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A New Theorem Generalization on Absolute Matrix Summability of Fourier Series

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ABSTRACT: We generalize a main theorem dealing with weighted mean summability of Fourier series to the $|A,\varphi_n;\delta|_k$ summability factors of Fourier series. Also, some new and known results are obtained.

Key Words: Weighted mean, matrix summability, summability factor, infinite series, Fourier series, Hölder inequality, Minkowski inequality, sequence space.

Contents

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). By u_n^α and t_n^α we denote the nth Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (na_n), respectively, that is (see [7])

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{j=0}^{n} A_{n-j}^{\alpha-1} s_j \quad \text{and} \quad t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{j=0}^{n} A_{n-j}^{\alpha-1} j a_j,$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0.$$

The series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geq 1$, if (see [9], [11])

$$\sum_{n=1}^{\infty} n^{k-1}\left|u_n^\alpha - u_{n-1}^\alpha\right|^k = \sum_{n=1}^{\infty}\frac{1}{n}\left|t_n^\alpha\right|^k < \infty.$$

If we take $\alpha = 1$, then $|C,\alpha|_k$ summability reduces to $|C,1|_k$ summability.

Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{j=0}^{n} p_j \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{j=0}^{n} p_j s_j$$

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defines the sequence (t_n) of the weighted mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [10]).

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty.$$

In the special case when $p_n = 1$ for all values of n, $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ summability.

Let $A = (a_{nj})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{j=0}^{n} a_{nj} s_j \quad (n = 0, 1, \ldots).$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if (see [18])

$$\sum_{n=1}^{\infty} n^{k-1} |\bar{\Delta} A_n(s)|^k < \infty,$$

and it is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [17])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\bar{\Delta} A_n(s)|^k < \infty,$$

and it is said to be summable $|A, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [13])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |\bar{\Delta} A_n(s)|^k < \infty,$$

and also, (φ_n) be a sequence of positive numbers. Then $\sum a_n$ is said to be summable $|A, \varphi_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [12], [16])

$$\sum_{n=1}^{\infty} (\varphi_n)^{\delta k + k - 1} |\bar{\Delta} A_n(s)|^k < \infty,$$

where $\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s)$.

In the special case for $\varphi_n = \frac{P_n}{p_n}$, the $|A, \varphi_n; \delta|_k$ summability reduces to $|A, p_n; \delta|_k$ summability. If we take $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, then $|A, \varphi_n; \delta|_k$ summability reduces to $|A, p_n|_k$ summability. If we take $\delta = 0$ and $\varphi_n = n$, then $|A, \varphi_n; \delta|_k$ summability reduces to $|A|_k$ summability. If we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nj} = \frac{p_j}{P_n}$, then $|A, \varphi_n; \delta|_k$ summability reduces to $|\bar{N}, p_n; \delta|_k$ summability (see [3]). Also if we take $\delta = 0$, $\varphi_n = \frac{P_n}{p_n}$ and $a_{nj} = \frac{p_j}{P_n}$, then $|A, \varphi_n; \delta|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability.

2. Known Result

The following theorem is dealing with $|\bar{N}, p_n|_k$ summability factors of infinite series.

Theorem 2.1 ([5]) *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \quad as \quad n \to \infty. \tag{2.1}$$

Let (X_n) be a positive monotonic nondecreasing sequence. If the sequences (X_n), (λ_n) and (p_n) satisfy the conditions

$$\lambda_m X_m = O(1) \quad as \quad m \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{m} nX_n |\Delta^2 \lambda_n| = O(1) \quad as \quad m \to \infty, \tag{2.3}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{2.4}$$

then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

For any sequence (λ_n), we write $\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1}$ and $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$. A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta \lambda_n| < \infty$. Let $f(t)$ be a periodic function with period 2π and Lebesgue integrable over $(-\pi, \pi)$. Write

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} C_n(x),$$

$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$ and $\phi_\alpha(t) = \frac{\alpha}{t^\alpha} \int_0^t (t-u)^{\alpha-1} \phi(u) du \quad (\alpha > 0)$.

It is well known that if $\phi(t) \in \mathcal{BV}(0, \pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the $(C, 1)$ mean of the sequence $(nC_n(x))$ (see [8]).

Many works have been done dealing with absolute summability factors of Fourier series (see [4]- [6], [14], [15], [20], [21]). Among them, in [5], Bor has proved the following theorem.

Theorem 2.2 ([5]) *If $\phi_1(t) \in \mathcal{BV}(0, \pi)$, (X_n) is a positive monotonic nondecreasing sequence. The sequence (p_n), (λ_n) satisfy conditions (2.1) – (2.3) and*

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n(x)|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{2.5}$$

then the series $\sum C_n(x) \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

The above theorem were extended for $|A, p_n; \delta|_k$ summability in the following:

Theorem 2.3 ([22]) *Let $A = (a_{nj})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \quad n = 0, 1, \ldots,$$

$$a_{n-1,j} \geq a_{nj} \quad for \quad n \geq j+1,$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right),$$

$$1 = O(n a_{nn}),$$

$$\sum_{j=1}^{n-1} |\hat{a}_{n,j+1}| a_{jj} = O(a_{nn}) \quad as \quad n \to \infty,$$

$$\sum_{n=j+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_j(\hat{a}_{nj})| = O\left(a_{jj} \left(\frac{P_j}{p_j}\right)^{\delta k}\right) \quad as \quad m \to \infty,$$

$$\sum_{n=j+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\hat{a}_{n,j+1}| = O\left(\left(\frac{P_j}{p_j}\right)^{\delta k}\right) \quad as \quad m \to \infty.$$

Suppose that $\phi_1(t) \in \mathcal{BV}(0,\pi)$ *and* (X_n) *be a positive monotonic nondecreasing sequence. If the sequences* (p_n) *and* (λ_n) *satisfy the conditions* $(2.2)-(2.3)$ *of Theorem 2.1 and also the condition*

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k} a_{nn}\frac{|t_n(x)|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m\to\infty$$

holds, then the series $\sum C_n(x)\lambda_n$ *is summable* $|A,p_n;\delta|_k$, $k\geq 1$ *and* $0\leq\delta\leq\frac{1}{k}$.

3. Main Results

We generalize Theorem 2.3 for $|A,\varphi_n;\delta|_k$ summability. Before stating the main theorem, we must first introduce some further notations.

With a normal matrix $A=(a_{nj})$, we associate two lower semimatrices $\bar{A}=(\bar{a}_{nj})$ and $\hat{A}=(\hat{a}_{nj})$ as follows:

$$\bar{a}_{nj}=\sum_{i=j}^{n}a_{ni}, \quad n,j=0,1,\ldots,$$
$$\hat{a}_{00}=\bar{a}_{00}=a_{00}, \quad \hat{a}_{nj}=\bar{a}_{nj}-\bar{a}_{n-1,j}, \quad n=1,2,\ldots.$$

We note that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. So, we have

$$A_n(s)=\sum_{j=0}^{n}a_{nj}s_j=\sum_{j=0}^{n}\bar{a}_{nj}a_j \quad \text{and} \quad \bar{\Delta}A_n(s)=\sum_{j=0}^{n}\hat{a}_{nj}a_j. \tag{3.1}$$

Theorem 3.1 *Let* (φ_n) *be a sequence of positive numbers and* $A=(a_{nj})$ *be a positive normal matrix such that*

$$\bar{a}_{n0}=1, \quad n=0,1,\ldots, \tag{3.2}$$
$$a_{n-1,j}\geq a_{nj} \quad for \quad n\geq j+1, \tag{3.3}$$
$$\varphi_n a_{nn}=O(1), \tag{3.4}$$
$$1=O(na_{nn}), \tag{3.5}$$
$$\sum_{j=1}^{n-1}|\hat{a}_{n,j+1}|a_{jj}=O(a_{nn}) \quad as \quad n\to\infty, \tag{3.6}$$
$$\sum_{n=j+1}^{m+1}(\varphi_n)^{\delta k}|\Delta_j(\hat{a}_{nj})|=O\left(a_{jj}\,(\varphi_j)^{\delta k}\right) \quad as \quad m\to\infty, \tag{3.7}$$
$$\sum_{n=j+1}^{m+1}(\varphi_n)^{\delta k}|\hat{a}_{n,j+1}|=O\left((\varphi_j)^{\delta k}\right) \quad as \quad m\to\infty. \tag{3.8}$$

Suppose that $\phi_1(t)\in\mathcal{BV}(0,\pi)$ *and* (X_n) *be a positive monotonic nondecreasing sequence. If the sequences* (p_n) *and* (λ_n) *satisfy the conditions* $(2.2)-(2.3)$ *of Theorem 2.1 and also the condition*

$$\sum_{n=1}^{m}(\varphi_n)^{\delta k}a_{nn}\frac{|t_n(x)|^k}{X_n^{k-1}}=O(X_m) \quad as \quad m\to\infty \tag{3.9}$$

are satisfied, then the series $\sum C_n(x)\lambda_n$ *is summable* $|A,\varphi_n;\delta|_k$, $k\geq 1$ *and* $0\leq\delta<\frac{1}{k}$.

Theorem 3.2 *Let* $\phi_1(t)\in\mathcal{BV}(0,\pi)$, (φ_n) *be a sequence of positive numbers and* (X_n) *be a positive monotonic nondecreasing sequence, the sequences* (p_n) *and* (λ_n) *satisfy the conditions* $(2.1)-(2.3)$ *of*

Theorem 2.1. Let $A = (a_{nj})$ *be a positive normal matrix by satisfying the conditions* (3.2) − (3.4), (3.7) *and* (3.9) *as in Theorem 3.1. If we replaced the condition* (3.5) − (3.6) *of Theorem 3.1 with*

$$|\hat{a}_{n,j+1}| = O(j|\Delta_j(\hat{a}_{nj})|),$$

then the series $\sum C_n(x)\lambda_n$ *is summable* $|A, \varphi_n; \delta|_k$, $k \geq 1$ *and* $0 \leq \delta < \frac{1}{k}$.

If we take $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, then we have a theorem on $|A, p_n|_k$ summability (see [19]). We need the following lemma for the proof of our theorem.

Lemma 3.1 ([2]) *Under the condition of Theorem 2.1 we have*

$$nX_n|\Delta\lambda_n| = O(1) \quad as \quad n \to \infty, \quad and \quad \sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty.$$

4. Proof of Theorem 3.1

Let $(I_n(x))$ denote the A-transform of the series $\sum_{n=1}^{\infty} C_n(x)\lambda_n$. Then, by (3.1), we have

$$\bar{\Delta} I_n(x) = \sum_{j=1}^{n} \hat{a}_{nj} C_j(x)\lambda_j.$$

Applying Abel's transformation to this sum, we have that

$$\begin{aligned}
\bar{\Delta} I_n(x) &= \sum_{j=1}^{n} \hat{a}_{nj} C_j(x)\lambda_j \frac{j}{j} = \sum_{j=1}^{n-1} \Delta_j\left(\frac{\hat{a}_{nj}\lambda_j}{j}\right) \sum_{r=1}^{j} rC_r(x) + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{r=1}^{n} rC_r(x) \\
&= \sum_{j=1}^{n-1} \Delta_j\left(\frac{\hat{a}_{nj}\lambda_j}{j}\right)(j+1)t_j(x) + \hat{a}_{nn}\lambda_n \frac{n+1}{n} t_n(x) \\
&= \sum_{j=1}^{n-1} \Delta_j(\hat{a}_{nj})\lambda_j t_j(x) \frac{j+1}{j} + \sum_{j=1}^{n-1} \hat{a}_{n,j+1}\Delta\lambda_j t_j(x) \frac{j+1}{j} + \sum_{j=1}^{n-1} \hat{a}_{n,j+1}\lambda_{j+1} \frac{t_j(x)}{j} + a_{nn}\lambda_n t_n(x) \frac{n+1}{n} \\
&= I_{n,1}(x) + I_{n,2}(x) + I_{n,3}(x) + I_{n,4}(x).
\end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} (\varphi_n)^{\delta k + k - 1} |I_{n,r}(x)|^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4.$$

Firstly, by applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, and by using $\sum_{j=1}^{n-1} |\Delta_j(\hat{a}_{nj})| \leq a_{nn}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} (\varphi_n)^{\delta k + k - 1} |I_{n,1}(x)|^k &\leq \sum_{n=2}^{m+1} (\varphi_n)^{\delta k + k - 1} \left\{ \sum_{j=1}^{n-1} \left|\frac{j+1}{j}\right| |\Delta_j(\hat{a}_{nj})| \, |\lambda_j||t_j(x)| \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} (\varphi_n)^{\delta k + k - 1} \sum_{j=1}^{n-1} |\Delta_j(\hat{a}_{nj})| \, |\lambda_j|^k |t_j(x)|^k \times \left\{ \sum_{j=1}^{n-1} |\Delta_j(\hat{a}_{nj})| \right\}^{k-1}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m+1}(\varphi_n)^{\delta k+k-1}\, a_{nn}^{k-1}\left\{\sum_{j=1}^{n-1}|\Delta_j(\hat{a}_{nj})|\,|\lambda_j|^k|t_j(x)|^k\right\}\\
&= O(1)\sum_{n=2}^{m+1}(\varphi_n)^{\delta k}\sum_{j=1}^{n-1}|\Delta_j(\hat{a}_{nj})|\,|\lambda_j|^k|t_j(x)|^k\\
&= O(1)\sum_{j=1}^{m}|\lambda_j|^{k-1}|\lambda_j||t_j(x)|^k\sum_{n=j+1}^{m+1}(\varphi_n)^{\delta k}\,|\Delta_j(\hat{a}_{nj})|\\
&= O(1)\sum_{j=1}^{m}|\lambda_j|^{k-1}|\lambda_j||t_j(x)|^k a_{jj}\,(\varphi_j)^{\delta k}\\
&= O(1)\sum_{j=1}^{m}|\lambda_j|\frac{|t_j(x)|^k}{X_j^{k-1}}\,(\varphi_j)^{\delta k}\,a_{jj}\\
&= O(1)\sum_{j=1}^{m-1}\Delta|\lambda_j|\sum_{r=1}^{j}(\varphi_r)^{\delta k}\,a_{rr}\frac{|t_r(x)|^k}{X_r^{k-1}}+O(1)|\lambda_m|\sum_{j=1}^{m}(\varphi_j)^{\delta k}\,a_{jj}\frac{|t_j(x)|^k}{X_j^{k-1}}\\
&= O(1)\sum_{j=1}^{m-1}|\Delta\lambda_j|X_j+O(1)|\lambda_m|X_m\\
&= O(1)\quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.1. Now, by using the condition (3.5) – (3.6) and Hölder's inequality, we have that

$$
\begin{aligned}
&\sum_{n=2}^{m+1}(\varphi_n)^{\delta k+k-1}\,|I_{n,2}(x)|^k \le \sum_{n=2}^{m+1}(\varphi_n)^{\delta k+k-1}\left\{\sum_{j=1}^{n-1}\left|\frac{j+1}{j}\right|\,|\hat{a}_{n,j+1}|\,|\Delta\lambda_j||t_j(x)|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}(\varphi_n)^{\delta k+k-1}\left\{\sum_{j=1}^{n-1}|\hat{a}_{n,j+1}|\,|\Delta\lambda_j||t_j(x)|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}(\varphi_n)^{\delta k+k-1}\sum_{j=1}^{n-1}|\hat{a}_{n,j+1}|\,|\Delta\lambda_j|^k|t_j(x)|^k a_{jj}^{1-k}\times\left\{\sum_{j=1}^{n-1}|\hat{a}_{n,j+1}|\,a_{jj}\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}(\varphi_n)^{\delta k+k-1}\,a_{nn}^{k-1}\left\{\sum_{j=1}^{n-1}|\hat{a}_{n,j+1}|\,|\Delta\lambda_j|^k|t_j(x)|^k a_{jj}^{1-k}\right\}\\
&= O(1)\sum_{n=2}^{m+1}(\varphi_n)^{\delta k}\sum_{j=1}^{n-1}|\hat{a}_{n,j+1}|\,|\Delta\lambda_j|^k|t_j(x)|^k\;a_{jj}\,j^k\\
&= O(1)\sum_{j=1}^{m}(j|\Delta\lambda_j|)^{k-1}\,(j|\Delta\lambda_j|)\,|t_j(x)|^k a_{jj}\sum_{n=j+1}^{m+1}|\hat{a}_{n,j+1}|\,(\varphi_n)^{\delta k}
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\sum_{j=1}^{m}\left(j|\Delta\lambda_j|\right)^{k-1}\left(j|\Delta\lambda_j|\right)|t_j(x)|^k a_{jj}\left(\varphi_j\right)^{\delta k}\\
&= O(1)\sum_{j=1}^{m}\left(j|\Delta\lambda_j|\right)\frac{|t_j(x)|^k}{X_j^{k-1}}\left(\varphi_j\right)^{\delta k} a_{jj}\\
&= O(1)\sum_{j=1}^{m-1}\Delta(j|\Delta\lambda_j|)\sum_{r=1}^{j}\left(\varphi_r\right)^{\delta k} a_{rr}\frac{|t_r(x)|^k}{X_r^{k-1}} + O(1)m|\Delta\lambda_m|\sum_{j=1}^{m}\left(\varphi_j\right)^{\delta k} a_{jj}\frac{|t_j(x)|^k}{X_j^{k-1}}\\
&= O(1)\sum_{j=1}^{m-1}|\Delta(j|\Delta\lambda_j|)|X_j + O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\sum_{j=1}^{m-1} jX_j|\Delta^2\lambda_j| + O(1)\sum_{j=1}^{m-1}X_j|\Delta\lambda_j| + O(1)m|\Delta\lambda_m|X_m\\
&= O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.1. Again, by using the condition (3.5) – (3.6), we have that

$$
\begin{aligned}
&\sum_{n=2}^{m+1}\left(\varphi_n\right)^{\delta k+k-1}|I_{n,3}(x)|^k \le \sum_{n=2}^{m+1}\left(\varphi_n\right)^{\delta k+k-1}\sum_{j=1}^{n-1}\left|\hat{a}_{n,j+1}\lambda_{j+1}\frac{t_j(x)}{j}\right|^k\\
&\le \sum_{n=2}^{m+1}\left(\varphi_n\right)^{\delta k+k-1}\left\{\sum_{j=1}^{n-1}|\hat{a}_{n,j+1}|\,|\lambda_{j+1}|\frac{|t_j(x)|}{j}\right\}^k\\
&\le \sum_{n=2}^{m+1}\left(\varphi_n\right)^{\delta k+k-1}\sum_{j=1}^{n-1}|\hat{a}_{n,j+1}|\,|\lambda_{j+1}|^k\frac{|t_j(x)|^k}{j^k}a_{jj}^{1-k}\times\left\{\sum_{j=1}^{n-1}|\hat{a}_{n,j+1}|\,a_{jj}\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\varphi_n\right)^{\delta k+k-1}a_{nn}^{k-1}\sum_{j=1}^{n-1}|\hat{a}_{n,j+1}|\,|\lambda_{j+1}|^k|t_j(x)|^k a_{jj}^{1-k}j^{-k}\\
&= O(1)\sum_{n=2}^{m+1}\left(\varphi_n\right)^{\delta k}\sum_{j=1}^{n-1}|\hat{a}_{n,j+1}|\,|\lambda_{j+1}|^k|t_j(x)|^k a_{jj}\\
&= O(1)\sum_{j=1}^{m}|\lambda_{j+1}|^{k-1}|\lambda_{j+1}||t_j(x)|^k a_{jj}\sum_{n=j+1}^{m+1}\left(\varphi_n\right)^{\delta k}|\hat{a}_{n,j+1}|\\
&= O(1)\sum_{j=1}^{m}|\lambda_{j+1}|\frac{|t_j(x)|^k}{X_j^{k-1}}\left(\varphi_j\right)^{\delta k}a_{jj} = O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.1. Finally, as in $I_{n,1}$, we have that

$$
\begin{aligned}
\sum_{n=1}^{m}|I_{n,4}(x)|^k &= O(1)\sum_{n=1}^{m}\left(\varphi_n\right)^{\delta k+k-1}\left(a_{nn}\right)^k|\lambda_n|^k|t_n(x)|^k\\
&= O(1)\sum_{n=1}^{m}\left(\varphi_n\right)^{\delta k}a_{nn}|\lambda_n|\frac{|t_n(x)|^k}{X_n^{k-1}} = O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of hypotheses of the Theorem 3.1 and Lemma 3.1. This completes the proof of Theorem 3.1.

5. Conclusion

If we take $\varphi_n = \frac{P_n}{p_n}$ in Theorem 3.1, then we have a result on $|A, p_n; \delta|_k$ summability factors of Fourier series. If we take $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$ in Theorem 3.1, then we have a result on $|A, p_n|_k$ summability factors of Fourier series. If we take $\delta = 0$ and $\varphi_n = n$ in Theorem 3.1 and Theorem 3.2, then we have results on $|A|_k$ summability factors of Fourier series. If we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nj} = \frac{p_j}{P_n}$ in Theorem 3.1, then we have a result on $|\bar{N}, p_n; \delta|_k$ summability factors of Fourier series. Also if we take $\delta = 0$, $\varphi_n = \frac{P_n}{p_n}$ and $a_{nj} = \frac{p_j}{P_n}$ in Theorem 3.1, then we have a result on $|\bar{N}, p_n|_k$ summability factors of Fourier series. It is noted that if we take $\varphi_n = \frac{P_n}{p_n}$ in Theorem 3.2, we obtain a theorem dealing with $|A, p_n; \delta|_k$ summability factors of Fourier series (see [22]). If we take $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$ in Theorem 3.2, we obtain a theorem dealing with $|A, p_n|_k$ summability factors of Fourier series (see [19]). If we take $\delta = 0$, $\varphi_n = \frac{P_n}{p_n}$ and $a_{nj} = \frac{p_j}{P_n}$ in Theorem 3.2, then we have a result on $|\bar{N}, p_n|_k$ summability factors of Fourier series (see [5]).

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