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A Space-Fractional Stefan Problem for Advection-Dispersion: Analytical and Computational Approaches

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ABSTRACT: This article presents an investigation of the space-fractional Stefan problem, a mathematical model formulated to describe phase transition phenomena in materials where heat transfer exhibits anomalous diffusion governed by spatial fractional derivatives of the Caputo type. The proposed formulation effectively captures nonlocal interactions and memory effects commonly observed in media with complex thermal properties, features that classical differential equations fail to accurately represent. The fractional Stefan problem finds broad applicability in modeling heat conduction in heterogeneous materials, solidification of alloys, cryopreservation of biological tissues, moisture transport in porous structures, and thermal evolution in geological formations. To facilitate the analysis, a similarity transformation is employed to reduce the governing space-fractional partial differential equation (PDE) to an equivalent fractional ordinary differential equation (ODE), preserving the essential characteristics of the anomalous diffusion process. The resulting fractional ODE is then solved using the Laplace Adomian Decomposition Method (LADM), which provides an efficient analytical approach for obtaining a series solution while satisfying the imposed initial and boundary conditions. The solution is expressed in terms of the three-parameter Mittag-Leffler function, thereby reflecting the inherent nonlocality of fractional diffusion. Furthermore, the derived solutions are shown to converge to their classical counterparts in the appropriate limiting cases, confirming the validity of the proposed approach. Numerical results are presented to verify the effectiveness and applicability of the proposed method.

Keywords: Fractional Stefan problem, explicit solution, self-similarity, Laplace transform, Rescaling techneque.

Contents

1. Introduction

This paper presents an analytical study for fractional space Stefan problem that represents a complex concept in mathematics, addressing the dynamics of phase transitions and thermal conduction in materials. A comprehensive understanding of this problem necessitates a solid foundation in thermodynamics and mathematical modeling. To formulate a fractional diffusion equation, we study a phase-change problem where heat flow is represented by fractional integrals. As the fractional order approaches unity, fractional diffusion equations—theoretical extensions of the classical dispersion-convection model—reduce to the classical case. In its simplest one-dimensional form, it can be expressed as:

$$\omega_t(x,t) = \omega_{xx}(x,t) + \omega_x(x,t) + G(x,t), \quad (x,t) \in \Omega \times (0,T);\ \Omega \subset \mathbb{R}, \tag{1.1}$$

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To engage the following fractional diffusion equation, it is presented for consideration., which incorporates a Caputo derivative concerning The Spatial Variable.

$$\omega_t(x,t) = \frac{\partial}{\partial x} {}_0^C\partial_x^\lambda \omega(x,t) + {}_0^C\partial_x^\lambda \omega(x,t), \quad \lambda \in (0,1);\ (x,t) \in \Omega \times (0,T), \tag{1.2}$$

It is important to note that the fractional Caputo derivative is a math idea that allows us to use derivatives with non-whole numbers. It helps in modeling different things in areas like physics and engineering. Specified order with respect to the spatial variable is represented by the notation ${}_0^C\partial_x^\lambda$

$${}_0^c\partial_x^\lambda \omega(x,t) = {}_oI^{I-\lambda}\left[\omega_x(.,t)\right](x,t) = \frac{1}{\Gamma(1-\lambda)} \int_0^x \frac{\omega_x(\rho,t)}{(x-\rho)^\lambda} d\rho \tag{1.3}$$

and $oI^{1-\lambda}$ The Riemann-Liouville fractional integral of a function θ with order $1-\lambda$ is a fundamental concept in fractional calculus, providing a generalization of the traditional integral. This integral plays a crucial role in various applications across mathematics, physics, and engineering, enabling the analysis of systems with memory and hereditary properties. Is defined

$${}_0I^\mu \omega(x,t) = \frac{1}{\Gamma(\mu)} \int_0^x \frac{\omega(\rho,t)}{(x-\rho)^{1-\mu}} d\rho, \mu \in (0,1) \tag{1.4}$$

Hereafter, the fractional order λ is assumed to lie within the interval $(0,1)$.
Furthermore, we will omit the subscript x in fractional integrals and derivatives when addressing one-variable functions, as previously indicated. Equation (1.2) provides a reliable mathematical framework for modeling anomalous diffusion, capturing memory and nonlocal effects. Moreover, it has been substantiated in reference [4] that this equation holds validity.

$$\omega_t(x,t) = {}_0^C\partial_x^{\lambda+1} \omega(x,t)$$

We propose an appropriate model for describing atypical diffusion, see [15]. Various fractional Stefan models have been proposed, as discussed in references [10,16,23], and [5]. The behavior of asymptotically solution and the interface has been clearly studied in [1,3,8]. A detailed study on self-similar solutions was conducted in [2], while explicit solutions were found in [17,18,19], and other related references. Space-fractional Stefan problems were introduced in [21], and research on this topic is still developing. Recently, K. Ryszewska analyzed a free boundary problem related to a space-fractional diffusion equation in [20]. It demonstrates that is possible to find a pair $\{\omega, \eta\}$ that meets the required conditions.

$$\omega_t(x,t) = \frac{\partial}{\partial x} {}_0^C\partial_x^{\lambda+1} \omega(x,t) + {}_0^C\partial_x^\lambda \omega(x,t) x \in (0,\eta(t)), t \in (0,T) \tag{1.5a}$$

$$\omega(x,0) = \varphi_0 x \in (0,\eta(0)) \tag{1.5b}$$

$$\omega_x(x,0) = \psi_0 x \in (0,\eta(0)) \tag{1.5c}$$

$$\omega(\eta(t),t) = 0 t \in (0,T) \tag{1.5d}$$

$$\dot{\eta}(t) = -\lim_{x \to \eta(t)^-} {}_0^C\partial_x^\lambda \omega(x,t) t \in (0,T) \tag{1.5e}$$

The problem (1.5) has a unique solution if the initial conditions φ_0 and ψ_0 are regular and $\eta(0) = b \in \mathbb{R}^+$. Consider the parabolic domain.

$$\Omega_{\eta,\ \mathrm{T}} = \{(\mathrm{x},\mathrm{t}) : 0 < x < \eta(\mathrm{t}), 0 < t < T\}$$

This document presents an analysis of two melting problems within the context of fractional space. The first problem is discussed in detail.

2. Physical Meaning and Zero-Time Phase Problem

Examine the phenomenon of rapid phase transition occurring when a semi-infinite slab undergoes melting. $(0 < \mathrm{x} < \infty)$ The material begins to melt at a specific temperature. It can effectively analyze heat flow at the fixed phase $x = 0$. The thermal properties are assumed to be constant. The symbols associated with heat transfer, along with their respective physical dimensions, are presented in [15]. Let $\omega = \omega(\mathrm{x}, \mathrm{t})$ Avoid mentioning the highlighted text. Instead, focus on improving the overall quality of the writing $\mathrm{F} = \mathrm{F}(\mathrm{x}, \mathrm{t})$ The heat flow of the material at a certain point x The content of the document pertains to the concept of time t .Let us define a function that represents the (unknown) position of the free boundary, also known as the phase change interface $x = \eta(\mathrm{t})$, at time t.In our model, the heat flux at a particular location is represented as a generalized weighted sum of all classical fluxes from the initial point to the current position. Here, nearby fluxes contribute more strongly than distant ones. Based on this weighting principle, we formulate the heat flux within the slab as follows.

$$\begin{aligned} F(x,t) &= -\delta_\lambda \tfrac{1}{\Gamma(1-\lambda)} \int_0^x k\omega_x(\rho,t)(x-\rho)^{-\lambda} d\rho \\ &= -\delta_\lambda k_0 I_x^{1-\lambda} \omega_x(x,t) \end{aligned} \tag{2.1}$$

Equation (2.1) can be reformulated using Caputo derivatives as follows.

$$F(x,t) = \omega_x(\rho,t)(x-\rho)^{-\lambda} d\rho \tag{2.2}$$

The symbol k represents thermal conductivity. This parameter is incorporated to ensure consistency in units with those presented in equation (2.1).

$$\lim_{\lambda \to 1} \delta_\lambda = 1. \tag{2.3}$$

From [2, Eq.(8)] we get

$$F = m/t^3, \tag{2.4}$$

then

$$\begin{aligned} \delta_\lambda I_x^{1-\lambda} \left(k\omega_x(x,t)\right) &= \delta_\lambda \left(\tfrac{1}{\Gamma(1-\lambda)} \int_0^x \tfrac{k\omega_x(x,t)}{(x-\rho)^\lambda} d\rho \right) \\ &= \delta_\lambda \tfrac{m}{t^3} m^{1-\lambda} \end{aligned} \tag{2.5}$$

Assume that $\delta_\lambda = m^{\lambda-1}$ and let us simplify the problem's two main equations. According to the first law of thermodynamics, we are aware that.

$$\sigma c \omega_x = -kF_x - F \tag{2.6}$$

By using equation (2.2) in the equation (2.6) where the flux function $F = F(x,t)$, we create the main equation that includes all important physical forces.

$$\sigma c \omega_t(x,t) = \delta_\lambda k \frac{\partial}{\partial x} \partial_x^\lambda \omega(x,t) + \delta_\lambda k \partial_x^\lambda \omega(x,t) \tag{2.7}$$

The fractional diffusivity constant is defined as

$$\mu_\lambda = \delta_\lambda \mu, \quad \mu = k/(\sigma c)$$

The highlighted section of the document is elaborated upon as follows.

$$\frac{\partial}{\partial t} \omega(x,t) = \mu_\lambda \frac{\partial}{\partial x} \partial_x^\lambda \omega(x,t) + \partial_x^\lambda \omega(x,t) \tag{2.8}$$

We are looking at a model where the solid part stays at a steady temperature. Based on the Rankine-Hugoniot conditions, there are certain rules that need to be followed.

$$[\![\boldsymbol{F}]\!]_l^\eta = -\sigma l \dot{\eta}(t) \tag{2.9}$$

The double-bracket notation $[\![.]\!]$ quantifies the flux discontinuity across solid-liquid interfaces, where l denotes the specific latent heat of fusion, see [7]. This fractional Stefan condition emerges naturally from the analysis of equations (2.2) and (2.9), leading to this expression:

$$\sigma l\dot{\eta}(t) = -\delta_\lambda k \lim_{x\to\eta(t)^-} \partial_x^\lambda \omega\left(x,t\right). \tag{2.10}$$

To keep it simple, this will be stated as.

$$\sigma l\dot{\eta}(t) = -\delta_\lambda k\partial_x^\lambda \omega(\eta(t),t). \tag{2.11}$$

In the quasi-stationary regime, the pair of solutions $\{\omega,\eta\}$ were shown to approach:

$$\omega(x,t) = 1 - \frac{x^\lambda}{[\Gamma(2+\lambda)]^{\frac{\lambda}{1+\lambda}} t^{\frac{\lambda}{1+\lambda}}}, \quad \eta(t) = [\Gamma(2+\lambda)]^{\frac{1}{1+\lambda}} t^{\frac{1}{1+\lambda}} \tag{2.12}$$

represent solutions to the following problem

$$\frac{\partial}{\partial x} {}_0^C\partial_x^\lambda \omega(x,t) = 0, \quad x\in(0,b), t\in(0,T) \tag{2.13a}$$

$$\omega(x,0) = 0, \quad x\in(0,b) \tag{2.13b}$$

$$\omega(\eta(t),t) = 0, \quad t\in(0,T) \tag{2.13c}$$

$$\dot{\eta}(t) = -{}_0^C\partial_x^\lambda \omega(\eta(t),t), \quad t\in(0,T) \tag{2.13d}$$

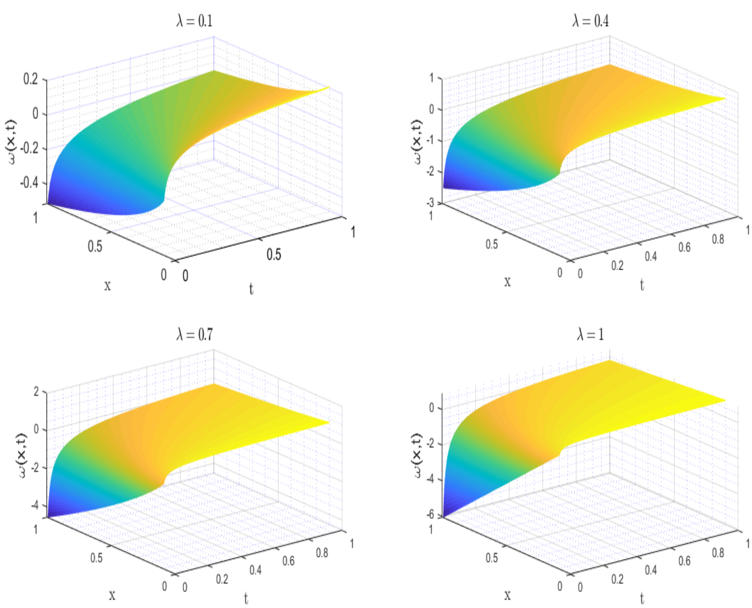

Figure 1: the approximate solution $\omega(x,t)$ for $\lambda = 0.1, 0.4, 0.7, 1$;

Lemma 2.1 *The pair $\{\omega,\eta\}$ from the formula (2.12) is satisfied the Stefan problem(2.13).*

Proof: First, we should consider the quasi-stationary case where $\omega_t = 0$ and Eq. (2.12) satisfies (2.13a) as follows:

$${}_0^C\partial_x^\lambda x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\lambda-\mu+1)} x^{\mu-\lambda},$$

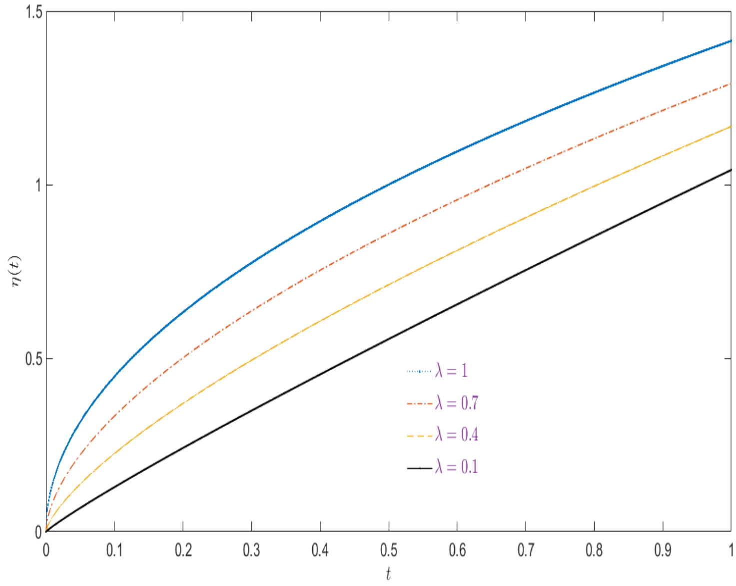

Figure 2: the interface η(t) for $\lambda = 0.1, 0.4, 0.7, 1$;

where $0 < x < \eta(t), 0 < t < T$. If $\mu = \lambda$, then ${}_0^C\partial_x^\lambda x^\mu = \Gamma(\lambda+1)$. Now, let us consider

$$\begin{aligned} {}_0^C\partial_x^\lambda \omega(x,t) &= {}_0^C\partial_x^\lambda[1] - {}_0^C\partial_x^\lambda\left(x^\lambda[\Gamma(2+\lambda)]^{-\frac{\lambda}{1+\lambda}}t^{-\frac{\lambda}{1+\lambda}}\right) \\ &= -[\Gamma(2+\lambda)]^{-\frac{\lambda}{1+\lambda}}t^{-\frac{\lambda}{1+\lambda}}\partial_x^\lambda x^\mu \\ &= -[\Gamma(2+\lambda)]^{\frac{1}{1+\lambda}}t^{-\frac{\lambda}{1+\lambda}}. \end{aligned}$$

Therefore, from above we get that $\frac{\partial}{\partial x}{}_0^C\partial_x^\lambda\omega(x,t) = 0$, and clearly the (2.13b) is satisfied. For part (2.13c), the function ω at the interface $\eta(t)$ has zero value ($\omega(\eta(t),t) = 0$). Finally, since

$${}_0^c\partial_x^\lambda\omega(x,t) = -\Gamma(\lambda+1)[\Gamma(2+\lambda)]^{-\frac{\lambda}{1+\lambda}}t^{-\frac{\lambda}{1+\lambda}} \tag{2.14}$$

And since $\eta(t) = [\Gamma(2+\lambda)]^{\frac{1}{1+\lambda}}t^{\frac{1}{1+\lambda}}$ then, $\eta'(t) = \frac{1}{1+\lambda}[\Gamma(2+\lambda)]^{\frac{1}{1+\lambda}}t^{\frac{-\lambda}{1+\lambda}}$. Let $m = 1+\lambda$ and since $\Gamma(m+1) = m\Gamma(m)$, then $\Gamma(2+\lambda) = (1+\lambda)\Gamma(1+\lambda)$,
Therefore,

$$\eta'(t) = \Gamma(\lambda+1)[\Gamma(2+\lambda)]^{-\frac{\lambda}{1+\lambda}}t^{-\frac{\lambda}{1+\lambda}} \tag{2.15}$$

by combining both (2.14) and (2.15), the Stefan condition (2.13d) is done. □

3. Fundamental Aspects of Fractional Transforms

Equation (1.4) provided the definition of the fractional Riemann-Liouville (FRL) integral, whereas equation (1.3) provided the definition of the Fractional Caputo (FC) derivative. It's crucial to keep in mind that the Riemann-Liouville derivative of order λ occurs for each

$${}_a^{RL}\partial_x^\lambda\theta(x) = \frac{d}{dx}{}_0I^{1-\lambda}\theta(x) \tag{3.1}$$

$$= \frac{1}{\Gamma(1-\lambda)}\frac{d}{dx}\int_0^x \theta(\rho)(x-\rho)^{-\lambda}d\rho \tag{3.2}$$

Lemma 3.1 *[22] The fractional integrals and derivatives with order $\lambda \in (0,1)$ have some features:*

1. *The FRL derivative is a left inverse operator of the FRL integral of the same order* $\lambda \in \mathbb{R}^+$. *If* $\theta \in L^1(0,1)$, *then*

$$ {}^{RL}_{\mathrm{a}}\partial^\lambda {}_{\mathrm{a}}I^\lambda \theta(x) = \theta(x) \quad a.e. \text{ in } (\mathrm{a},\mathrm{b}) $$

2. *The FRL integral is generally not considered a left inverse operator for the FRL derivative.*

Particularly, ${}_{\mathrm{a}}\mathrm{I}^\lambda \left({}^{\mathrm{RL}}_{\mathrm{a}}\partial^\lambda \theta\right)(\mathrm{x}) = \theta(\mathrm{x}) - \frac{\mathrm{a}^{1-\lambda}\theta(\mathrm{a}^+)}{\Gamma(\lambda)(\mathrm{x}-\mathrm{a})^{1-\lambda}}$, *for every* $\mathrm{x} \in [\mathrm{a},\mathrm{b}]$.

3. *If there exist some function* $\varphi \in L^1(0,1)$ *such that* $\theta = {}_{\mathrm{a}}I^\lambda \phi$, *then* ${}_{\mathrm{a}}I^{\lambda RL}{}^{\lambda}_{\mathrm{a}}\theta(x) = \theta(x), \forall x \in [\mathrm{a},\mathrm{b}]$
4. *If* $\theta \in AC[\mathrm{a},\mathrm{b}]$, *then*

$$ {}^{RL}_{\mathrm{a}}\partial^\lambda \theta(x) = \frac{\theta(a)}{\Gamma(1-\lambda)}(x-\mathrm{a})^{-\lambda} + {}^{C}_{\mathrm{a}}\partial^\lambda \theta(x) \ a.e. \text{ in } (\mathrm{a},\mathrm{b}) $$

5. *For every* $\theta \in AC[a,b]$ *such that* ${}_{\mathrm{a}}I^{1-\lambda}\theta' \in AC[\mathrm{a},\mathrm{b}]$ *it holds that*

$$ \frac{d}{dx}{}^{C}_{\mathrm{a}}\partial^\lambda \theta(x) = {}^{RL}_{\mathrm{a}}\partial^\lambda (\theta')(x), \ a.\ e. \text{ in } (\mathrm{a},\mathrm{b}) $$

Lemma 3.2 *[9] The following limits are applicable:*

1. *If we consider the identity operator* ${}_{\mathrm{a}}I^0 = Id$, *then* $\forall \theta \in L^1(0,1)$,

$$ \lim_{\mathrm{a}>0} I^\lambda \theta(x) = {}_{\mathrm{a}}I^0 \theta(x) = \theta(x), \quad a.e. \text{ in } (\mathrm{a},\mathrm{b}) $$

2. $\forall \theta \in \mathrm{AC}[\mathrm{a};\mathrm{b}]$ *the limits*

$$ \overset{RL}{\lim_{a>1}} {}^{\lambda}_{a}\partial^\lambda \theta(x) = \theta'(x), \quad \lim_{a>1} {}^{RL}_{a}\partial^\lambda \theta(x) = \theta'(x) \quad \text{and} \ \lim_{a>1} {}^{C}_{a}\partial^\lambda \theta(x) = \theta'(x) $$

hold a.e. in (a,b). *If additionally, there exists* $\theta'(0^+)$, *The statement is very important.*

$$ \overset{C}{\lim_{a\triangleright 1}} {}^{\lambda}_{a}\partial^\lambda \theta(x) = \theta'(x) - \theta'\left(0^+\right).. $$

There are few functions for which we can directly compute their fractional integral or derivative. The fractional integral and derivative of power functions are defined by gamma function as follows:

$$ {}_{a}I^\lambda\left((x-a)^\mu\right) = \frac{\Gamma(\mu+1)(x-a)^{\mu+\lambda}}{\Gamma(\mu+\lambda+1)}, -1<\mu \tag{3.3} $$

and

$$ {}^{RL}_{a}\partial^\lambda\left((x-a)^\mu\right) = \begin{cases} \frac{\Gamma(\mu+1)(x-a)^{\mu+\lambda}}{\Gamma(\mu-\lambda+1)}; & \mu \neq \lambda-1 \\ 0; & \mu = \lambda-1 \end{cases} \tag{3.4} $$

Besides, in (see [7]: Theorem 2 and Theorem 4) Some integral and differential properties related to special cases of the three-parameters Mittag-Leffler function were proven.

Definition 3.3 *[12] The 3-parameters Mittag-Leffler function* $E_{\lambda,m,l}(\zeta)$ *is defined by*

$$ E_{\lambda,m,l}(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^n, \tag{3.5} $$

with

$$ c_0 = 1, c_n = \prod_{j=0}^{n-1} \frac{\Gamma(\lambda(jm+l)+1)}{\Gamma(\lambda(jm+l+1)+1)}, (n=1,2,3,\ldots), $$

where $\lambda(jm+l) \neq -1,-2,-3,\ldots j = (0,1,2,3,\ldots); \lambda>0, m>0,$ *and* l *are integers.*

Example 3.4 *Consider these two particular examples for 3-parametric Mittag-Leffler function*

1. $E_{1,1,0}(\zeta) = e^\zeta$ *and for* $m=1$ *and* $l=0$; *then* $E_{\lambda,1,0}(\zeta) = E_\lambda(\zeta)$.
2. $E_{1,2,1}\left(-\zeta^2/2\right) = e^{-\zeta^2/2}$.

4. Solving the Stefan Problem via the Self-Similarity Approach

This section offers a simple solution to the identified issues. To keep things simple, all thermophysical parameters are assigned a constant value of one. To find a self-similar solution [3,6,13,14], we apply the method of similarity variables. Starting from a solution $\omega = \omega(x,t)$ of (1.2) (space fractional diffusion equation), we introduce a rescaling ω_μ parameterized by $\mu > 0$.

Theorem 4.1 *Let $\omega(x,t)$ be a solution to (1.2). This solution is asymptotically equal to the solution of the following equation*

$$\omega_t(x,t) = \frac{\partial}{\partial x}{}_0^c\partial_x^\lambda \omega(x,t), \qquad in\ \widetilde{\Omega} \times (0,\widetilde{T}) \tag{4.1}$$

For all $\mu^{\alpha_i} > 0, i = 1,2,3$ and $\alpha = 1+\lambda$ where $\widetilde{\Omega} = \frac{1}{2}\Omega, (0,\widetilde{T}) = \left(0, \frac{T}{\mu^\alpha}\right)$.

Proof: Suppose that $\omega_\mu(x,t) = \mu^{\alpha_1}\omega\left(\mu^{\alpha_2}x, \mu^{\alpha_3}t\right)$ represents the rescaling function where $\bar{x} = \mu^{\alpha_2}x$ and $\bar{t} = \mu^{\alpha_3}t$ then by the following calculation

$$\frac{\partial}{\partial t}\omega_\mu(x,t) = \mu^{\alpha_1+\alpha_3}\omega_{\bar{t}}(\bar{x},\bar{t}) \tag{4.2}$$

$$\frac{\partial}{\partial x}\omega_\mu(x,t) = \mu^{\alpha_1+\alpha_2}\omega_{\bar{x}}(\bar{x},\bar{t}) \tag{4.3}$$

To change the variable $\bar{\rho} = \mu^{\alpha_2}\rho$, the fractional derivative becimes

$$\begin{aligned}
{}_0^{c_0}\partial_x^\lambda \omega_\mu(x,t) &= \frac{1}{\Gamma(1-\lambda)}\int_0^{\bar{x}} \frac{\mu^{\alpha_1+\alpha_2}\omega_{\bar{\rho}}(\bar{\rho},\bar{t})}{\left(\mu^{-\alpha_2}\bar{x} - \mu^{-\alpha_2}\bar{\rho}\right)^\lambda}\left[\mu^{-\alpha_2}d\bar{\rho}\right] \\
&= \frac{1}{\Gamma(1-\lambda)}\int_0^{\bar{x}} \frac{\mu^{\alpha_1}\omega_{\bar{\rho}}(\bar{\rho},\bar{t})}{\mu^{-\lambda\alpha_2}(\bar{x}-\bar{\rho})^\lambda}d\bar{\rho} \\
&= \mu^{\alpha_1+\lambda\alpha_2}\frac{1}{\Gamma(1-\lambda)}\int_0^{\bar{x}} \frac{\omega_{\bar{\rho}}(\bar{\rho},\bar{t})}{(\bar{x}-\bar{\rho})^\lambda}d\bar{\rho} \\
&= \mu^{\alpha_1+\lambda\alpha_2}{}_0^c\partial_{\bar{x}}^\lambda \omega(\bar{x},\bar{t})
\end{aligned}$$

$$\frac{\partial}{\partial x}{}_0^c\partial_x^\lambda \omega_\mu(x,t) = \mu^{\alpha_1+(\lambda+1)\alpha_2}\frac{\partial}{\partial \bar{x}}{}_0^c\partial_{\bar{x}}^\lambda \omega(\bar{x},\bar{t}) \tag{4.4}$$

Now, from (4.2) and (4.4). If $\alpha_3 = (\lambda+1)\alpha_2$ then we get

$$\begin{aligned}
\frac{\partial}{\partial t}\omega_\mu(x,t) - \frac{\partial}{\partial x}{}_0^c\partial_x^\lambda \omega_\mu(x,t) &= \mu^{\alpha_1+(\lambda+1)\alpha_2}\left(\omega_{\bar{t}}(\bar{x},\bar{t}) - \frac{\partial}{\partial \bar{x}}{}_0^c\partial_{\bar{x}}^\lambda \omega(\bar{x},\bar{t})\right) \\
&= \mu^{\alpha_1+(\lambda+1)\alpha_2}{}_0^c\partial_{\bar{x}}^\lambda \omega(\bar{x},\bar{t}).
\end{aligned}$$

Since ${}_0^c\partial_x^\lambda \omega_\mu(x,t) = \mu^{\alpha_1+\lambda\alpha_2}{}_0^c\partial_{\bar{x}}^\lambda \omega(\bar{x},\bar{t})$, thus

$$\begin{aligned}
{}_0^c\partial_{\bar{x}}^\lambda \omega(\bar{x},\bar{t}) &= \mu^{-\alpha_1-\lambda\alpha_2}{}_0^c\partial_x^\lambda \omega_\mu(x,t) \\
\frac{\partial}{\partial t}\omega_\mu(x,t) - \frac{\partial}{\partial x}{}_0^c\partial_x^\lambda \omega_\mu(x,t) &= \mu^{\alpha_2}{}_0^c\partial_{\mathrm{x}}^\lambda \omega_\mu(\mathrm{x},\mathrm{t}).
\end{aligned}$$

Under the conditions $\alpha_1 = \alpha_2 < 0, \alpha_3 = (\lambda+1)\alpha_2$ and by taking

$$\lim_{\mu\to\infty}\omega_\mu(x,t) = \omega(x,t)$$

Therefore, the equation (4.1) is satisfied. □

Theorem 4.2 *The function ω is a solution of equation (4.1) if and only if a function ϑ defined by*

$$t^{\dfrac{-1}{\lambda+1}}\vartheta(z) := \omega(\mathrm{x},\mathrm{t}), \tag{4.5}$$

where z is similarity variable defined by

$$z := xt^{\frac{-1}{\lambda+1}} \tag{4.6}$$

is a solution to the equation

$$\frac{\partial}{\partial z}{}_0^c\partial_z^\lambda\vartheta(z) + \frac{1}{\lambda+1}\left(z\vartheta'(z)+\vartheta(z)\right) = 0 \tag{4.7}$$

Proof: Suppose that define the function $\omega_\mu(x,t) = \mu^{\alpha_1}\omega\left(\mu^{\alpha_2}x, \mu^{\alpha_3}t\right)$. Using the same technique in proof of theorem 4.1, and let $\alpha_1 = -1/((\lambda+1)\alpha_2), \alpha_2 = \alpha_3/(\lambda+1)$

$$\omega(x,t) = t^{\frac{-1}{\lambda+1}}\omega\left(t^{\frac{-1}{\lambda+1}}x, 1\right) \tag{4.8}$$

$$\omega(x,t) = t^{\frac{-1}{\lambda+1}}\vartheta(z). \tag{4.9}$$

Thus, the variable function from (4.6) and the shape function si defined as $\vartheta(z) := \omega(x,t)$, with the variable $z = t^{\frac{-1}{\lambda+1}}x$. By applying the Chain Rule, we transform the expression into an ordinary fractional differential equation describing the function $\vartheta = \vartheta(z)$.

$$\begin{aligned}\frac{\partial}{\partial t}\omega(x,t) &= t^{\frac{-1}{\lambda+1}}\left(\frac{\partial\vartheta}{\partial z}\times\frac{\partial z}{\partial t}\right) - \frac{1}{\lambda+1}t^{\frac{-1}{\lambda+1}-1}\frac{\partial\vartheta}{\partial z}\\ &= -\frac{1}{\lambda+1}t^{\frac{-1}{\lambda+1}-1}\left(z\vartheta'(z)+\vartheta'(z)\right)\\ &= -\frac{1}{\lambda+1}t^{\frac{-1}{\lambda+1}-1}\left(z\vartheta'(z)+\vartheta'(z)\right)\end{aligned} \tag{4.10}$$

Also, by making the substitution $\zeta = \rho t^{\frac{-1}{1+\lambda}}$ it follows that $d\rho = t^{\frac{1}{1+\lambda}}\mathrm{d}\zeta$ and doing some calculation to get

$$\omega_\rho(\rho,t) = \mathrm{t}^{\frac{-2}{1+\lambda}}\theta'(\zeta) \tag{4.11}$$

$$\begin{aligned}{}_0^C\partial_x^\lambda\omega(x,t) &= \frac{1}{\Gamma(1-\lambda)}\int_0^x\frac{\omega_\rho(\rho,t)}{(x-\rho)^\lambda}d\rho\\ &= \frac{1}{t}{}_0^C\partial_\zeta^\lambda\vartheta(\zeta)\end{aligned} \tag{4.12}$$

Then By taking the derivative of the fractional derivative (4.12), we get

$$\begin{aligned}\frac{\partial}{\partial x}{}_0^C\partial_x^\lambda\omega_\mu(x,t) &= \frac{\partial}{\partial x}\left(\frac{1}{t}{}_0^C\partial_\zeta^\lambda\vartheta(\zeta)\right)\\ &= \frac{\partial}{\partial\zeta}\left(\frac{1}{t}{}_0^c\partial_\zeta^\lambda\vartheta(\zeta)\right)\frac{\partial\zeta}{\partial x}\\ &= \frac{\partial}{\partial\zeta}\left(\frac{1}{t}{}_0^c\partial_\zeta^\lambda\vartheta(\zeta)\right)t^{\frac{-1}{1+\lambda}}\end{aligned}$$

from above calculation we get

$$\frac{\partial}{\partial x}{}_0^c\partial_x^\lambda\omega_\mu(x,t) = t^{\frac{-1}{1+\lambda}-1}\frac{\partial}{\partial\zeta}\left({}_0^c\partial_\zeta^\lambda\vartheta(\zeta)\right) \tag{4.13}$$

From (4.10) and (4.13), we get:

$$-t^{\frac{-1}{1+\lambda}-1}\left(\frac{1}{\lambda+1}\left(\zeta\vartheta'(\zeta)+\vartheta(\zeta)\right)+\frac{\partial}{\partial\zeta}{}_0^c\partial_\zeta^\lambda\vartheta(\zeta)\right) = 0; t>0$$

Therefore, if ϑ is a solution to (4.7), then it confirms that ω is indeed a solution of (4.1). $\square$

5. Analytical Discussion for the Fractional ODEs by using LADM

Let us consider the fractional ordinary differential equation (4.7) with the initial conditions

$$\vartheta(0) = \varphi_0 \tag{5.1}$$
$$\vartheta'(0) = \psi_0 \tag{5.2}$$

Let integrate both sides with respect to z to eliminate the outer derivative:

$$^{c}\partial_z^{\lambda}\vartheta(z) + \frac{1}{\ell+1}\int (z\vartheta'(z) + \vartheta(z))\, dz = \text{ Const}$$

From the initial conditions(5.1)-(5.2), we get Const $= 0$ and $\ell > -1$.
Apply the Laplace Adomian Decomposition Method (LADM) to the equation (4.7), from [2,11], then we have

$$\mathcal{L}\left\{^{C}\partial_z^{\lambda}\vartheta(z)\right\} + \frac{1}{\ell+1}\mathcal{L}\left\{\int (z\vartheta'(z) + \vartheta(z))\, dz\right\} = 0$$

Using the Laplace transform of the Caputo derivative:

$$s^{\lambda}F(s) - \sum_{k=0}^{m-1} s^{\lambda-1-k}\vartheta^{(k)}(0) + \frac{1}{\ell+1}\mathcal{L}\left\{\int (z\vartheta'(z) + \vartheta(z))\, dz\right\} = 0$$
$$s^{\lambda}F(s) - \sum_{k=0}^{m-1} s^{\lambda-1-k}\vartheta^{(k)}(0) + \frac{1}{\lambda+1}\mathcal{L}\{\mathrm{z}\vartheta(\mathrm{z})\} = 0$$

Since the solution as an infinite series

$$\vartheta(z) = \sum_{n=0}^{\infty}\vartheta_n(z)$$
$$F(s) = \sum_{n=0}^{\infty}F_n(s)$$

Now apply the decomposition:

$$\sum_{n=0}^{\infty}F_n(s) = \frac{1}{s^{\lambda+1}}\vartheta(0) + \frac{1}{s^{\lambda+2}}\vartheta'(0) - \frac{1}{s^{\lambda+2}(\ell+1)}\mathcal{L}\left\{z\sum_{n=0}^{\infty}\vartheta_n(z)\right\}$$
$$F_0(s) = \frac{1}{s^{\lambda+1}}\vartheta(0) + \frac{1}{s^{\lambda+1}}\vartheta'(0), \quad n = 0$$
$$F_n(s) = -\frac{1}{s^{\lambda+2}(\ell+1)}\mathcal{L}\left\{z\vartheta_{n-1}(z)\right\}, \quad \mathrm{n} \geq 1$$

Taking inverse Laplace:

$$\vartheta_0(z) = \varphi_0 + \psi_0 z,$$
$$\vartheta_1(z) = -\frac{\varphi_0 z^{\alpha+1}}{(\alpha+1)!(\lambda+1)} - \frac{\psi_0 z^{\alpha+2}}{(\alpha+2)!(\lambda+1)}$$
$$\vartheta_2(z) = \frac{\varphi_0 z^{2\alpha+1}}{(2\alpha+1)!(\lambda+1)^2} + \frac{\psi_0 z^{2\alpha+2}}{(2\alpha+2)!(\lambda+1)^2}$$
$$\vdots$$

Then to take general term formula

$$\vartheta_n(z)=(-1)^n\frac{\varphi_0 z^{\alpha n+1}}{(\alpha n+1)!(\lambda+1)^n}+(-1)^n\frac{\psi_0 z^{\alpha n+2}}{(\alpha n+2)!(\lambda+1)^n}$$

Thus the series solution is:

$$\vartheta(z)=\varphi_0\sum_{n=0}^{\infty}\frac{(-1)^n z^{\alpha n+1}}{(\alpha n+1)!(\lambda+1)^n}+\psi_0\sum_{n=0}^{\infty}\frac{(-1)^n z^{\alpha n+2}}{(\alpha n+2)!(\lambda+1)^n}$$

$$\vartheta(z)=\varphi_0 z\sum_{n=0}^{\infty}\frac{(-1)^n z^{\alpha n}}{(\alpha n+1)!(\lambda+1)^n}+\psi_0 z^2\sum_{n=0}^{\infty}\frac{(-1)^n z^{\alpha n}}{(\alpha n+2)!(\lambda+1)^n}$$

The analytic solution ϑ(z) is finally approximated using the formula:

$$\vartheta(z)=\varphi_0 zE_{\alpha,m,r}\left(\frac{-z^2}{\lambda+1}\right)+\psi_0 z^2E_{\alpha,m+1,r}\left(\frac{-z^2}{\lambda+1}\right)\tag{5.3}$$

here, $0<\alpha<1,\ \mathrm{m}=2,\mathrm{r}=1$.From Eq.(5.3) and the self-similar formula (4.5), we get

$$\omega(x,t)=\mathrm{t}^{-\frac{1}{\lambda+1}}\left[\varphi_0 zE_{\lambda,m,r}\left(\frac{-x^2}{\ell+1}\mathrm{t}^{\frac{2}{\lambda+1}}\right)+\psi_0 z^2E_{\lambda,m+1,r}\left(\frac{-x^2}{\ell+1}\mathrm{t}^{\frac{2}{\lambda+1}}\right)\right]\tag{5.4}$$

is a solution of the problem (1.5).

6. Numerical Results and Discussion

Consider the parabolic fractional diffusion-convection equation from the Stefan problem (1.5) with the initial conditions

$$\omega(x,0)=\omega_x(x,0)=0; 0<x<1\tag{6.1}$$

which has asymptotically solution to the Eq.(4.7) with initials that can be chosen $\vartheta(0)=\vartheta'(0)=1$ without loss of generality, and applying the LADM and with $\ell=2$,

$$\vartheta_0(z)=1+z$$
$$\vartheta_1(z)=-\frac{z^{\lambda+1}}{(\lambda+1)!(\ell+1)}-\frac{z^{\lambda+2}}{(\lambda+2)!(\ell+1)},$$
$$\vdots$$

Then to take general term formula

$$\vartheta_n(z)=\frac{(-1)^n z^{n\lambda+1}}{(\ell+1)^n}\left[\frac{1}{(n\lambda+1)!}+\frac{\mathrm{z}}{(n\lambda+2)!}\right]$$

Thus the series solution is defined as

$$\vartheta(z)=\sum_{n=0}^{\infty}\frac{(-1)^n z^{n\lambda+1}}{(\ell+1)^n}\left[\frac{1}{(n\lambda+1)!}+\frac{\mathrm{z}}{(n\lambda+2)!}\right]$$

The analytic solution ϑ(z) is finally approximated to Mittag-Leffler formula as follows:

$$\vartheta(z)=zE_{\lambda,2,1}\left(\frac{-z^2}{\ell+1}\right)+z^2E_{\lambda,3,1}\left(\frac{-z^2}{\ell+1}\right)\tag{6.2}$$

where, $0<\lambda<1$. Therefore, the solution $\omega(x,t)$ from the Eq.(5.4) should be satisfied

$$\omega(x,t)=\mathrm{t}^{-\frac{1}{\lambda+1}}\left[zE_{\lambda,2,1}\left(\frac{-x^2}{\ell+1}\mathrm{t}^{\frac{2}{\lambda+1}}\right)+z^2E_{\lambda,3,1}\left(\frac{-x^2}{\ell+1}\mathrm{t}^{\frac{2}{\lambda+1}}\right)\right]\tag{6.3}$$

The numerical solution of the Stefan problem represented by the space-fractional diffusion-advection with the initial condition (6.1), can be described and illustrated in the above application example. The numerical values in Figure 1 represent the approximate solution $\vartheta(z)$ of the Eq.(40) in the example through different values of z with values $\lambda = 0.4, 0.6, 0.8$, and 1 . We observe that the approximate solution for decreases when z increases. Then it is clear to get numerical solution $\omega(x,t)$ of the Stefan problem (1.5) through different values of t and x ; with values $\lambda = 0.4, 0.6, 0.8$, and 1 ; in Figure 2, we observe that the approximate solution for decreases when t increases for fixed values of x.

We can determine from the preceding reasoning and the numerical results that are supported to our analytical results. The suggested LADM is very successful in providing the analytical solutions for the space-fractional diffusion-convection problem.

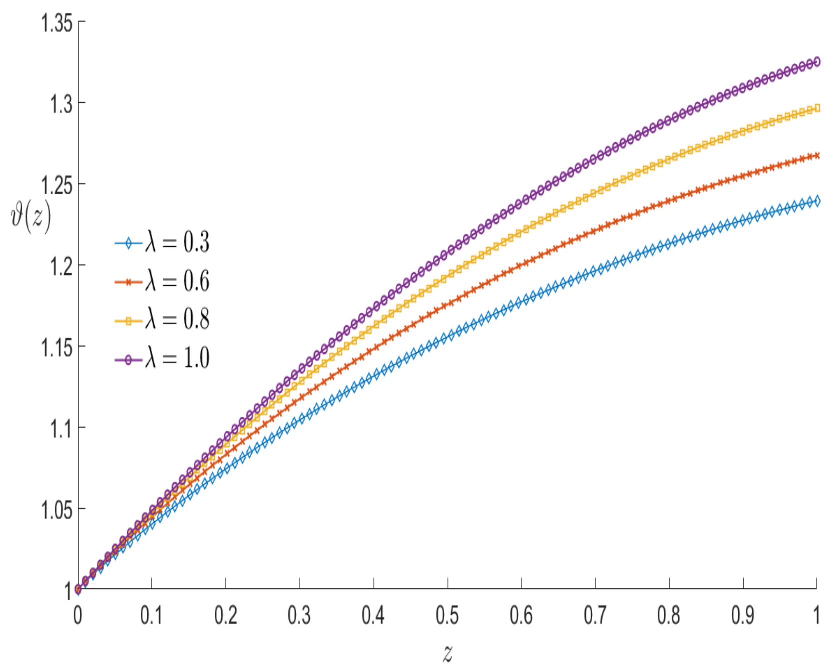

Figure 3: The graph of the approximate solution $\vartheta(z)$ to the Fractional ODE(4.7) among different values of z ; for $\lambda = 0.4, 0.6, 0.8$, and 1 .

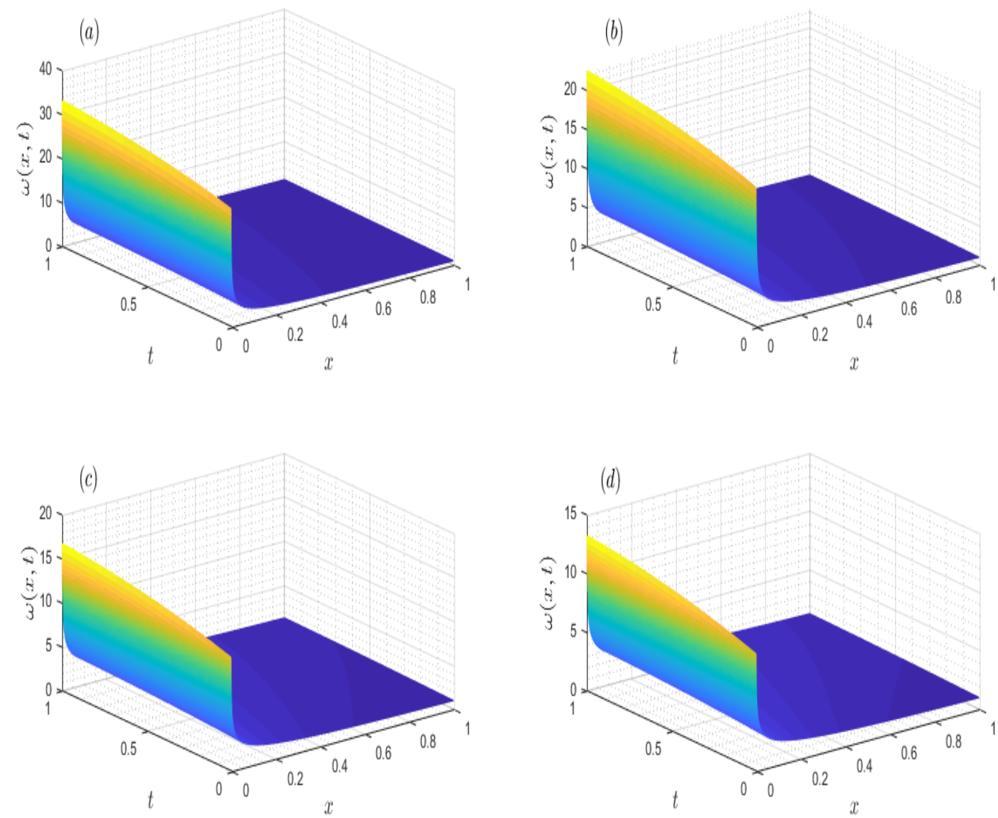

Figure 4: The graph of the approximate solution $\omega(x,t)$ to the fractional PDE from the Stefan problem (1.5) with (a) $\lambda = 0.3$; (b) $\lambda = 0.6$; (c) $\lambda = 0.8$; (d) $\lambda = 1.0$; and among different values of x and t.

7. Conclusion

This study has developed and analyzed a space-fractional Stefan problem to model phase-change phenomena in media where heat transfer is governed by anomalous diffusion. By incorporating Caputo-type spatial fractional derivatives of order $0 < \lambda < 1$, the model successfully captures the nonlocal interactions and memory effects that classical formulations overlook. Through a similarity transformation, the governing fractional PDE was reduced to a fractional ODE, which was efficiently solved using the Laplace Adomian Decomposition Method (LADM). The resulting analytical solutions, expressed via the three-parameter Mittag–Leffler function, accurately describe the intrinsic characteristics of fractional diffusion. Numerical simulations further validated the accuracy and applicability of the proposed approach across different boundary conditions. Importantly, the model was shown to recover the classical Stefan problem as a limiting case, ensuring theoretical consistency. These findings provide a comprehensive mathematical framework for studying anomalous heat conduction and phase transitions in complex materials, with potential applications ranging from materials science to geophysics and bioengineering. Future research may extend this work to multi-dimensional settings, multi-phase systems, and more general forms of fractional operators.

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