have:

$$\forall (X, Y) \in \mathbb{R}^n \times \mathbb{R}^n \quad \langle X \dot{\ltimes} B, Y \dot{\ltimes} B \rangle = \langle X, Y \rangle_c$$

Thus, we obtain a global deficiency of an orthonormed basis of an Euclidean space. Where this time the coordinate matrix became visible.

4.2. A logical formulation for duality formulas

Let E^* be the dual space of a vector space E.
Consider a basis $\mathcal{B} = (u_1, u_2, ..., u_n)$ of E and note $\mathcal{B}^* = (u_1^*, u_2^*, ..., u_n^*)$ the dual basis of $\mathcal{B}$.
Consider $\varphi : E \to \mathbb{K}$ a linear form of E. So we have the decomposition of φ in $\mathcal{B}^*$:

$$\varphi = \sum_{i=1}^{n} \varphi(u_i) u_i^*$$

By definition:

$$M_B(\varphi) = (\varphi(u_1), \varphi(u_2), ..., \varphi(u_n))$$

Hence the logical formulation of this formula:

$$\forall \varphi \in E^* \quad \varphi = M(\varphi, \mathcal{B}) \dot{\ltimes} \mathcal{B}^*$$

Note this freedom of vision, we can look at φ as a vector of E^* in which case its coordinate matrix in $\mathcal{B}^*$ is the matrix $M(\varphi, \mathcal{B})$ which is at the same time its matrix as a linear application of E in $\mathbb{K}$.

4.3. Global definition of a transition matrix in the general case

Let's put ourselves in a theoretical situation where E is any vector space of dimension n. Let $\mathcal{B}$ be a basis of E, therefore a family $\mathcal{C}$ of E is uniquely written:

$$\mathcal{C} = (C_1 \dot{\ltimes} \mathcal{B}, C_2 \dot{\ltimes} \mathcal{B}, ..., C_n \dot{\ltimes} \mathcal{B})$$

Recall that we have coordinated isomorphism (or better *erasure* of the basis $\mathcal{B}$)

$$\begin{aligned} M_{\mathcal{B}} : E &\to \mathbb{K}^m \\ x = X \dot{\ltimes} \mathcal{B} &\mapsto X \end{aligned} \tag{4.2}$$

By definition of the image of a family by an application, $M_B(\mathcal{C}) = (C_1, C_2, ..., C_n)$ is a family of $\mathbb{K}^n$, which is confused with the matrix of $M_{m,n}(\mathbb{K})$ whose columns are the column vectors $C_1, C_2, ..., C_n$. Note that the classical notation $M_{\mathcal{B}}(\mathcal{C})$ coincides with its natural construction.

Visual coding: The matrix of the family $\mathcal{C} = (C_1 \dot{\ltimes} \mathcal{B}, C_2 \dot{\ltimes} \mathcal{B}, ..., C_n \dot{\ltimes} \mathcal{B})$ in the basis $\mathcal{B}$ is obtained by **erase** from the basis $\mathcal{B}$

$$M_B(\mathcal{C}) = (C_1, C_2, ..., C_n) \in M_{m,n}(\mathbb{K})$$

Now let's recall that if $m = n$ and $\mathcal{C}$ is a basis of E, then the matrix $M_{\mathcal{B}}(\mathcal{C})$ denoted $P_{\mathcal{B},\mathcal{C}}$, called the transition matrix from $\mathcal{B}$ to $\mathcal{C}$.

4.4. Definition of the matrix of a linear map

Instead of using the usual language found in textbooks,

"Let $\mathcal{B} = (u_1, u_2, ..., u_n)$ be a basis of E, and $\mathcal{C} = (v_1, v_2, ..., v_m)$ a basis of F, and $f : E \to F$ a linear map"

we can simply say that $f : (E, \mathcal{B}) \to (F, \mathcal{C})$ is an oriented linear map.

We know that, each time a basis $\mathcal{B}$ of E is fixed, giving a linear map $f : E \to F$ is equivalent to giving the image family $f(\mathcal{B}) = (f(u_1), f(u_2), ..., f(u_n))$ in F.

Now, we can express the family $f(\mathcal{B}) = (f(u_1), f(u_2), ..., f(u_n))$ of F in the basis $\mathcal{C}$ of F, and we write

$$\forall j \in [1, n] : f(u_j) = \sum_{i=1}^{m} a_{i,j} v_i = C_j \dot{\ltimes} C$$

where $C_j = (a_{1,j}, a_{2,j}, ..., a_{n,j})^t$ denotes the matrix (or column vector) of coordinates of the vector $f(u_j)$ in the basis $\mathcal{C}$. This leads us to represent the oriented linear map $f : (E, \mathcal{B}) \to (F, \mathcal{C})$ by a matrix in $M_{m,n}(F)$, which we denote by $M(\mathcal{C}, f, \mathcal{B})$, that is, the matrix of the family $f(\mathcal{B})$ expressed in the basis $\mathcal{C}$. We say that $M(\mathcal{C}, f, \mathcal{B})$ is the matrix of f relative to the two bases $\mathcal{B}, \mathcal{C}$.

$$M(\mathcal{C}, f, \mathcal{B}) = \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \\ \downarrow & \downarrow & & \downarrow \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \begin{array}{l} \leftarrow v_1 \\ \leftarrow v_2 \\ \vdots \\ \leftarrow v_m \end{array}$$

Note that in this notation "$M(\mathcal{C}, f, \mathcal{B})$", if $\mathcal{C} \neq \mathcal{B}$, the order of the bases is very important, and if $\mathcal{C} = \mathcal{B}$, one simply writes $M(f, \mathcal{B})$ instead of $M(\mathcal{B}, f, \mathcal{B})$.

4.5. Matrix representation of a linear map

Let $\mathcal{B}$ be a basis of E. Then every vector x in E can be uniquely written as $x = X \cdot \mathcal{B}$ (4.1). We emphasize that in such a notation, it is implicit that $X_{\mathcal{B}} = M_{\mathcal{B}}(x)$ is the coordinate matrix of the vector x with respect to the basis $\mathcal{B}$. Our goal is to determine the coordinate matrix of the vector $f(x)$.

Theorem 4.1 *Let $f : (E, \mathcal{B}) \to (F, \mathcal{C})$ be an oriented linear map, and let us denote $A = M(\mathcal{C}, f, \mathcal{B})$. We have*

$$\forall x \in E : \quad M_{\mathcal{C}}(f(x)) = A \times X_{\mathcal{B}}$$

Proof: this is equivalent to showing that

$$\forall x \in E : \quad f(x) = (A \times X_{\mathcal{B}}) \dot{\ltimes} \mathcal{C}$$

To this end, let $x = X_{\mathcal{B}} \dot{\ltimes} \mathcal{B}$. Since f is linear, we have

$$f(x) = X_{\mathcal{B}} \dot{\ltimes} f(\mathcal{B}),$$

and let $f(\mathcal{B}) = (C_1 \dot{\ltimes} \mathcal{C}, C_2 \dot{\ltimes} \mathcal{C}, \ldots, C_n \dot{\ltimes} \mathcal{C})$ so that $A = M(\mathcal{C}, f, \mathcal{B}) = (C_1, C_2, \ldots, C_n)$.

On the other hand, by definition,

$$X_{\mathcal{B}} \dot{\ltimes} f(\mathcal{B}) = \sum_{j=1}^{n} x_j (C_j \dot{\ltimes} \mathcal{C}) = \sum_{j=1}^{n} (x_j C_j) \dot{\ltimes} \mathcal{C}),$$

and from (3.1) we have

$$A \times X = \sum_{j=1}^{n} x_j C_j$$

To provide an interpretation of Theorem 4.1, consider the following definition:

Definition 4.1 *let $A \in M_{m,n}(\mathbb{K})$ be a matrix. The linear map associated with the matrix A is the map denoted by f_A defined by*

$$\begin{aligned} f_A : \mathbb{K}^n &\to \mathbb{K}^m \\ X &\mapsto AX \end{aligned}$$

Example 4.1 for $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in M_3(\mathbb{R})$ we have $AX = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}$

then the map associated with the matrix A is simply the circular permutation endomorphism

$$\begin{aligned} f_A : \mathbb{R}^3 &\to \mathbb{R}^3 \\ (x_1, x_2, x_3) &\mapsto (x_2, x_3, x_1) \end{aligned}$$

As a result of (4.2) and 4.1, we have the following commutative diagram.

$$\begin{array}{ccc} (E, \mathcal{B}) & \xrightarrow{f} & (F, \mathcal{C}) \\ M_{\mathcal{B}} \downarrow & & \downarrow M_{\mathcal{C}} \\ (\mathbb{K}^n, \mathcal{B}_n) & \xrightarrow{f_A} & (\mathbb{K}^m, \mathcal{B}_m) \end{array}$$

5. The External Scalar Product of a Matrix by a Family of Vectors and Applications

5.1. Definition and propositions

In this part, we will define the scalar product of a matrix by a family of vectors, from the scalar product of a column vector by a family of vectors.

Definition 5.1 *Let $A \in M_{m,n}(\mathbb{K})$ a matrix that we present by its columns:*

$$A = (C_1, C_2, ..., C_n)$$

we call the scalar product of the matrix A by the family $\mathcal{F} \in E^m$, the family $A \dot{\ltimes} \mathcal{F}$ of E, defined by :

$$A \dot{\ltimes} \mathcal{F} = (C_1 \dot{\ltimes} \mathcal{F}, C_2 \dot{\ltimes} \mathcal{F}, ..., C_n \dot{\ltimes} \mathcal{F})$$

Example 5.1 We find the generalization of the usual matrix product.

1. If $A = X \in M_{n,1}(\mathbb{K})$ We find the product. $X \dot{\ltimes} \mathcal{F}$
2. If $\mathcal{F} = B$ a matrix of $M_{n,m}(\mathbb{K})$, then $X \dot{\ltimes} \mathcal{F}$ is equal to the usual matrix product $B \times A$. We get the symmetry relationship: $A \dot{\ltimes} B = B \times A$
3. Note the perfect match, the application: $\begin{aligned} \dot{\ltimes} : M_{m,n}(\mathbb{K}) \times E^m &\to E^n \\ (A, \mathcal{F}) &\mapsto A \dot{\ltimes} \mathcal{F} \end{aligned}$ is $\mathbb{K}$-bilinear

The external scalar product $\dot{\ltimes}$ is associative, in this sense we have the following theorem:

Theorem 5.1 *Let $A \in M_{n,m}(\mathbb{K})$, $B \in M_{m,n}(\mathbb{K})$ and $\mathcal{F} \in E^n$, we have $(A \times B) \dot{\ltimes} \mathcal{F} = B \dot{\ltimes} (A \dot{\ltimes} \mathcal{F})$*

Proof. Let us first show the following lemma:

Lemma 5.1 $(\forall X \in \mathbb{K}^m) \quad (\forall \mathcal{F} \in E^n) \quad (\forall A \in M_{n,m}(\mathbb{K})) : \qquad (AX) \dot{\ltimes} \mathcal{F} = X \dot{\ltimes} (A \dot{\ltimes} \mathcal{F})$

Proof: Since both expressions are linear in X, just show the relationship to $X = E_i$, where $i \in [[1, m]]$.
Let $\mathcal{F} = (u_1, u_2, ..., u_n) \in E^n$.
We have

$$\begin{cases} (AX) \dot{\ltimes} \mathcal{F} = (A.E_i) \dot{\ltimes} \mathcal{F}) = C_i(A) \dot{\ltimes} \mathcal{F} \\ \\ X \dot{\ltimes} (A \dot{\ltimes} \mathcal{F}) = E_i \dot{\ltimes} (A \dot{\ltimes} \mathcal{F}) = C_i(A) \dot{\ltimes} \mathcal{F} \end{cases}$$

Then

$$\begin{aligned} (A \times B) \dot{\ltimes} \mathcal{F} &= (C_1(A \times B) \dot{\ltimes} \mathcal{F}, ..., C_n(A \times B) \dot{\ltimes} \mathcal{F}) \\ &= ((A \times C_1(B)) \dot{\ltimes} \mathcal{F}, ..., (A \times C_n(B)) \dot{\ltimes} \mathcal{F}) \\ &= (C_1(B) \dot{\ltimes} (A \dot{\ltimes} \mathcal{F}), C_2(B) \dot{\ltimes} (A \dot{\ltimes} \mathcal{F}), ..., C_n(B) \dot{\ltimes} (A \dot{\ltimes} \mathcal{F})) \\ &= B \dot{\ltimes} (A \dot{\ltimes} \mathcal{F}) \end{aligned}$$

5.2. The notions of basis and invertible matrix are equivalent

Let us note $\mathcal{B}(E)$ the set of the bases of the vector space E. Subsequently, it should be noted that the basis of a vector space and an invertible matrix are equivalent. This means that each time one chooses a basis $\mathcal{B}_0$ of space E, the basis erase bijection : $\begin{aligned} M_B : E &\to \mathbb{K}^m \\ x = X \dot{\ltimes} \mathcal{B} &\mapsto X \end{aligned}$ identifies a basis and an invertible matrix.

More precisely, we have the following theorem:

Theorem 5.2 *Let $\mathcal{B}(E)$ the set of bases of E and $\mathcal{B}_0$ a basis of E. we have a bijection of sets:*

$$\begin{aligned} \sigma_{\mathcal{B}_0} : \mathcal{GL}_n(\mathbb{K}) &\to \mathcal{B}(E) \\ P &\mapsto P \dot{\ltimes} \mathcal{B}_0 \end{aligned}$$

Proof: The reciprocal bijection is given by isomorphism deletion of the base $\mathcal{B}_0$:

$$P_{\mathcal{B}_0} : \mathcal{B}(E) \to \mathcal{GL}_n(\mathbb{K})$$
$$\mathcal{B} = (C_1 \dot{\ltimes} \mathcal{B}_0, C_2 \dot{\ltimes} \mathcal{B}_0, ..., C_n \dot{\ltimes} \mathcal{B}_0) \mapsto P_{\mathcal{B}_0, B} = (C_1, ..., C_n)$$

Note that $det_{\mathcal{B}_0}(P \dot{\ltimes} \mathcal{B}_0) = detP$. Such that $P \dot{\ltimes} \mathcal{B}_0$ is a basis if, and only if, the matrix P is invertible.

let us now consider $(E, \langle ., . \rangle)$ Euclidean space, $\mathcal{B}_0$ an orthonormal basis of $(E, \langle ., . \rangle)$ and let noted by $\mathcal{BON}(E)$ the set of the orthonormal bases of this Euclidean space. As a consequence of 5.2, we have that in an Euclidean space the notion of orthonormed basis and orthogonal matrix are equivalent.

With these notations, we have:

Corollary 5.1 *The application:* $\begin{array}{c} \sigma : \mathcal{O}_n(\mathbb{R}) \to \mathcal{BON}(E) \\ P \mapsto P \dot{\ltimes} \mathcal{B}_0 \end{array}$ *is a bijection of sets.*

Proof: With global notations , $\mathcal{B}_0$ is an orthonormed basis of $(E, \langle ., . \rangle)$ means that:

$$\forall i, j \in [[1, n]] \quad \langle C_i \dot{\ltimes} \mathcal{B}_0, C_j \dot{\ltimes} \mathcal{B}_0 \rangle = \langle C_i, C_j \rangle_c = \delta_{i,j}$$

It follows that the family $P \dot{\ltimes} \mathcal{B}_0 = (C_1 \dot{\ltimes} \mathcal{B}_0, C_2 \dot{\ltimes} \mathcal{B}_0, ..., C_n \dot{\ltimes} \mathcal{B}_0)$ is an orthonormed basis, if and only if, the matrix $P = (C_1, ..., C_n)$ is orthogonal.

From this we deduce the set bijection of 5.2 induces another bijection σ.

6. Practical Determination of the Dual or Antedual Basis with Global Notations

Recall that in general, if F is a $\mathbb{K}$ -vector space, we note $\mathcal{B}(F)$ the set of bases of F. It is well known that we have a bijection of sets: $\begin{array}{c} \sigma_* : \mathcal{B}(E) \to \mathcal{B}(E^*) \\ \mathcal{B} \mapsto \mathcal{B}^* \end{array}$, where $\mathcal{B}^*$ is the dual basis of $\mathcal{B}$.

Let $\mathcal{B}$ be a basis of E.

By definition of the basis $\mathcal{B}_0$, there is a unique inverse matrix P such as $\mathcal{B} = P \dot{\ltimes} \mathcal{B}_0$.

Recall that $P^* =^t P^{-1}$ is called the dual matrix of P.

Thanks to the global notations, there is only one practical formula for the determination of the dual or antedual basis.

With these notations, we have:

Theorem 6.1 $(\forall B \in \mathcal{B}(E))$: $\quad B^* = (P \dot{\ltimes} \mathcal{B}_0)^* = P^* \dot{\ltimes} \mathcal{B}_0^*$ *(where* $P^* =^t P^{-1}$*)*

Let's introduce group involutive autopmorphism

$$\begin{array}{c} t : (\mathcal{GL}_n(\mathbb{K}), \times) \to (\mathcal{GL}_n(\mathbb{K}), \times) \\ P \mapsto^t P^{-1} \end{array} \tag{6.1}$$

We represent the above ((6.1)) relationship as follows:

Theorem 6.2 *Let* $\mathcal{B}_0$ *be a basis of* E*,then we have the following diagram:*

$$\begin{array}{ccc} \mathcal{GL}_n(\mathbf{K}) & \xrightarrow{t} & \mathcal{GL}_n(\mathbf{K}) \\ \sigma_{\mathcal{B}_0} \downarrow & & \downarrow \sigma_{\mathcal{B}_0^*} \\ \mathcal{B}(E) & \xrightarrow{\sigma_*} & \mathcal{B}(E^*) \end{array}$$

Proof: Note that thanks to the global notations we have replaced in the commutative diagram the set bijection σ_* by the automorphism of groups t, and especially the automorphism t is involutive, has a deep meaning in duality, modulo the identification of a vector space of finite dimension to its bidual E^{**}, the basis $(B^*)^* = B^{**}$ identifies with the basis B. Thus, automorphism t clearly shows that the notion of dual and antedual basis are equivalent.

We notice that with these notations, the classical relationship

$$P_{\mathcal{B}_0^*, \mathcal{B}^*} =^t P_{\mathcal{B}_0, \mathcal{B}}^{-1}$$

Is written

$$P_{\mathcal{B}_0^*} \circ \sigma = t \circ P_{\mathcal{B}_0}$$

Remark 6.1 How to determine the antedual basis from the 6.2 relationship?

Let $\mathcal{C}$ be a basis of E^* and we look for $\mathcal{B}$ a basis of E such as $\mathcal{B}^* = \mathcal{C}$.

Such as $\mathcal{B}_0^*$ is a basis of E^*, Then $\exists Q \in \mathcal{GL}_n(\mathbb{K})$ such as $C = Q \dot{\ltimes} \mathcal{B}_0^*$

Or $Q = Q^{**}$

Hence $C = (Q^*)^* \dot{\ltimes} \mathcal{B}_0^*$.

According to 6.2 : $\mathcal{C} = (Q^* \dot{\ltimes} \mathcal{B}_0)^*$

So by definition of the antedual basis of $\mathcal{C}$ is $Q^* \dot{\ltimes} \mathcal{B}_0$

Conclusion: To unearth the anterior basis, it is enough to determine $Q^* =^t Q^{-1}$

References

1. Cajori, F., *A History of Mathematical Notations. I. Notations in Elementary Mathematics. II. Notations mainly in Higher Mathematics*, Open Court Publishing Company; Dover, Chicago; New York (1993).
2. Cohen, H., Hurst, R., *Mathematical Methods in Engineering*, Journal of Engineering Mathematics, 112(1), 1–25 (2017).
3. Dağdeviren, A., Kürüz, F., *On Dual Type Octonions and Their Properties*, Turk. J. Math. Comput. Sci., 17(1)(2025), 93–101.
4. Dragomir, S. S., et al., *Some Tensorial and Hadamard Product Inequalities for Convex Functions of Selfadjoint Operators in Hilbert Spaces*, Turk. J. Math. Comput. Sci., 17(1)(2025), 47–58.
5. Heeffer, A., *On the nature and origin of algebraic symbolism*, New Perspectives on Mathematical Practices: Essays in Philosophy and History of Mathematics, World Scientific, Singapour, pp. 1–27 (2012).
6. Hoffman, K., Kunze, R., *Linear algebra*, 2nd ed., Prentice-Hall (1971).
7. Larcombe, P.J., *MATHEMATICAL MUSINGS II Cassius J. Keyser: A Doctrine of Ideals*, Palestine Journal of Mathematics, Vol. 13(1), 358–360 (2024).
8. Rensaa, R.J., Hogstad, N.M., Monaghan, J., *Perspectives and reflections on teaching linear algebra*, Teaching Mathematics and Its Applications, 39(4), 296–310 (2020). DOI: 10.1093/teamat/hraa008.
9. Rogalski, M., *Les rapports entre local et global : mathématiques, rôle en physique élémentaire, questions didactiques*, in Laurence Viennot (éd.), Didactique, épistémologie et histoire des sciences: Penser l'enseignement, pp. 61–87, Presses Universitaires de France, Paris (2008).
10. Strang, G., *III. Gilbert Strang: What to Teach and How?*, in "The Best Writing on Mathematics", Princeton University Press, pp. 214–217 (2016).
11. Zhang, S., Wang, J., *A New Approach to Linear Algebra*, International Journal of Mathematics and Mathematical Sciences, Article ID 2345634 (2018).

Rafik Bouifden,
Department of Mathematics,
Faculty of Sciences of Tetouan,
Morocco.
E-mail address: rafik.bouifden@etu.uae.ac.ma

and

Aziz Haddi,
Department of Mathematics,
Faculty of Sciences of Tetouan,
Morocco.
E-mail address: ahaddi@uae.ac.ma