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Asymptotics of Solutions to p-Laplacian Equations Involving Convection and Reaction Terms

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ABSTRACT: The purpose of this work is to investigate a nonlinear p-Laplacian equation that incorporates both convection and reaction effects. The model under consideration takes the form

$$\operatorname{div}(|\nabla U|^{p-2}\nabla U) + \lambda x \nabla(|U|^{q-1}U) + \theta U = 0 \quad \text{in} \quad \mathbb{R}^N,$$

with parameters $N \geq 1$, $p > 2$, $q > 1$, $\lambda > 0$, and $\theta > 0$. Our main results concern the existence of global radial solutions, which are shown to be strictly positive under suitable assumptions. In addition, we examine the qualitative properties of these solutions and describe their asymptotic profile as $|x| \to \infty$.

Key Words: Asymptotic behavior, entire solution, existence of positive solutions, radial solution.

Contents

1. Introduction

Nonlinear partial differential equations (PDEs) involving the p-Laplacian operator arise in various physical, biological, and geometric contexts, including non-Newtonian fluid mechanics, reaction-diffusion processes, and mathematical physics [10,18]. Among these, equations incorporating convection and reaction terms play a crucial role in modeling transport phenomena, population dynamics, and combustion theory [13,17]. This paper investigates the existence and qualitative properties of solutions to the nonlinear equation

$$\operatorname{div}(|\nabla U|^{p-2}\nabla U) + \lambda x \nabla(|U|^{q-1}U) + \theta U = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

where $N \geq 1$, $p > 2$, $q > 1$, $\lambda > 0$ and $\theta > 0$.

The presence of the convection term $x\nabla(|U|^{q-1}U)$ introduces additional mathematical challenges, as it influences both the existence and the asymptotic behavior of solutions. Several studies have investigated p-Laplacian equations with reaction and convection terms under various settings. In particular, the term $x\nabla(|U|^{q-1}U)$ can be physically interpreted as representing a radial drift or flow, either outward from or inward toward a central point, driven by a velocity field that is proportional to both the distance from the origin and the magnitude of the solution.

For instance, in [11], the authors established existence results and studied the *Emden-Fowler equation* with a convection term. In [5], the authors investigated the structure of radial solutions and studied the asymptotic behavior of positive solutions near infinity. Moreover, Bouzelmate, Gmira, and Reyes in [6] investigated radial self-similar solutions of the *Ornstein-Uhlenbeck equation*. For further details, we refer the reader to [1,2,8,12,16].

When $p = 2$, Chipot and Weissler [7] analyzed the one-dimensional form of the nonlinear parabolic equation with a gradient term. Subsequently, Serrin and Zou published two important papers [14,15],

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which focus on the existence of radial ground states and the introduction of novel energy functions. More recently, Bidaut-Véron and Véron [3] also examined this equation and demonstrated the existence of a positive solution with a nonnegative measure μ as boundary data. For more details, see [4,9,19,20].

Our primary aim is to establish the existence of entire radial solutions and to determine conditions under which these solutions remain strictly positive. Using appropriate functional and analytical techniques, we also investigate the qualitative properties of solutions, including their asymptotic decay at infinity. Our approach combines methods from nonlinear analysis, such as fixed-point theorems, and energy estimates, to derive meaningful insights into the solution structure. More specific, we study the following Cauchy problem:

Problem (P): Find a function u defined on $[0,+\infty[$ such that $|u'|^{p-2}u' \in C^1([0,+\infty[)$ and that satisfies

$$\begin{cases} \left(|u'|^{p-2}u'\right)' + \dfrac{N-1}{r}|u'|^{p-2}u' + \lambda r(|u|^{q-1}u)' + \theta u = 0, \quad r > 0, \\ u(0) = A, \quad u'(0) = 0, \end{cases} \tag{1.2}$$

where $N \geq 1,\ p > 2,\ q > 1,\ \lambda > 0$ and $\theta > 0$.

The paper is structured as follows. In Section 2, we establish the existence of entire radial solutions u to problem (P). Section 3 is devoted to studying qualitative properties of these solutions. Finally, Section 4 deals with their asymptotic behavior at infinity.

2. Existence of Entire Solutions

We establish in this section the existence of global solutions u to problem (P) by applying the Banach Fixed Point Theorem.

Theorem 2.1 *Problem (P) admits a unique global solution u. Furthermore, it satisfies*

$$(|u'|^{p-2}\, u')'(0) = -\frac{A\theta}{N} < 0. \tag{2.1}$$

Proof:

1) Existence and uniqueness of a local solution.

Multiply equation (1.2) with r^{N-1}, we derive

$$\left(r^{N-1}\,|u'|^{p-2}\,u' + \lambda r^N |u|^{q-1}u\right)' = r^{N-1}u(-\theta + \lambda N|u|^{q-1}). \tag{2.2}$$

By integrating equation (2.2) twice from 0 to r , we get

$$u(r) = A - \int_0^r G(F[u](s))ds, \tag{2.3}$$

where

$$G(s) = |s|^{\frac{2-p}{p-1}} s, \quad s \in \mathbb{R}, \tag{2.4}$$

and F is the following function

$$F[u](s) = \lambda s|u|^{q-1}u(s) + s^{1-N}\int_0^s \sigma^{N-1}\left(\theta u(\sigma) - \lambda N|u|^{q-1}u(\sigma)\right) d\sigma. \tag{2.5}$$

We consider the corresponding complete metric space

$$V_{A,\delta,R} = \{v \in C([0,R]) \text{ such that } \|v - A\|_0 \leqslant \delta\}.$$

Moreover, we introduce the operator Ψ on $V_{A,\delta,R}$ as follows

$$\Psi[v](r) = A - \int_0^r G(F[v](s))ds. \tag{2.6}$$

i) Ψ maps $V_{A,\delta,R}$ into itself for some small δ and $R > 0$.

It is obvious that $\Psi[v] \in C([0,R])$. Given that $\|v - A\|_0 \le \delta$, we conclude that $v \in [A-\delta, A+\delta]$. It is easy to show that $F[v]$ has a constant sign in $[0,R]$ for every $v \in V_{A,\delta,R}$. Moreover, there exists $m > 0$ such that

$$F[v](s) \ge ms \quad \text{for all } s \in [0,R], \tag{2.7}$$

where $m = \dfrac{A}{N}$.

Since $\dfrac{G(r)}{r}$ is decreasing on $(0,+\infty)$, then

$$|\Psi[v](r) - A| \le \int_0^r \frac{G(F[v](s))}{F[v](s)} |F[v](s)| ds \le \int_0^r \frac{G(ms)}{ms} |F[v](s)| ds,$$

for $r \in [0,R]$. Hence

$$|F[v](s)| \le Cs,$$

where $C = \dfrac{\theta}{N}(A+\delta) + 2\lambda(A+\delta)^q$. Then

$$|\Psi[v](r) - A| \le \frac{p-1}{p} C r^{\frac{p}{p-1}} m^{\frac{2-p}{p-1}}.$$

Select R sufficiently small so that

$$|\Psi[v](r) - A| \le \delta, \quad v \in V_{A,\delta,R}.$$

Consequently, $\Psi[v] \in V_{A,\delta,R}$, which confirming i).

ii) Ψ is a contraction in some interval $[0,R]$.

For any $v, w \in V_{A,\delta,R}$, we have

$$|\Psi[v](r) - \Psi[w](r)| \le \int_0^r |G(F[v](s)) - G(F[w](s))| ds, \tag{2.8}$$

where $F[v]$ is defined by (2.5). Next, we introduce the function

$$\Phi(s) = \min(F[v](s), F[w](s)).$$

Using estimate (2.7), for $0 \le s \le r < R$, we derive

$$\Phi(s) \ge ms.$$

It follows that

$$|G(F[v](s)) - G(F[w](s))| \le \frac{G(\Phi(s))}{\Phi(s)} |F[v](s) - F[w](s)| \le \frac{G(ms)}{ms} |F[v](s) - F[w](s)|. \tag{2.9}$$

Having in mind that

$$|F[v](s) - F[w](s)| \le C' \|v - w\|_0 s, \tag{2.10}$$

where $C' = \dfrac{\theta}{N} + q\lambda(A+\delta)^{q-1}$, and using (2.7) – (2.10), we find

$$|\Psi[v](s) - \Psi[w](s)| \le \frac{p-1}{p} C' r^{\frac{p}{p-1}} m^{\frac{2-p}{p-1}} \|v - w\|_0. \tag{2.11}$$

When R is chosen small enough, Ψ becomes a contraction. According to the Banach Fixed Point Theorem, there is a unique fixed point of Ψ in $V_{A,\delta,R}$, which constitutes a solution to equation (2.3) and thus to problem (P). Consequently, $u \in C^1((0,R])$. Now, we will show that $u \in C^1$ in $r=0$. By integrate equation (2.2) on $(0,r)$, we get

$$\frac{|u'|^{p-2}\,u'(r)}{r} = -\lambda|u|^{q-1}u + r^{-N}\int_0^r s^{N-1}(-\theta u(s) + \lambda N|u|^{q-1}u(s))ds. \tag{2.12}$$

Using L'Hopital's rule, we derive

$$(|u'|^{p-2}u')'(0) = \lim_{r\to 0}\frac{|u'|^{p-2}\,u'(r)}{r} = -\frac{A\theta}{N}.$$

Using (1.2), we find

$$\lim_{r\to 0}(|u'|^{p-2}\,u')'(r) = -\frac{A\theta}{N}.$$

We conclude that problem (1.2) admits a unique solution u on an interval $[0, R_{max}[$, with $0 < R_{max} \le +\infty$.

2) Existence of global solution.
Consider

$$H(r) = \frac{p-1}{p}\,|u'|^p + \frac{\theta}{2}|u|^2. \tag{2.13}$$

By (1.2), we get

$$H'(r) = -ru'^2\left(\frac{N-1}{r}|u'|^{p-2} + \lambda q|u|^{q-1}\right). \tag{2.14}$$

Since H is positive and decreasing, it follows that H is bounded. Consequently, both u and u' are also bounded for all $r \ge 0$, which allows that necessarily $R_{max} = +\infty$ and the solution u will be extended to the whole $\mathbb{R}^+$.

□

3. Structure of Radial Solutions

We analyze in this section the structure of the radial solutions. More specifically, we establish that the solution remains strictly positive under certain conditions, we also determine when the solution changes sign. This characterization is obtained through a scaling transformation.

Lemma 3.1 *Let u be a solution of (P) and define $S_u := \{r > 0, u(r) > 0\}$. Then, for every $r \in S_u$, one has $u'(r) < 0$.*

Proof: We proceed by contradiction. Suppose $r_0 > 0$ is the first zero of u'. From (2.1), we know that $u'(r) < 0$ for small r. Since u' is continuous, there exists an interval $]r_0 - \varepsilon, r_0[$ for some $\varepsilon > 0$ where u' is strictly increasing and negative. This implies that $(|u'|^{p-2}\,u')'(r) > 0$ in this region, leading to $(|u'|^{p-2}u')'(r_0) \ge 0$. Yet, from (1.2), we derive $(|u'|^{p-2}\,u')'(r_0) = -\theta u(r_0) < 0$, which contradicts the previous inequality. Hence, the proof is complete. □

Proposition 3.1 *Let $u > 0$ be a solution to (P). Then, for any $r > 0$, it holds that*

$$0 < |u'(r)| < \left(\frac{p\theta}{2(p-1)}\right)^{\frac{1}{p}} A^{\frac{2}{p}}. \tag{3.1}$$

Moreover,

$$A - \left(\frac{p\theta}{2(p-1)}\right)^{\frac{1}{p}} A^{\frac{2}{p}} r < u(r) < A. \tag{3.2}$$

Proof: We have $u > 0$ and $u' < 0$ on $(0, +\infty)$, applying (2.13) and (2.14) yields $H' < 0$ for all $r \geq 0$. As a result, we obtain $H(r) < H(0) = \dfrac{\theta}{2}A^2$. This implies that $0 \leq \dfrac{p-1}{p}|u'|^p \leq \dfrac{\theta}{2}A^2$. Thus, (3.1) holds. Furthermore, from (3.1), we derive $u'(r) > -\left(\dfrac{p\theta}{2(p-1)}\right)^{\frac{1}{p}} A^{\frac{2}{p}}$. By integrating from 0 to r, we derive $A - \left(\dfrac{p\theta}{2(p-1)}\right)^{\frac{1}{p}} A^{\frac{2}{p}} r < u < u(0) = A$. □

Theorem 3.1 *There exists a constant $A_* > 0$ such that, for every $A > A_*$, the solution $u > 0$ of (P) is strictly positive.*

Proof:
We define the transformation

$$u(r) = AY(\eta), \quad \text{where} \quad \eta = A^{\frac{q+1-p}{p}} r.$$

Therefore, Y satisfies the problem

$$\begin{cases} (|Y'|^{p-2} Y')' + \dfrac{N-1}{\eta} |Y'|^{p-2} Y' + q\lambda\eta |Y|^{q-1} Y' + A^{1-q}\theta Y = 0, & \text{for } \eta > 0, \\ Y(0) = 1, \quad Y'(0) = 0. \end{cases} \tag{3.3}$$

In view of (3.1) and (3.2), we derive

$$|Y(\eta)| \leq 1 \quad \text{and} \quad |Y'(\eta)| \leq \left(\frac{p\theta}{2(p-1)}\right)^{1/p} A^{\frac{1-q}{p}}. \tag{3.4}$$

Hence, since $q > 1$, then for large A, problem (3.3) is a perturbation of the following problem

$$\begin{cases} (|Z'|^{p-2} Z')' + \dfrac{N-1}{\eta} |Z'|^{p-2} Z' + q\lambda\eta |Z|^{q-1} Z' = 0, & \text{for } \eta > 0, \\ Z(0) = 1, \quad Z'(0) = 0. \end{cases} \tag{3.5}$$

We claim that Z of (3.5) is strictly positive. Assume, for contradiction, that η_0 is the first zero of Z, implying $Z'(\eta_0) \leq 0$. Multiplying (3.5) by η^{N-1} and integrating over $(0, \eta_0)$, we obtain

$$\eta_0^{N-1} |Z'|^{p-2} Z'(\eta_0) = \lambda N \int_0^{\eta_0} s^{N-1} Z^q(s) ds. \tag{3.6}$$

This results in a contradiction, proving that Z must be strictly positive. Consequently, Y is also strictly positive, leading to the conclusion that u is strictly positive as well.

□

Theorem 3.2 *There exists $A_0 > 0$ such that for all $A \in (0, A_0)$, the solution u of (P) changes the sign.*

Proof:
We introduce the following transformation

$$u(r) = AY(\eta), \quad \text{where} \quad \eta = A^{\frac{2-p}{p}} r.$$

Then, Y satisfies the problem

$$\begin{cases} (|Y'|^{p-2} Y')' + \dfrac{N-1}{\eta} |Y'|^{p-2} Y' + qA^{q-1}\lambda\eta |Y|^{q-1} Y' + \theta Y = 0, & \text{for } \eta > 0, \\ Y(0) = 1, \quad Y'(0) = 0. \end{cases} \tag{3.7}$$

Thanks to (3.1) and (3.2), we derive

$$|Y(\eta)| \leq 1 \quad \text{and} \quad |Y'(\eta)| \leq \left(\frac{p\theta}{2(p-1)}\right)^{1/p}. \tag{3.8}$$

Hence, when A is taken sufficiently small, problem (3.7) can be regarded as a perturbation of the following problem

$$\begin{cases} (|Z'|^{p-2} Z')' + \dfrac{N-1}{\eta} |Z'|^{p-2} Z' + \theta Z = 0, \quad \text{for } \eta > 0, \\ Z(0) = 1, \quad Z'(0) = 0. \end{cases} \tag{3.9}$$

Equation (3.9) can be reformulated in the following way

$$\left(\eta^{N-1} |Z'|^{p-2} Z'\right)' = -\theta \eta^{N-1} Z(\eta). \tag{3.10}$$

Integrating this last equality over $(0, \eta)$, we have

$$\eta^{N-1} |Z'|^{p-2} Z' = -\theta \int_0^{\eta} s^{N-1} Z(s) ds. \tag{3.11}$$

If Z is strictly positive, then $Z' < 0$, which implies that

$$\eta^{N-1} |Z'|^{p-2} Z' < -\frac{\theta}{N} \eta^N Z(\eta).$$

This gives

$$\left(Z^{\frac{p-2}{p-1}}\right)' \leq -\frac{p-2}{p} \left(\frac{\theta}{N}\right)^{\frac{1}{p-1}} \left(\eta^{\frac{p}{p-1}}\right)'.$$

by integrating over $(0, \eta)$, we get

$$Z^{\frac{p-2}{p-1}}(\eta) \leq 1 - \frac{p-2}{p} \left(\frac{\theta}{N}\right)^{\frac{1}{p-1}} \eta^{\frac{p}{p-1}}.$$

Letting $\eta \to +\infty$ leads to a contradiction. Let η_0 be the first zero of Z. Then, from (3.11), we have

$$\eta_0^{N-1} |Z'|^{p-2} Z'(\eta_0) = -\theta \int_0^{\eta_0} s^{N-1} Z(s) ds.$$

Since $\theta > 0$ and $Z > 0$ on $(0, \eta_0)$, we get $Z'(\eta_0) < 0$. Thus, Z changes sign, and consequently, u does as well.

□

4. Asymptotic Behavior Near Infinity

Our focus here is on describing how the positive solution of (P) behaves as r becomes large.

Theorem 4.1 *Suppose u is a solution to (P). If $N > 1$ or $N = 1$ and $u > 0$, then*

$$\lim_{r \to +\infty} u(r) = \lim_{r \to +\infty} u'(r) = 0. \tag{4.1}$$

Proof:

- **1)** $N > 1$. We have $H \geq 0$ and $H' \leq 0$ for all $r > 0$, it follows the existence of a constant $L \geq 0$ such that $\lim_{r\to+\infty} H(r) = L$. If $L > 0$, we can find some $r_1 > 0$ such that

$$H(r) \geq \frac{L}{2} \quad \text{for} \quad r \geq r_1. \tag{4.2}$$

Consider the function

$$B(r) = H(r) + \frac{N-1}{2r}|u'|^{p-2}u'u + \frac{q\lambda(N-1)}{2(q+1)}|u|^{q+1} + q\lambda \int_0^r s|u(s)|^{q-1}u'(s)^2\, ds. \tag{4.3}$$

Then

$$B'(r) = -\frac{N-1}{2r}\left[|u'|^p + \frac{N}{r}|u'|^{p-2}u'u + \theta u^2\right]. \tag{4.4}$$

Due to the boundedness of H, u and u' are bounded, so we get

$$\lim_{r\to+\infty} \frac{|u'|^{p-2}u'u}{r} = 0.$$

According to (2.13) and (4.2), we get for $r \geq r_1$ that

$$\frac{L}{2} \leq H(r) = \frac{p-1}{p}|u'|^p + \frac{\theta}{2}u^2 \leq |u'|^p + \theta u^2.$$

As a result, there are two constants $c > 0$ and $r_2 \geq r_1$ such that

$$B'(r) \leq -\frac{c}{r} \quad \text{for} \quad r \geq r_2.$$

It follows that

$$B(r) \leq B(r_2) - c\ln\left(\frac{r}{r_2}\right) \quad \text{for} \quad r \geq r_2,$$

which implies that $\lim_{r\to+\infty} B(r) = -\infty$. Having in mind that

$$H(r) + \frac{N-1}{2r}|u'|^{p-2}u'u \leq B(r),$$

then, $\lim_{r\to+\infty} H(r) = -\infty$, which is impossible. Therefore, the conclusion follows.

- **2)** $N = 1$ and $u > 0$. Let

$$\psi(r) = |u'|^{p-2}u' + \lambda r|u|^{q-1}u. \tag{4.5}$$

From (1.2), we get

$$\psi'(r) = u(\lambda|u|^{q-1} - \theta). \tag{4.6}$$

Since $u(r) > 0$, the function is strictly decreasing, which implies $\lim_{r\to+\infty} u(r) \in [0, +\infty[$. Assume that $\lim_{r\to+\infty} u(r) = \ell > 0$. Because the energy function H defined in (2.13) has a finite limit, we then have $\lim_{r\to+\infty} u'(r) = 0$. Hence, $\lim_{r\to+\infty} \psi(r) = +\infty$. Applying L'Hopital's rule, we derive

$$\lim_{r\to+\infty} \psi'(r) = \lim_{r\to+\infty} \frac{\psi(r)}{r}.$$

Therefore

$$\ell(\lambda\ell^{q-1} - \theta) = \lambda\ell^q.$$

Hence, $-\theta\ell = 0$, which contradicts the assumption that $\ell > 0$. As a result, $\lim_{r\to+\infty} u(r) = 0$.

□

To establish additional results, we first reformulate equation (1.2) in an equivalent form. For any $c > 0$, we introduce the function $h_c(r)$ by

$$h_c(r) = cu(r) + ru'(r), \quad r > 0. \tag{4.7}$$

This leads to the relation

$$(r^c u(r))' = r^{c-1} h_c(r), \quad r > 0. \tag{4.8}$$

Consequently, applying (1.2), we obtain an alternative formulation valid for all $r > 0$ where $u'(r) \neq 0$.

$$(p-1)|u'(r)|^{p-2}h_c'(r) \quad = ru(r)\left[-\theta - q\lambda r|u|^{q-3}uu' + (p-1)\left(c - \frac{N-p}{p-1}\right)\frac{|u'|^{p-2}u'}{ru}\right]. \tag{4.9}$$

In view of (4.7) and (4.9), when $h_c(r_0) = 0$ for some $r_0 > 0$, we have

$$(p-1)|u'(r_0)|^{p-2}h_c'(r_0) \quad = r_0 u(r_0)\left[-\theta + q\lambda c\, u(r_0)^{q-1} + c^{p-1}(p-1)\left(\frac{N-p}{p-1} - c\right)\frac{1}{r_0^p u(r_0)^{2-p}}\right]. \tag{4.10}$$

Proposition 4.1 *Let $u > 0$ be a solution of (P). Then the function $r^c u$ is strictly monotone for large r.*

Proof: Let r_0 be a sufficiently large point where $h_c(r_0) = 0$. Given that $p > 2$, $\lim\limits_{r\to+\infty} u(r) = 0$ and by using equation (4.10), we derive

$$(p-1)|u'(r_0)|^{p-2}h_c'(r_0) \underset{+\infty}{\sim} -\theta r_0 u(r_0). \tag{4.11}$$

This implies that $h_c'(r_0) < 0$. Thus $h_c(r)$ does not vanish for large r, which gives that $r^c u(r)$ is strictly monotone for large r. □

Theorem 4.2 *Let $u > 0$ be a solution of (P). Then $h_c(r) < 0$ for sufficiently large r, and $\lim\limits_{r\to+\infty} r^c u(r) = 0$.*

Proof: From Proposition 4.1, we know that for $c > 0$, the function $h_c(r)$ does not vanish for sufficiently large r. Assume that $h_c(r) > 0$ when r is large. Then, by using (4.7) together with the fact that $u'(r) < 0$, we deduce

$$|u'(r)| < \frac{cu(r)}{r} \quad \text{for large } r. \tag{4.12}$$

From (1.2), we derive

$$(|u'|^{p-2}u')'(r) < u\left[-\theta + q\lambda u(r)^{q-1} + c^{p-1}(N-1)\frac{u(r)^{p-2}}{r^p}\right]. \tag{4.13}$$

Since $\theta > 0$, $u(r) > 0$ and $\lim\limits_{r\to+\infty} u(r) = 0$, then $(|u'|^{p-2}u')'(r) < 0$ for large r. Given that $u'(r) < 0$, we deduce $\lim\limits_{r\to+\infty} u'(r) \in [-\infty, 0[$, which leads to a contradiction. Hence, $h_c(r) < 0$ for large r. By (4.8), this implies $\lim\limits_{r\to+\infty} r^c u(r) \in [0, +\infty[$. If $\lim\limits_{r\to+\infty} r^c u(r) = \ell > 0$, then for small enough $\varepsilon > 0$, one has $\lim\limits_{r\to+\infty} r^{c+\varepsilon} u(r) = +\infty$, which contradicts the fact that $h_{c+\varepsilon}(r) < 0$ for large r. We therefore get $\lim\limits_{r\to+\infty} r^c u(r) = 0$. □

5. Conclusion

By applying the Banach Fixed Point Theorem, we proved the existence and uniqueness of a global solution. Furthermore, by using a change of scale, we proved the existence of two types of solutions: a strictly positive solutions and a sign-changing solutions under specific conditions, and we derived several qualitative properties of the solution u. We also investigated the asymptotic behavior of the positive solution to problem (P) at infinity, and showed that for sufficiently large r, $\lim_{r \to +\infty} r^c u(r) = 0$, $\forall c > 0$.

These results naturally give rise to an open question: what is the exact asymptotic equivalent of the solution near infinity? This problem is still unresolved and will be addressed in future research.

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