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On Generalized Weakly (Ricci) ϕ-Symmetric (LCS)$_n$ Manifold

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ABSTRACT: The present paper introduce the notions of generalized weakly ϕ-symmetric and generalized weakly Ricci ϕ-symmetric (LCS)$_n$ manifold. We further investigate some applications of generalized weakly ϕ-symmetric $(CS)_4$-space time.

Keywords: Generalized weakly ϕ-symmetric, generalized weakly Ricci ϕ-symmetric, η-Einstein, Lorentzian Para Sasakian manifold.

Contents

1. Introduction

In our manuscript, we shall mark the Levi-Civita connection, curvature tensor, Ricci tensor and Ricci operator by the symbols ∇, R(or $\bar{R}$), S and Q respectively. As a weaker class of local symmetry [17], Takahashi [37] began the investigation on locally ϕ-symmetric manifold. Further exploration to weaken such notion has been caried out by many authors. For details, we refer to [4], [12], [19], [20], [23], [24], [25], [34], [35], [36] and the references therein.

A semi-Riemannian (or Riemannian) manifold of dimension n is said to be a generalized weakly ϕ-symmetric if it fit the equation

$$\begin{aligned} & \phi^2(\nabla_X R)(Y,U,V,Z) \\ = \; & A_1(X)R(Y,U,V,Z) + B_1(Y)R(X,U,V,Z) + B_1(U)R(Y,X,V,Z) \\ & + D_1(V)R(Y,U,X,Z) + D_1(Z)R(Y,U,V,X) + A_2(X)G(Y,U,V,Z) \\ & + B_2(Y)G(X,U,V,Z) + B_2(U)\, G(Y,X,V,Z) + D_2(V)\, G(Y,U,X,Z) \\ & + D_2(Z)G(Y,U,V,X) \end{aligned} \tag{1.1}$$

where

$$G(Y,U,V,\ W) = g(U,V)g(Y,W) - g(Y,\ V)g(U,W) \tag{1.2}$$

(ϕ being a $(1,1)$ tensor) for any vector fields X,Y,U and the 1-forms $A_i = g(,X_{A_i})$, $B_i = g(,X_{B_i})$ and $D_i = g(,X_{D_i})$ for $i = 1,2$.

The charm of generalized ϕ-weakly symmetric space is that it has the spice of

(i) locally ϕ-symmetric space [37] (for $X_{A_i} = X_{B_i} = X_{D_i} = 0$) $i = 1,2$,

(ii) locally ϕ-recurrent space [20] (for $X_{A_1} \neq 0$, $X_{A_2} = X_{B_i} = X_{D_i} = 0$) $i = 1,2$,

(iii) generalized ϕ-recurrent space in the sense of [21] (for $X_{A_i} \neq 0$, $X_{B_i} = X_{D_i} = 0$) $i = 1,2$,

(iv) quasi ϕ-recurrent space in the sense [29] (for $X_{A_i} \neq 0$, $X_{B_1} = X_{D_1} = 0$, $X_{B_2} = X_{D_2} = (\beta - \gamma)X_{A_2}$) $i = 1,2$

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(v) pseudo ϕ-symmetric space in the sense of [24] (for $\frac{1}{2}X_{A_1} = X_{B_1} = X_{D_1} \neq 0$, $X_{A_2} = X_{B_2} = X_{D_2} = 0$),

(vi) generalized pseudo ϕ-symmetric space in the sense of [5] (for $\frac{1}{2}X_{A_i} = X_{B_i} = X_{D_i} \neq 0$) $i = 1, 2$,

(vii) semi-pseudo ϕ-symmetric space in the sense of [39] ($X_{A_i} = X_{B_2} = X_{D_2} = 0, X_{B_1} = X_{D_1} \neq 0$),

(viii) generalized semi-pseudo ϕ-symmetric space in the sense of [8] ($X_{A_i} = 0, X_{B_i} = X_{D_i} \neq 0$),

(ix) almost pseudo ϕ-symmetric space in the sense of [18] (for $X_{A_1} = H_1 + K_1, X_{B_1} = X_{D_1} = H_1 \neq 0$ and $X_{A_2} = X_{B_2} = X_{D_2} = 0$),

(x) almost generalized pseudo ϕ-symmetric space in the sense of [8]) ($X_{A_i} = H_i + K_i, X_{B_i} = X_{D_i} = H_i \neq 0$), $i = 1, 2$,

(xi) weakly ϕ-symmetric space in the sense of [38] (for $X_{A_1}X_{B_1}X_{D_1} \neq 0$, $X_{A_2} = X_{B_2} = X_{D_2} = 0$).

Analogously, a semi-Riemannian (or Riemannian) manifold (M^n, g) is said to be generalized weakly Ricci ϕ-symmetric, if it satisfies the condition

$$\begin{aligned} \phi^2(\nabla_X Q)(U) &= A^*(X)QU + B^*(U)QX + S(U,X)\varrho^* \\ &\quad + \alpha^*(X)U + \beta^*(U)X + g(U,X)\sigma^* \end{aligned} \tag{1.3}$$

for any vector fields X, U and V and the 1-forms $A^* = g(,\pi_1^*)$, $\alpha^* = g(,\pi_2^*)$, $D^* = g(,\varrho^*)$, $B^* = g(,\delta_1^*)$, $\beta^* = g(,\delta_2^*)$ and $\gamma = g(,\sigma^*)$.

Einstein's equation in general relativity is given by:

$$S(X,Y) - (r/2)g(X,Y) + \lambda g(X,Y) = kT(X,Y) \tag{1.4}$$

for all vector fields X, Y, where λ is the cosmological constant, k is the gravitational constant and T is the energy momentum tensor of type (0,2).

We present our manuscript as follows: Section 2 is concerned with some basic results of an $(LCS)_n$-manifold. In section 3, we have investigated generalized weakly ϕ-symmetric $(LCS)_n$ manifold. It is found that such a manifolds may be considered as nearly η-Einstein (nearly quasi-Einstein) manifold. We observe that generalized weakly symmetric $(LCS)_n$ manifold is η-Einstein. Section 4 deal with generalized weakly Ricci ϕ-symmetric $(LCS)_n$ manifolds. We prove that each of (i) Ricci ϕ-symmetric, (ii) Ricci ϕ-recurrent, (iii) generalized Ricci ϕ-recurrent, (iv) pseudo Ricci ϕ-symmetric, (v) generalized pseudo Ricci ϕ-symmetric, (vi) semi-pseudo Ricci ϕ-symmetric, (vii) generalized semi-pseudo Ricci ϕ-symmetric, (viii) almost pseudo Ricci ϕ-symmetric, (ix) almost generalized pseudo Ricci ϕ-symmetric, (x) weakly Ricci ϕ-symmetric $(LCS)_n$ manifold is quasi-Einstein. Finally, we discuss some applications of generalized weakly ϕ-symmetric $(CS)_4$-spacetime.

2. Some Known results on $(LCS)_n$-manifold

Let $M^n(\phi, \eta, \xi, g)$ be an $(LCS)_n$-manifold. In an $(LCS)_n$-manifold, the following relations hold [1,3,6,14,15,16,31]:

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\} \qquad (\alpha \neq 0) \tag{2.1}$$

$$\nabla_X \alpha = (X\alpha) = \rho\eta(X), \tag{2.2}$$

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.3}$$

$$\phi X = X + \eta(X)\xi, \tag{2.4}$$

$$\begin{aligned} \phi \circ \xi &= 0, \ \ \eta(\xi) = -1, && (2.5) \\ \eta(\phi X) &= 0, \qquad g(\phi X, \phi Y) - g(X,Y) = \eta(X)\eta(Y), && (2.6) \\ \eta(R(X,Y)Z) &= (\alpha^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], && (2.7) \\ R(X,Y)\xi &= (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], && (2.8) \\ S(X,\xi) &= (n-1)(\alpha^2 - \rho)\eta(X), \quad Q\xi = (n-1)(\alpha^2 - \rho)\xi && (2.9) \end{aligned}$$

for any vector fields X, Y, Z.

It is to be noted that 4-dimensional Lorentzian concircular structure spacetime is termed as $(CS)_4$-spacetime([13], [33], [32]).

Definition 2.1. *An $(LCS)_n$ manifolds is said to nearly η-Einstein (nearly quasi-Einstein) manifold if Ricci tensor is of the form $S = a\eta \odot \eta + bg + E$ where a, b are scalar functions and E being a tensor of type $(0,2)$.*

3. Generalized weakly ϕ-symmetric $(LCS)_n$ manifold

In this section, we consider a generalized weakly ϕ-symmetric $(LCS)_n$ manifold. Using equation (2.4) in equation (1.1) we have

$$\begin{aligned}
& (\nabla_X R)(Y,U)V + \eta\left((\nabla_X R)(Y,U)V\right)\xi \\
= \; & A_1(X)R(Y,U)V + H_1(Y)R(X,U)V + H_1(U)R(Y,X)V \\
& +H_1(V)R(Y,U)X + g(R(Y,U)V,\ X)X_{H_1} + A_2(X)G(Y,U)V \\
& +H_2(Y)G(X,U)V + H_2(U)\ G(Y,X)V + H_2(V)\ G(Y,U)X \\
& \quad + g(G(Y,U)V,\ X)X_{H_2}
\end{aligned} \tag{3.1}$$

where $H_i = \frac{B_i + D_i}{2}$ for $i = 1, 2$. The foregoing equation can also be written as

$$\begin{aligned}
& g((\nabla_X R)(Y,U)V, W) + \eta((\nabla_X R)(Y,U)V)\eta(W) \\
= \; & A_1(X)g(R(Y,U)V,W) + H_1(Y)g(R(X,U)V,W) + H_1(U)g(R(Y,X)V,W) \\
& +H_1(V)g(R(Y,U)X,W) + H_1(W)g(R(Y,U)V,\ X) + H_2(X)g(G(Y,U)V,W) \\
& +H_2(Y)g(G(X,U)V,W) + H_2(U)\ g(G(Y,X)V,W) + H_2(V)g(G(Y,U)X,W) \\
& +H_2(W)\ g(G(Y,U)V,X\ ).
\end{aligned} \tag{3.2}$$

which yield

$$\begin{aligned}
& (\nabla_X S)(U,V) + \eta((\nabla_X R)(\xi,U))(V) \\
= \; & A_1(X)S(U,V) + H_1(R(X,U)V) + H_1(U)S(X,V) + H_1(V)S(U,X) \\
& +H_1(R(X,V)U) + (n-1)H_2(X)g(U,V) + H_2(G(X,U)V) \\
& +(n-1)H_2(U)g(X,V) + (n-1)H_2(V)g(U,X) + H_2(G(X,V)U)
\end{aligned} \tag{3.3}$$

after contraction. Making use of (2.9), (2.3) and (2.6), we infer

$$\eta((\nabla_X R)(\xi,U))(V) = -(2\alpha\rho - \beta)\eta(X)\left\{g(U,V) + \eta(U)\eta(V)\right\}. \tag{3.4}$$

In view of (3.3) and (3.4), we get

$$\begin{aligned}
& (\nabla_X S)(U,V) \\
= \; & (2\alpha\rho - \beta)\eta(X)\left\{g(U,V) + \eta(U)\eta(V)\right\} + A_1(X)S(U,V) \\
& +H_1(R(X,U)V) + H_1(U)S(X,V) + H_1(V)S(U,X) \\
& +H_1(R(X,V)U) + (n+1)H_2(X)g(U,V) \\
& +(n-3)H_2(U)g(X,V) + (n-1)H_2(V)g(U,X).
\end{aligned} \tag{3.5}$$

This motivate us to state the following.

Theorem 3.1. *A generalized weakly ϕ-symmetric $(LCS)_n$-manifold reduces to $(GWRS)_n$-manifold if*

$$H_1(U) = -\ 3\eta(U)H_1(\xi). \tag{3.6}$$

Again from (2.8), we have

$$\begin{aligned}
(\nabla_X R)(Y,U)\xi \;&=\; \nabla_X R(Y,U)\xi - R(\nabla_X Y,U)\xi - R(Y,\nabla_X U)\xi - R(Y,U)\nabla_X \xi \\
&=\; -\left[(2\alpha\rho - \beta - \alpha) + (\alpha^2 - \rho)\right]\eta(X)\left\{\eta(Y)U - \eta(U)Y\right\} \\
&\quad\; +(\alpha^2 - \rho)\left\{g(X,U)Y - g(X,Y)U\right\} - \alpha R(Y,U)X.
\end{aligned} \tag{3.7}$$

Now, setting $V=\xi$ in (3.1) and using (3.7), we find

$$\begin{aligned}
& [\alpha+H_1(\xi)]\tilde{R}(Y,U,W,X) \\
= & \left[(2\alpha\rho-\beta-\alpha)+(\alpha^2-\rho)\right]\{\eta(X)\eta(Y)g(U,W)-\eta(X)\eta(U)g(Y,W)\} \\
& +\left[H_2(\xi)-(\alpha^2-\rho)\right]\{g(X,U)g(Y,W)-g(X,Y)g(U,W)\} \\
& +[A_1(X)(\alpha^2-\rho)+A_2(X)]\{\eta(U)g(Y,W)-\eta(Y)g(U,W)\} \\
& +\left[H_2(Y)+H_1(Y)(\alpha^2-\rho)\right]\{\eta(U)g(X,W)-\eta(X)g(U,W)\} \\
& +\left[H_2(U)+H_1(U)(\alpha^2-\rho)\right]\{\eta(X)g(Y,W)-\eta(Y)g(X,W)\} \\
& +\left[\{(\alpha^2-\rho)-\alpha\}\eta(W)+(\alpha^2-\rho)H_1(W)+H_2(W)\right]\{g(X,Y)\eta U) \\
& -g(X,U)\eta(Y)\} \qquad (3.8)
\end{aligned}$$

which gives

$$\begin{aligned}
& [\alpha+H_1(\xi)]S(X,Y) \\
= & [(n-1)(2\alpha\rho-\beta)-(n-2)\alpha+(n-2)(\alpha^2-\rho)]\eta(X)\eta(Y) \\
& +[(n-2)(\alpha^2-\rho)+\alpha+(\alpha^2-\rho)H_1(\xi)-(n-2)H_2(\xi)]g(X,Y) \\
& +E(X,Y) \qquad (3.9)
\end{aligned}$$

after contraction, where

$$\begin{aligned}
E(X,Y) = & (1-n)(\alpha^2-\rho)A_1(X)\eta(Y)+(1-n)A_2(X)\eta(Y) \\
& +(2-n)H_2(Y)\eta(X)+(2-n)(\alpha^2-\rho)H_1(Y)\eta(X) \\
& -2H_2(X)\eta(Y)-2(\alpha^2-\rho)H_1(X)\eta(Y) \qquad (3.10)
\end{aligned}$$

Thus we can state that

Theorem 3.2. *A generalized weakly ϕ-symmetric $(LCS)_n$ manifolds may be considered as nearly η-Einstein (nearly quasi-Einstein) manifold provided algebraic equation involving 1-form $\alpha+H_1(\xi)\neq 0$.*

Remark 1. *We note that $\alpha+H_1(\xi)=0$ gives the relation between the 1-form.*

4. Generalized weakly Ricci ϕ-symmetric $(LCS)_n$ manifolds

In this section we consider a generalized weakly Ricci ϕ-symmetric $(LCS)_n$ manifolds. Then by the virtue of (2.5) and (1.3) we have

$$\begin{aligned}
& (\nabla_X Q)(U)+\eta((\nabla_X Q)(U))\xi \\
= & A^*(X)QU+B^*(U)QX+S(U,X)\varrho^* \\
& +\alpha^*(X)U+\beta^*(U)X+g(U,X)\sigma^*. \qquad (4.1)
\end{aligned}$$

Taking inner product with V on both side, we get

$$\begin{aligned}
& g(\nabla_X Q(U),V)-S(\nabla_X U,V)+\eta((\nabla_X Q)(U))\eta(V) \\
= & A^*(X)S(U,V)+B^*(U)S(V,X)+D^*(V)S(U,X) \\
& +\alpha^*(X)g(U,V)+\beta^*(U)g(V,X)+\gamma^*(V)g(U,X). \qquad (4.2)
\end{aligned}$$

Putting $U=\xi$ in (4.2) and using (2.3),(2.6) we get

$$\begin{aligned}
& (n-1)\alpha(\alpha^2-\rho)g(X,V)-\alpha S(X,V) \\
= & (n-1)(\alpha^2-\rho)A^*(X)\eta(V)+(n-1)(\alpha^2-\rho)D^*(V)\eta(X) \\
& +\alpha^*(X)\eta(V)+\beta^*(\xi)g(V,X)+B^*(\xi)S(V,X)+\gamma^*(V)\eta(X). \qquad (4.3)
\end{aligned}$$

Again, setting $X = V = \xi$, $X = \xi$ and $V = \xi$ successively in (4.3), we get

$$\begin{aligned} & (n-1)(\alpha^2-\rho)\{A^*(\xi)+B^*(\xi)+D^*(\xi)\} \\ = \; & -\{\alpha^*(\xi)+\beta^*(\xi)+\gamma^*(\xi)\}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} & (n-1)(\alpha^2-\rho)D^*(V)+\gamma^*(V) \\ = \; & [(n-1)(\alpha^2-\rho)\{A^*(\xi)+B^*(\xi)\}+\alpha^*(\xi)+\beta^*(\xi)]\eta(V), \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} & (n-1)(\alpha^2-\rho)A^*(X)+\alpha^*(X) \\ = \; & [(n-1)(\alpha^2-\rho)\{B^*(\xi)+D^*(\xi)\}+\beta^*(\xi)+\gamma^*(\xi)]\eta(X). \end{aligned} \tag{4.6}$$

respectively. Using (4.4), (4.5) and (4.6) in (4.3), we get

$$\begin{aligned} & S(X,V) \\ = \; & \left(\frac{(n-1)\alpha(\alpha^2-\rho)-\beta^*(\xi)}{\alpha+B^*(\xi)}\right) g(V,X) \\ & -\left(\frac{(n-1)\alpha(\alpha^2-\rho)+\beta^*(\xi)}{\alpha+B^*(\xi)}\right)\eta(X)\eta(V). \end{aligned} \tag{4.7}$$

$$[\alpha+B^*(\xi)]r = (n-1)[(n+1)\alpha(\alpha^2-\rho)-\beta^*(\xi)]$$

This leads to the following:

Theorem 4.1. *Every generalized weakly Ricci ϕ-symmetric $(LCS)_n$ manifold is quasi-Einstein provided that $B_1(\xi) \neq -\alpha$.*

Corollary 4.1. *Each of (i) Ricci ϕ-symmetric, (ii) Ricci ϕ-recurrent, (iii) generalized Ricci ϕ-recurrent, (iv) pseudo Ricci ϕ-symmetric, (v) generalized pseudo Ricci ϕ-symmetric, (vi) semi-pseudo Ricci ϕ-symmetric, (vii) generalized semi-pseudo Ricci ϕ-symmetric, (viii) almost pseudo Ricci ϕ-symmetric, (ix) almost generalized pseudo Ricci ϕ-symmetric, (x) weakly Ricci ϕ-symmetric $(LCS)_n$ manifold is quasi-Einstein.*

5. Generalized weakly (Ricci) ϕ-symmetric $(CS)_4$-spacetime

Definition 5.1. *[30] The Ricci tensor S of spacetime is said to a timelike convergence condition if it admits the following*

$$S(U,U) > 0, \; U \text{ being timelike vector field.}$$

Proposition 5.1. *The timelike vector field ξ of a generalized weakly Ricci ϕ-symmetric $(CS)_4$-spacetime possesses convergence condition if $\alpha(\alpha^2-\rho) < 0$.*

In view of (1.4) and (4.7), we have

$$\begin{aligned} & kT(X,Z) \\ = \; & \left(\lambda - \frac{r}{2} + \frac{(n-1)\alpha(\alpha^2-\rho)-\beta^*(\xi)}{\alpha+B^*(\xi)}\right) g(V,X) \\ & -\left(\frac{(n-1)\alpha(\alpha^2-\rho)+\beta^*(\xi)}{\alpha+B^*(\xi)}\right)\eta(X)\eta(V). \end{aligned} \tag{5.1}$$

Now, for the choice of the vector field ξ to be Killing, we have

$$(\pounds_\xi g)(X,Z) = 0. \tag{5.2}$$

The equation (5.2) implies that $\alpha = 0$ and hence $\rho = 0$ (see Theorem 3.7, page 419 [13]). Consequently, $\alpha^2 - \rho = 0$ and hence equation (4.7) yields $r = 0$ provided $B_1(\xi) \neq 0$, i.e., ξ is not orthogonal to ρ_1. Consequently, (5.1) yields

$$(\pounds_\xi T)(X,Z) = 0.$$

Definition 5.2. *A spacetime M is said to admit a matter collineation, if the Lie derivative of the energy momentum tensor with respect to the characteristic vector field ξ vanishes identically, that is,*

$$(\pounds_\xi T)(X,Y) = 0 \text{ for any } X, Y \in \chi(M). \tag{5.3}$$

Thus we can state the following:

Theorem 5.1. *If the characteristic vector field ξ of a generalized weakly Ricci ϕ-symmetric $(CS)_4$-spacetime with Einstein equation and $B_1(\xi) \neq 0$ is Killing, then it admits matter collineation.*

Again suppose that α is constant. Then $\alpha^2 - \rho =$ constant and hence it follows from equation (5.1) that r is constant. Consequently, (5.3) yields

$$k(\pounds_\xi T)(X,Z) = \left[\lambda - \frac{r}{2} + \frac{(n-1)\alpha(\alpha^2-\rho) - \beta^*(\xi)}{\alpha + B^*(\xi)}\right](\pounds_\xi g)(X,Z). \tag{5.4}$$

Again, if (5.3) holds then (5.4) implies that

$$(\pounds_\xi g)(X,Z) = 0 \text{ as } r \neq 2\left[\lambda + \left(\frac{(n-1)\alpha(\alpha^2-\rho) - \beta^*(\xi)}{\alpha + B_1(\xi)}\right)\right] \text{ by (3.6)}.$$

Therefore ξ is a Killing vector field and hence by previous argument the spacetime is of vanishing scalar curvature. Thus we can state the following:

Theorem 5.2. *If a generalized weakly Ricci ϕ-symmetric $(CS)_4$-spacetime with Einstein equation and $B_1(\xi) \neq 0$ admits a matter collineation, then the characteristic vector field ξ of the spacetime is a Killing vector field and the spacetime is of vanishing scalar curvature.*

Combining Theorem 3.7 and Theorem 3.8, we can state the following:

Theorem 5.3. *If a generalized weakly Ricci-symmetric $(CS)_4$-spacetime with $B_1(\xi) \neq 0$ satisfies Einstein equation, then the characteristic vector field ξ of the spacetime is a Killing vector field if and only if it admits matter collineation.*

Definition 5.3. *A spacetime M is said to admit a curvature collineation ([22]) if the Lie derivative of the curvature tensor with respect to the characteristic vector field ξ vanishes identically, that is,*

$$(\pounds_\xi R)(X,Y)Z = 0.$$

If ξ is a Killing vector field then (5.2) holds, which gives after covariant differentiation

$$(\nabla_X \pounds_\xi g)(Y,Z) = 0. \tag{5.5}$$

By Yano [40], we also have

$$(\pounds_\xi \nabla_X g - \nabla_X \pounds_\xi g - \nabla_{[\xi,X]} g)(Y,Z) = -g((\pounds_\xi \nabla)(X,Y),Z) - g((\pounds_\xi \nabla)(X,Z),Y). \tag{5.6}$$

In view of of the parallelism of the Lorentzian metric g, it follows from the above relation that

$$(\nabla_X \pounds_\xi g)(Y,Z) = g((\pounds_\xi \nabla)(X,Y),Z) + g((\pounds_\xi \nabla)(X,Z),Y). \tag{5.7}$$

Because

$$(\pounds_\xi \nabla)(X,Y) = (\pounds_\xi \nabla)(Y,X), \tag{5.8}$$

it follows from (5.7) that

$$2g((\pounds_\xi \nabla)(X,Y),Z) = (\nabla_X \pounds_\xi g)(Y,Z) + (\nabla_Y \pounds_\xi g)(Z,X) - (\nabla_Z \pounds_\xi g)(X,Y). \tag{5.9}$$

Making use of (5.5) and (5.9), we have

$$(\pounds_\xi \nabla)(X,Y) = 0. \tag{5.10}$$

Taking the covariant derivative of the above equation along an arbitrary vector field we get

$$(\nabla_X \pounds_\xi \nabla)(Y,Z) = 0. \tag{5.11}$$

Next, by using the above equation in the following formula (see Yano [40]])

$$(\pounds_\xi R)(X,Y)Z = (\nabla_X \pounds_\xi \nabla)(Y,Z) - (\nabla_Y \pounds_\xi \nabla)(X,Z), \tag{5.12}$$

we obtain

$$(\pounds_\xi R)(X,Y)Z = 0 \tag{5.13}$$

for any X, Y, $Z \in \chi(M)$. Contracting X in (5.13), we get

$$(\pounds_\xi S)(Y,Z) = 0. \tag{5.14}$$

Using (5.13) and (5.14) in (5.12), we have

$$(\pounds_\xi \omega)(X,Y,Z) = 0 \tag{5.15}$$

where ω is well known quasi conformal like curvature tensor([10], [11], [9], [7]). This leads to the following:

Theorem 5.4. *If the characteristic vector field ξ of a generalized weakly Ricci ϕ-symmetric $(CS)_4$-spacetime with $B_1(\xi) \neq 0$ obeying Einstein equation is a Killing vector field, then such spacetime admits (i) curvature collineation, (ii) conformal collineation, (iii) conharmonic collineation, (iv) concircular collineation, (v) projective collineation, (vi) m-projective collineation.*

Again, if the vector field ξ is a conformal Killing, then

$$(\pounds_\xi g)(X,Z) = 2\mu g(X,Z) \tag{5.16}$$

where μ is a scalar function. In view of (5.2), (5.4) and (5.16), we find

$$(\pounds_\xi T)(X,Z) = 2\mu T(X,Z) \tag{5.17}$$

provided α is constant and $3(\alpha^2 - \rho)B_1(\xi) + B_2(\xi) = 0$. Thus we can state that

Theorem 5.5. *If the characteristic vector field ξ of a generalized weakly Ricci ϕ-symmetric $(CS)_4$-spacetime with $B_1(\xi) \neq 0$ obeying Einstein equation is conformal Killing, then the energy momentum tensor is also conformal Killing.*

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