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On the Existence of Renormalized Solution for Some Nonlinear Parabolic Problems in Musielak-Orlicz Spaces

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ABSTRACT: In this paper, we will prove in Musielak–Orlicz spaces, the existence of renormalized solution for nonlinear parabolic problems of Leray-Lions type, in the case where the Musielak–Orlicz function φ doesn't satisfy the Δ_2-condition while the right hand side f belongs to $L^1(Q_T)$.

Key Words: Nonlinear parabolic problems, Musielak-Orlicz space, renormalized solution, existence.

Contents

1. Introduction and Basic Hypothesis

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$ $(N \geq 2)$, and let φ be a Musielak-Orlicz function that satisfies the log-Hölder condition, its Young conjugate function is denoted by $\overline{\varphi}$ and verifying Δ_2-condition. Let T be a positive constant, and we set $Q_T = \Omega \times (0,T)$.
In this paper, we consider the following strongly nonlinear parabolic problem

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} + \mathcal{A}(u) + H(x,t,u,\nabla u) = f + \operatorname{div}(\phi(x,t,u)) & \text{in} \quad Q_T, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(t=0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $b : \mathbb{R} \longmapsto \mathbb{R}$ is a strictly increasing $\mathcal{C}^1(\mathbb{R})$-function, and satisfying the following conditions :

$$\forall s \in \mathbb{R}, \quad b_0 < b'(s) < b_1 \quad \text{and} \quad b(0) = 0. \tag{1.2}$$

The mapping

$$\mathcal{A} : D(\mathcal{A}) \subset W_0^{1,x} L_\varphi(Q_T) \longmapsto W^{-1,x} L_{\overline{\varphi}}(Q_T),$$

defined by $\mathcal{A}(u) = -\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray-Lions operator, where $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \longmapsto \mathbb{R}^N$ is a Carathéodory function that satisfying the following conditions :

$$|a(x,t,r,\xi)| \leq \rho_1 \left(a_0(x,t) + \overline{\varphi}_x^{-1}(\psi(x,\rho_2|r|)) \right) + \overline{\varphi}_x^{-1}(\varphi(x,\rho_3|\xi|)) \Big), \tag{1.3}$$

$$(a(x,t,r,\xi) - a(x,t,r,\xi^*)) \cdot (\xi - \xi^*) > 0, \tag{1.4}$$

$$a(x,t,r,\xi) \cdot \xi \geq \alpha\varphi(x,|\xi|), \tag{1.5}$$

where ψ is a Musielak-Orlicz function such that $\psi \prec\prec \varphi$, $a_0(\cdot,\cdot) \in E_{\overline{\varphi}}(Q_T)$, $\alpha > 0$ and $\rho_i > 0$ for $i = 1,2,3$, such that for a.e $(x,t) \in Q_T$ and for all $r \in \mathbb{R}$ and $\xi, \xi^* \in \mathbb{R}^N$ with $\xi \neq \xi^*$. The Carathéodory function ϕ that satisfying the following condition :

$$|\phi(x,t,r)| \leq c_0(x,t)\overline{\varphi}_x^{-1}\varphi(x, \frac{\alpha_0}{\lambda}|b(r)|) \quad \text{for a.e. } (x,t) \in Q_T \text{ and for all } r \in \mathbb{R}, \tag{1.6}$$

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where $0 < \alpha_0 < \min\left(1, \frac{1}{b_1}\right)$ and $\lambda = \operatorname{diam}(Q_T)$, with $\|c_0(.,.)\|_{L^\infty(Q_T)} < \frac{\alpha}{(\alpha_0 b_1 + 1)}$.
Let $H : Q_T \times \mathbb{R} \times \mathbb{R}^N \longmapsto \mathbb{R}$ is a Carathéodory function such that for a.e. $(x,t) \in Q_T$, and all $r \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$:

$$|H(x,t,r,\xi)| \leq h(x,t) + d(|r|)\varphi(x,|\xi|) \tag{1.7}$$

where $d : \mathbb{R}^+ \longmapsto \mathbb{R}^+$ is a continuous positive function which belongs to $L^\infty(\mathbb{R}^+)$, $h(x,t) \in L^1(Q_T)$ and H satisfies the classical sign condition

$$H(x,t,r,\xi)r \geq 0. \tag{1.8}$$

$$f \in L^1(Q_T), \tag{1.9}$$

$$u_0 \in L^1(\Omega). \tag{1.10}$$

Under these assumptions, we establish an existence theorem for renormalized solutions of the problem (1.1).
In the setting of classical Sobolev spaces $L^p(0,T;W^{1,p}(\Omega))$ Porretta has proved in [22] the existence of solutions to problem (1.1), with $b(u) = u$ and H being a nonlinearity satisfying a natural growth condition.
In the case where $H = 0$, the existence and uniqueness of renormalized solutions for parabolic problems of type (1.1) in the Orlicz space framework has been proved by Aberqi et al. [1], while f belongs to $L^1(Q_T)$.
In the Musielak-Orlicz framework, Benkirane et al. in [12] have studied the existence of entropy solutions for a nonlinear elliptic problem of the type :

$$\mathcal{A}(u) + H(x,u,\nabla u) = \operatorname{div}(\phi(x,u)) + \mu \quad \text{in} \quad \Omega,$$

where μ is assumed to belong to $L^1(\Omega) + W^{-1}E_{\overline{\varphi}}(\Omega)$. Many papers deals the existence of solutions of elliptic and parabolic problems under different hypotheses in order to get the fundamental results, we refer the reader to [2], [3], [6],[7], [8], [9], [10], [11], [16], [18] and [19]. The paper is organized as follows : In section 2, we give some preliminaries results. Section 3 is devoted to some auxiliary lemmas which can be used to our result. Finally, in section 4, we present the sense of renormalized solution associated with the parabolic problem (1.1). Moreover, we will prove the existence result.

2. Preliminaries

Let Ω be a domain of $\mathbb{R}^N$, and let $\varphi(x,t) : \Omega \times \mathbb{R}^+ \longmapsto \mathbb{R}^+$ be a function such that :

(*i*) $\varphi(x,\cdot)$ is an N-function for all $x \in \Omega$, i.e. convex, continuous, strictly increasing with $\varphi(x,0) = 0$, $\varphi(x,t) > 0$ for all $t > 0$ and such that

$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = \infty. \tag{2.1}$$

(*ii*) $\varphi(\cdot,t)$ is a measurable function for all $t \geq 0$.

A function $\varphi(x,t)$ which satisfies the conditions (i) and (ii) is called a Musielak-Orlicz function.
The Musielak-orlicz function φ is said to satisfy the Δ_2-condition, if there exist $k > 0$, and a nonnegative function $\Theta(\cdot) \in L^1(\Omega)$ such that

$$\varphi(x,2t) \leq k\varphi(x,t) + \Theta(x) \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad t \geq 0. \tag{2.2}$$

Let $\varphi(x,t)$ be a Musielak-Orlicz function such that $\varphi_x(t) = \varphi(x,t)$, and let φ_x^{-1} be the nonnegative reciprocal function satisfies

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi\left(x, \varphi_x^{-1}(t)\right) = t.$$

We say that ψ grows essentially less rapidly than φ at 0 (resp. near infinity), and we write $\psi \prec\prec \varphi$, if for every positive constant δ, we have

$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{\psi(x,\delta t)}{\varphi(x,t)} = 0 \quad \left(\text{resp.} \quad \lim_{t\to\infty} \sup_{x\in\Omega} \frac{\psi(x,\delta t)}{\varphi(x,t)} = 0\right).$$

Remark 2.1 *If $\psi \prec\prec \varphi$ near infinity, then for all $\varepsilon > 0$ there exists a nonnegative function $h \in L^1(\cdot)$, such that*

$$\psi(x,t) \leq \varphi(x,\varepsilon t) + h(x) \quad \textit{for all} \;\; t \geq 0 \quad \textit{and for a.e.} \;\; x \in \Omega. \tag{2.3}$$

Let φ be a Musielak-Orlicz function, and $u : \Omega \longmapsto \mathbb{R}$ be a measurable function. We define the modular

$$\Phi_{u,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)\, dx,$$

and the convex set

$$K_\varphi(\Omega) = \{u : \Omega \longmapsto \mathbb{R} \text{ measurable } / \ \Phi_{\varphi,\Omega}(u) < +\infty\}$$

The set $K_\varphi(\Omega)$ is called the Musielak-Orlicz class. We define the Musielak-Orlicz space $L_\varphi(\Omega)$ by the vector space

$$L_\varphi(\Omega) = \left\{u : \Omega \longmapsto \mathbb{R} \text{ measurable } / \ \frac{u}{\lambda} \in K_\varphi(\Omega) \;\text{ for some } \lambda > 0\right\}.$$

For a Musielak-Orlicz function φ we pose :

$$\overline{\varphi}(x,s) = \sup_{t>0}(st - \varphi(x,s))$$

$\overline{\varphi}$ is the Musielak-Orlicz function conjugate of φ in the sense of Young with respect to the variable s. In the space $L_\varphi(\Omega)$ we present the two norms :

$$\|u\|_{\varphi,\Omega} = \inf\left\{\lambda > 0 \;:\; \int_\Omega \varphi\left(x, \frac{|u(x)|}{\lambda}\right) dx \leq 1\right\},$$

which is named the Luxemburg norm and the so-called Orlicz norm by :

$$\|u\|_{\varphi,\Omega} = \sup_{\|w\|_{\overline{\varphi}} \leq 1} \int_\Omega |u(x)w(x)|\, dx.$$

These two norms are equivalent (see [21]).
The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in Ω is denoted by $E_\varphi(\Omega)$. It is a separable space (see [21], Theorem 7.10). We say that the sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n \longrightarrow \infty} \Phi_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0.$$

For φ and her conjugate function $\overline{\varphi}$, the following inequality is named the Young's inequality (see [21]) :

$$rs \leq \varphi(x,r) + \overline{\varphi}(x,s), \quad \forall r,s \geq 0, \text{ a.e. } x \in \Omega. \tag{2.4}$$

Let $u \in L_\varphi(\Omega)$ and $v \in L_{\overline{\varphi}}(\Omega)$, thus we have :

$$\left|\int_\Omega u(x)v(x)\, dx\right| \leq \|u\|_{\varphi,\Omega}\|v\|_{\overline{\varphi},\Omega} \quad \text{(the Hölder inequality (see [21]))} \tag{2.5}$$

We define the Musielak-Orlicz-Sobolev space as

$$W^1L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : D^\alpha u \in L_\varphi(\Omega), \quad \text{for all } |\alpha| \leq 1\}.$$

The space $L_\varphi(\Omega)$ is endowed with the norm

$$\|u\|_{1,\varphi,\Omega} = \inf\{\lambda > 0 : \sum_{|\alpha| \leq 1} \Phi_{\varphi,\Omega}\left(\frac{D^\alpha u}{\lambda}\right) \leq 1\}.$$

Let φ be a Musielak-Orlicz function for all $x \in \mathbb{R}^N$, we denote by ∇_x^α the distributional derivative of u on Q_T for the order $\alpha \in \mathbb{N}^N$. We define the inhomogeneous Musielak-Orlicz-Sobolev spaces by

$$W^{1,x}L_\varphi(Q_T) = \{u \in L_\varphi(Q_T) : \nabla_x^\alpha u \in L_\varphi(Q_T), \quad \text{for all } \alpha \in \mathbb{N}^N, |\alpha| \leq 1\}.$$

$$W^{1,x}E_\varphi(Q_T) = \{u \in E_\varphi(Q_T) : \nabla_x^\alpha u \in E_\varphi(Q_T), \quad \text{for all } \alpha \in \mathbb{N}^N, |\alpha| \leq 1\}.$$

The space $W_0^{1,x}E_\varphi(Q_T)$ is defined as the closure of $\mathcal{D}(Q_T)$ for the norm topology in $W^{1,x}L_\varphi(Q_T)$. When Ω has the segment property, and φ satisfies the log-Hölder continuity, then the closure of $D(Q_T)$ with respect to the weak $\sigma(\Pi L\varphi, \Pi E_{\overline{\varphi}})$-topology denoted by $W_0^{1,x}L_\varphi(Q_T)$, is also the closure of $\mathcal{D}(Q_T)$ for the modular convergence in $W^{1,x}L_\varphi(Q_T)$.
The dual space of $W_0^{1,x}E_\varphi(Q_T)$ is given by

$$W^{-1,x}L_{\overline{\varphi}}(Q_T) = \left\{ F = \sum_{|\alpha|\leq 1} \nabla_x^\alpha F_\alpha : \quad F_\alpha \in L_{\overline{\varphi}}(Q_T) \right\}.$$

The space $W^{-1,x}L_{\overline{\varphi}}(Q_T)$ is endowed with the norm

$$\|F\| = \inf \sum_{|\alpha|\leq 1} \|F_\alpha\|_{\overline{\varphi},Q_T}.$$

3. Some Auxiliary Lemmas

We will use the following technical lemmas.

Lemma 3.1 (see [5]) *(Approximation theorem)) Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ $(N \geq 2)$, and let φ and $\overline{\varphi}$ be two complementary Musielak-Orlicz functions which satisfy the following conditions :*

1. *There exists a constant $\delta > 0$ such that $\inf_{x\in\Omega} \varphi(x,1) > \delta$,*
2. *There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$, we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \quad \textit{for all} \quad t \geq 1.$$

3. *If $E \subset \Omega$ is a bounded measurable set, then $\int_E \varphi(x,1)\,dx < \infty$.*
4. *There exists a constant $\mu > 0$ such that $\overline{\varphi}(x,1) \leq \mu$ a.e. in Ω.*

Under these assumptions, $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, and $\mathcal{D}(\Omega)$ is dense in $W_0^1 L_\varphi(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W_0^1 L_\varphi(\Omega)$ for the modular convergence.

Remark 3.1 *Let Ω be a bounded subset of $\mathbb{R}^N$ $(N \geq 2)$. Then, the condition 4 in Lemma 3.1 implies that the embedding $W^1L_\varphi(\Omega) \hookrightarrow W^{1,1}(\Omega)$ is continuous. Since the embedding $W^{1,1}L(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$ is compact, thus we have the following compact embedding*

$$W^1L_\varphi(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$$

Lemma 3.2 (see [14]) *(Modular Poincaré inequality) Under the assumptions of Lemma 3.1, and by assuming that $\varphi(x,\cdot)$ decreases with respect to one of coordinate of x, there exists a constant $\lambda > 0$ which depends only on Ω such that*

$$\int_\Omega \varphi(x,|u|)dx \leq \int_\Omega \varphi(x,\lambda|\nabla u|)\,dx \quad \textit{for all} \quad u \in W_0^1 L_\varphi(\Omega). \tag{3.1}$$

Lemma 3.3 (see [13], Lemma 4) *Let $G : \mathbb{R} \longmapsto \mathbb{R}$ be uniformly Lipschitz function, with $G(0) = 0$. If $u \in W_0^1 L_\varphi(\Omega)$, then $G(u) \in W_0^1 L_\varphi(\Omega)$. Moreover, if the set K of discontinuity points of $G'(\cdot)$ is finite, then*

$$\frac{\partial}{\partial x_i} G(u) = \begin{cases} G'(u) \dfrac{\partial u}{\partial x_i} & a.e \ in \ \{x \in \Omega : u(x) \notin K\}, \\ 0 & a.e \ in \ \{x \in \Omega : u(x) \in K\}. \end{cases} \tag{3.2}$$

For any $k > 0$, we define the truncation function by

$$T_k(r) = \begin{cases} r & if \quad |r| \leq k, \\ k \dfrac{r}{|r|} & if \quad |r| > k. \end{cases}$$

Remark 3.2 Let $k > 0$, it's clear that the function $T_k(\cdot)$ verifying the assumptions of the Lemma 3.3, then $T_k(u) \in W_0^1 L_\varphi(\Omega)$ for any $u \in W_0^1 L_\varphi(\Omega)$. Moreover, we have

$$\frac{\partial T_k(u)}{\partial x_i} = \begin{cases} \dfrac{\partial u}{\partial x_i} & \text{for} \quad |s| < k, \\ 0 & \text{for} \quad |s| \geq k. \end{cases}$$

Lemma 3.4 *Suppose that $\upsilon_n, \upsilon \in L^1(\Omega)$ such that*

i) $\upsilon_n \geq 0 \quad a.e \ in \quad \Omega,$

ii) $\upsilon_n \longrightarrow \upsilon \quad a.e \ in \quad \Omega,$

iii) $\displaystyle\int_\Omega \upsilon_n(x)\, dx \longrightarrow \int_\Omega \upsilon(x)\, dx.$

Then $\upsilon_n \longrightarrow \upsilon$ strongly in $L^1(\Omega)$.

Lemma 3.5 (see [13], Lemma 1) *Let $u \in L_\varphi(\Omega)$ and $(u_n)_n$ be a uniformly bounded sequence in $L_\varphi(\Omega)$. If $u_n \longrightarrow u$ a.e. in Ω, then $u_n \rightharpoonup u$ weakly in $L_\varphi(\Omega)$ for $\sigma\,(L_\varphi(\Omega), E_{\overline{\varphi}}(\Omega))$.*

Lemma 3.6 (see [11]) *Let $a < b \in \mathbb{R}$ and Ω be a bounded domain of $\mathbb{R}^N$ with the segment property, then*

$$\left\{ u \in W_0^{1,x} L_\varphi(\Omega \times]a,b[) : \frac{\partial u}{\partial t} \in W_0^{-1,x} L_\varphi(\Omega \times]a,b[) + L^1(\Omega \times]a,b[) \right\} \subset \mathcal{C}(]a,b[, L^1(\Omega)).$$

Lemma 3.7 (see [17]) *Under assumptions (1.2)-(1.10), and let $(u_n)_n$ be a sequence in $W_0^{1,x} L_\varphi(Q_T)$ such that:*

$$u_n \rightharpoonup u \quad weakly \ in \ W_0^{1,x} L_\varphi(Q_T) \quad for \ \ \sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}}),$$

$$(a(x,t,u_n,\nabla u_n))_n \ \ is \ uniformly \ bounded \ in \ (L_{\overline{\varphi}}(Q_T))^N,$$

$$\lim_{n\to\infty} \int_{Q_T} (a(x,t,u,\nabla u_n) - a(x,t,u,\nabla u \chi_r)) \cdot (\nabla u_n - \nabla u \chi_r)\, dx\, dt = 0,$$

where χ_r is the characteristic function of $Q^r = \{(x,t) \in Q_T : |\nabla u| \leq r\}$. Then,

$$\nabla u_n \longrightarrow \nabla u \quad a.e \quad in \ Q_T,$$

$$\lim_{n\to\infty} \int_{Q_T} a(x,t,u_n,\nabla u_n) \cdot \nabla u_n\, dx\, dt = \int_{Q_T} a(x,t,u,\nabla u) \cdot \nabla u\, dx\, dt,$$

$$\varphi(x, |\nabla u_n|) \to \varphi(x, |\nabla u|) \quad strongly \ in \quad L^1(Q_T).$$

In order to deal with the time derivative, we introduce a time mollification of a function $u \in W_0^{1,x}L_\varphi(Q_T)$.
Thus we define, for all $\mu > 0$ and all $(x,t) \in Q_T$

$$u_\mu(x,t) = \int_{-\infty}^{t} \widetilde{u}(x,s)\exp(\mu(s-t))\,ds,$$

where $\widetilde{u}(x,s) = u(x,s)\chi_{[0,T]}(s)$ is the zero extension of u.

Lemma 3.8 (see [4]) *If $u \in L_\varphi(Q_T)$ then u_μ is measurable in Q_T and $\dfrac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and if $u \in K_\varphi(Q_T)$ then*

$$\int_{Q_T} \varphi(x,u_\mu)\,dx\,dt \le \int_{Q_T} \varphi(x,u)\,dx\,dt$$

Lemma 3.9 (see [23])

1. *If $u \in L_\varphi(Q_T)$ then $u_\mu \longrightarrow u$ for the modular convergence in $L_\varphi(Q_T)$ as $\mu \longrightarrow \infty$.*
2. *If $u \in W_0^{1,x}L_\varphi(Q_T)$ then $u_\mu \longrightarrow u$ for the modular convergence in $W_0^{1,x}L_\varphi(Q_T)$ as $\mu \longrightarrow \infty$.*

4. Main Result

Before we state our main result, we give the definition of a renormalized solution of (1.1).

Definition 4.1 *A mesurable function u defined on Q_T is a renormalized solution of problem (1.1), if satisfies the following conditions :*

$$T_k(u) \in W_0^{1,x}L_\varphi(Q_T), \quad \text{for all } k > 0, \tag{4.1}$$

$$b(u) \in L^\infty(0,T;L^1(\Omega)), \tag{4.2}$$

$$\int_{\{(x,t)\in Q_T \,:\, l\le |u| \le l+1\}} a(x,t,u,\nabla u)\cdot\nabla u\,dx\,dt \longrightarrow 0 \quad \text{as} \quad l \longrightarrow +\infty, \tag{4.3}$$

and, for every function $S \in W^{2,\infty}(\mathbb{R})$ wich S' has a compact support, we have

$$\begin{aligned} &\frac{\partial B_S(u)}{\partial t} - \operatorname{div}(S'(u)(a(x,t,u,\nabla u))) + S''(u)a(x,t,u,\nabla u)\cdot\nabla u \\ &+ H(x,t,u,\nabla u)S'(u) = fS'(u) + \operatorname{div}(S'(u)\phi(x,t,u))) \\ &- S''(u)\phi(x,t,u)\cdot\nabla u \quad \text{in} \ \ \mathcal{D}'(Q_T), \end{aligned} \tag{4.4}$$

where $B_S(t) = \displaystyle\int_0^t b'(\omega)S'(\omega)d\omega$ and

$$B_S(u)(t=0) = B_S(u_0) \quad \text{in } \Omega. \tag{4.5}$$

Theorem 4.1 *Under the assumptions (1.2)-(1.10), problem (1.1) admits at least one renormalized solution.*

Proof:

Step 1 : Approximate problem

We define the following approximations :

$$b_n(r) = b(T_n(r)) \quad \text{for any } r \in \mathbb{R}, \tag{4.6}$$

$$a_n(x,t,r,\xi) = a(x,t,T_n(r),\xi) \quad \text{a.e. } (x,t) \in Q_T, \forall r \in \mathbb{R}, \forall \xi \in \mathbb{R}^N, \tag{4.7}$$

$$\phi_n(x,t,r) = \phi(x,t,T_n(r)) \quad \text{a.e. } (x,t) \in Q_T, \forall r \in \mathbb{R}, \tag{4.8}$$

$$H_n(x,t,r,\xi) = T_n(H(x,t,r,\xi)) \quad \text{a.e. } (x,t) \in Q_T, \forall r \in \mathbb{R}, \forall \xi \in \mathbb{R}^N, \tag{4.9}$$

$$f_n = T_n(f) \quad \text{implies that} \quad f_n \longrightarrow f \quad \text{strongly in } L^1(Q_T), \tag{4.10}$$

$$u_{0,n} = T_n(u_0) \quad \text{implies that} \quad u_{0,n} \longrightarrow u_0 \quad \text{strongly in} \quad L^1(\Omega). \tag{4.11}$$

Also, we consider the approximate problem :

$$\begin{cases} \dfrac{\partial b_n(u_n)}{\partial t} - \operatorname{div}(a_n(x,t,u_n,\nabla u_n)) + H_n(x,t,u_n,\nabla u_n) = f_n + \operatorname{div}(\phi_n(x,t,u_n)) & \text{in } Q_T, \\ u_n(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(t=0) = u_{0,n}(x) & \text{in } \Omega. \end{cases} \tag{4.12}$$

It is clear that the Carathéodory function $a_n + \phi_n$ satisfying the assumptions (11),(12) and (13) in [18]. Then, in view of Theorem 4 in [18], the approximate problem (4.12) admit at least one weak solution $u_n \in W_0^{1,x}L_\varphi(Q_T)$.

Remark 4.1 *The explicit dependence in x and t of the functions a, ϕ and H will be omitted so that $a(x,t,u,\nabla u) = a(u,\nabla u)$, $\phi(x,t,u) = \phi(u)$ and $H(x,t,u,\nabla u) = H(u,\nabla u)$.*

Step 2 : A priori estimates

Let $k > 0$ and $\tau \in (0,T)$, by taking $T_k(u_n)\chi_{(0,\tau)}$ as a test function for the approximate problem (4.12), we get

$$\begin{aligned} &\int_\Omega B_k^n(u_n(\tau))\,dx + \int_{Q_\tau} a_n(u_n,\nabla u_n)\nabla T_k(u_n)\,dx\,dt \\ &+ \int_{Q_\tau} H_n(u_n,\nabla u_n)T_k(u_n)\,dx\,dt = \int_{Q_\tau} f_n T_k(u_n)\,dx\,dt \\ &- \int_{Q_\tau} \phi_n(u_n)\nabla T_k(u_n)\,dx\,dt + \int_\Omega B_k^n(u_n(0))\,dx, \end{aligned} \tag{4.13}$$

where $B_k^n(r) = \displaystyle\int_0^r \frac{\partial b_n(s)}{\partial s} T_k(s)ds$. By the definition of $B_k^n(r)$, we deduce that

$$\int_\Omega B_k^n(u_n(\tau))\,dx \geq 0 \quad \text{and} \quad \int_\Omega B_k^n(u_n(0))\,dx \leq k||b(u_0)||_{L^1(\Omega)}.$$

By (1.6) and using Young inequality, we have

$$\begin{aligned} \int_{Q_\tau} \phi_n(u_n)\nabla T_k(u_n)\,dx\,dt \leq ||c_0(.,.)||_{L^\infty(Q_T)}[\alpha_0 b_1 \int_{Q_\tau} \varphi(x, \frac{|T_k(u_n)|}{\lambda})\,dx\,dt \\ + \int_{Q_\tau} \varphi(x,|\nabla T_k(u_n)|)\,dx\,dt], \end{aligned}$$

thanks to (3.1), we obtain

$$\int_{Q_\tau} \phi_n(u_n)\nabla T_k(u_n)\,dx\,dt \leq ||c_0(.,.)||_{L^\infty(Q_T)}(\alpha_0 b_1 + 1) \int_{Q_\tau} \varphi(x,|\nabla T_k(u_n)|)\,dx\,dt.$$

By (1.8), one has

$$\int_{Q_\tau} H_n(u_n, \nabla u_n) T_k(u_n)\, dx\, dt \geq 0.$$

Returning to (4.13) and using (1.5), we get

$$\int_{Q_T} a_n(u_n, \nabla u_n) \nabla T_k(u_n)\, dx\, dt \leq ||c_0(.,.)||_{L^\infty(Q_T)} \frac{(\alpha_0 b_1 + 1)}{\alpha} \int_{Q_T} a_n(u_n, \nabla u_n) \nabla T_k(u_n)\, dx\, dt$$
$$+ k \left(||f_n||_{L^1(Q_T)} + ||b(u_0)||_{L^1(\Omega)} \right),$$

so by (1.6), we get

$$\int_{Q_T} a_n(x, t, u_n, \nabla u_n) \nabla T_k(u_n)\, dx\, dt \leq k C_1. \tag{4.14}$$

By (1.5), we obtain

$$\int_{Q_T} \varphi(x, |\nabla T_k(u_n)|)\, dx\, dt \leq k C_2. \tag{4.15}$$

Moreover, we have

$$\begin{aligned} \inf_{x \in \Omega} \varphi(x, \frac{k}{\lambda}) \text{meas}\{|u_n| > k\} &\leq \int_{\{|u_n| > k\}} \varphi\left(x, \frac{|T_k(u_n)|}{\lambda}\right) dx\, dt \\ &\leq \int_{Q_T} \varphi(x, |\nabla T_k(u_n)|)\, dx\, dt \\ &\leq k C_2. \end{aligned} \tag{4.16}$$

Finally

$$\text{meas}\{|u_n| > k\} \leq \frac{k C_2}{\inf_{x \in \Omega} \varphi(x, \frac{k}{\lambda})} \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty. \tag{4.17}$$

For every $\eta > 0$, we have

$$\text{meas}\{|u_n - u_m| > \eta\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\}$$
$$+ \text{meas}\{|T_k(u_n) - T_k(u_m)| > \eta\}.$$

From (4.15) we conclude that $(T_k(u_n))_n$ is bounded in $W_0^{1,x} L_\varphi(Q_T)$, then by using Remark 3.1 there exists a measurable function v_k, such that $T_k(u_n) \longrightarrow v_k$ strongly in $L^1(Q_T)$ as $n \to \infty$ for a subsequence, and thanks to the reciprocal Lebesgue's theorem we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Q_T . Consequently, for any $\varepsilon, \eta > 0$ there exists $n_0(\varepsilon, \eta) > 0$ such that

$$\text{meas}\{|u_n - u_m| > \eta\} \leq \varepsilon \quad \text{for all } n, m > n_0(\varepsilon, \eta).$$

This proves that (u_n) is a Cauchy sequence in measure in Q_T. Then

$$u_n \longrightarrow u \quad \text{a.e in} \;\; Q_T. \tag{4.18}$$

Finally, for all $k > 0$ we have

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in} \quad W_0^{1,x} L_\varphi(Q_T) \quad \text{for} \;\; \sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}}). \tag{4.19}$$

Let $\Gamma_k \in W^{2,\infty}(\mathbb{R})$ such that Γ'_k has a compact support $supp(\Gamma'_k) \subset [-k, k]$. We multiply the Eq. (4.12) by $\Gamma'_k(u_n)$, to obtain in $\mathcal{D}'(Q_T)$,

$$\begin{aligned} \frac{\partial B^n_{\Gamma_k}(u_n)}{\partial t} &= \text{div}(a_n(u_n, \nabla u_n) \phi_n(u_n)) \Gamma'_k(u_n)) - a_n(u_n, \nabla u_n) \Gamma''_k(u_n) \nabla u_n \\ &+ \text{div}(\Gamma'_k(u_n) \phi_n(u_n)) - \Gamma''_k(u_n) \phi_n(u_n) \nabla u_n - H_n(u_n, \nabla u_n) \Gamma'_k(u_n) \\ &+ f_n \Gamma'_k(u_n), \end{aligned} \tag{4.20}$$

where $B^n_{\Gamma_k}(r)=\displaystyle\int_0^r \Gamma'_k(s)\frac{\partial b_n(s)}{\partial s}ds.$
First we have

$$|\nabla B^n_{\Gamma_k}(u_n)|\le b_1|\nabla T_k(u_n)|\|\Gamma_k\|_{L^\infty(\mathbb{R})}\quad \text{a.e in } Q_T,$$

by using (4.15), we obtain

$$\left(B^n_{\Gamma_k}(u_n)\right)\quad \text{is bounded in } W^{1,x}_0L_\varphi(Q_T), \tag{4.21}$$

since $supp(\Gamma'_k)\subset[-k,k]$ and $supp(\Gamma''_k)\subset[-k,k]$,u_n may be replaced by $T_k(u_n)$ in each of these terms. As a consequence, each in right hand side of (4.20) is bounded either in $W^{-1,x}L_{\overline{\varphi}}(Q_T)$ or in $L^1(Q_T)$, we conclude that

$$\left(\frac{\partial B^n_{\Gamma_k}(u_n)}{\partial t}\right)\quad \text{is bounded in } L^1(Q_T)+W^{-1,x}L_{\overline{\varphi}}(Q_T), \tag{4.22}$$

Now we will prove that

$$b(u)\in L^\infty(0,T;L^1(\Omega))$$

By using (4.13) and Fatou's lemma, we deduce that

$$\frac{1}{k}\int_\Omega B_k(u(\tau))\,dx\le C_1,$$

for almost any τ in $(0,T)$. By definition of $B_k(r)$ and since $\frac{1}{k}B_k(u(\tau))$ converges pointwise to $b(u)$, as $k\longrightarrow 0$, we conclude that $b(u)\in L^\infty(0,T;L^1(\Omega))$.

Step 3 : Boundedness of $a_n(T_k(u_n),\nabla T_k(u_n))$ in $(L_{\overline{\varphi}}(Q_T))^N$

Let $\vartheta\in(E_\varphi(Q_T))^N$ such that $\|\vartheta\|_{\varphi,Q_T}=1$. By using (1.4), we have

$$\begin{aligned}
&\int_{Q_T} a_n\left(T_k(u_n),\nabla T_k(u_n)\right)\cdot\vartheta\,dx\,dt\\
&\quad\le\int_{Q_T} a_n\left(T_k(u_n),\nabla T_k(u_n)\right)\cdot\nabla T_k(u_n)\,dx\,dt-\int_{Q_T} a_n\left(T_k(u_n),\vartheta\right)\cdot\left(\nabla T_k(u_n)-\vartheta\right)\,dx\,dt\\
&\quad\le kc-\int_{Q_T} a_n\left(T_k(u_n),\vartheta\right)\cdot\nabla T_k(u_n)\,dx\,dt+\int_{Q_T} a_n\left(T_k(u_n),\vartheta\right)\cdot\vartheta\,dx\,dt,
\end{aligned}$$

For the two last terms on the right-hand side, in view of Young's inequality we have

$$\begin{aligned}
\int_{Q_T} a_n\left(T_k(u_n),\nabla T_k(u_n)\right)\cdot\vartheta\,dx\,dt&\le kc+6\rho_1\int_{Q_T}\overline{\varphi}\left(x,\frac{a_n\left(T_k(u_n),\vartheta\right)}{3\rho_1}\right)dx\,dt\\
&\quad+3\rho_1\int_{Q_T}\varphi(x,|\nabla T_k(u_n)|)\,dx\,dt+3\rho_1\int_{Q_T}\varphi(x,|\vartheta|)\,dx\,dt.
\end{aligned}$$

By using (1.3), (2.1) and the convexity of $\overline{\varphi}$, we obtain

$$\begin{aligned}
\overline{\varphi}\left(x,\frac{|a_n\left(T_k(u_n),\vartheta\right)|}{3\rho_1}\right)&\le\frac{1}{3}\left(\overline{\varphi}(x,a_0(x,t))+\psi(x,\rho_2|T_k(u_n)|)+\varphi(x,\rho_3|\vartheta|)\right)\\
&\le\frac{1}{3}\left(\overline{\varphi}(x,a_0(x,t))+\psi(x,\rho_2k)+\varphi(x,\rho_3|\vartheta|)\right)\in L^1(Q_T),
\end{aligned}$$

we deduce that $a_n(T_k(u_n),\vartheta)$ is bounded in $L_{\overline{\varphi}}(Q_T)^N$. This implies from (4.15) that

$$\int_{Q_T} a_n\left(T_k(u_n),\nabla T_k(u_n)\right)\cdot\vartheta\,dx\,dt\le C(k,\vartheta),\quad\forall\vartheta\in(E_\varphi(\Omega))^N\text{ with }\|\vartheta\|_{\varphi,\Omega}=1,$$

where $C(k,\vartheta)$ is a constant depending on k and ϑ.
Consequently, by using the uniform boundedness principe we conclude that

$$a_n\left(T_k(u_n),\nabla T_k(u_n)\right)\quad\text{is bounded in }\left(L_{\overline{\varphi}}(Q_T)\right)^N. \tag{4.23}$$

Step 4 : Some regularity results

Multiplying the equation (4.12) by the function $Z_l(u_n) = T_1(u_n - T_l(u_n))$ and by applying the same argument as in Step 2, we obtain

$$\int_{\{l\leq|u_n|\leq l+1\}} a_n(u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt + \int_{\{|u_n|>l+1\}} |H_n(u_n,\nabla u_n)|\,dx\,dt$$

$$\leq C\left[\int_{Q_T} f_n Z_l(u_n)\,dx\,dt + \int_{\{|u_{n0}|>l\}} |b_n(u_{0,n})|\,dx\right], \tag{4.24}$$

by combining (4.10) and (4.18), we get

$$\lim_{n\to\infty}\int_{\{l\leq|u_n|\leq l+1\}} a_n(u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt + \lim_{n\to\infty}\int_{\{|u_n|>l+1\}} |H_n(u_n,\nabla u_n)|\,dx\,dt$$

$$\leq C\left[\int_{Q_T} f Z_l(u)\,dx\,dt + \int_{\{|u_0|>l\}} |b(u_0)|\,dx\right].$$

Applying Lebesgue's Dominated Convergence Theorem and passing $l\to\infty$, we obtain

$$\lim_{l\to\infty}\limsup_{n\to\infty}\int_{\{l\leq|u_n|\leq l+1\}} a(u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt = 0, \tag{4.25}$$

and

$$\lim_{l\to+\infty}\limsup_{n\to+\infty}\int_{\{|u_n|>l+1\}} |H_n(u_n,\nabla u_n)|\,dx\,dt = 0. \tag{4.26}$$

Step 5 : Almost everywhere convergence of the gradients

Let $k>0$, since $T_k(u)\in W_0^{1,x}L\varphi(Q_T)$ and $\mathcal{D}(Q_T)$ dense modularly in $W_0^{1,x}L\varphi(Q_T)$. Then, there exists a sequence $(v_j)_j\subset\mathcal{D}(Q_T)$ converges modularly to $T_k(u)$ in $W_0^{1,x}L\varphi(Q_T)$ as $j\to\infty$, which implies that

$$T_k(v_j)\longrightarrow T_k(u)\quad\text{modularly in } W_0^{1,x}L\varphi(Q_T), \tag{4.27}$$

and

$$T_k(v_j)\longrightarrow T_k(u)\quad\text{almost everywhere in } W_0^{1,x}L\varphi(Q_T)\quad\text{for a subsequence.} \tag{4.28}$$

Let $l>0$, we define the Lipschitz real function $S_l(s)$ as follows

$$S_l(s) = 1 - |T_{l+1}(s) - T_l(s)|\,.$$

Let $n\geq\mu\geq j\geq l\geq\beta\geq 2k$, we denote by $\varepsilon(n,\mu,j,\beta,l)$ the real function that satisfies

$$\lim_{l\to\infty}\lim_{\beta\to\infty}\lim_{j\to\infty}\lim_{\mu\to\infty}\lim_{n\to\infty}\varepsilon(n,\mu,j,\beta,l) = 0.$$

We set $W^{n,j}_{\mu,\beta} = (T_\beta(T_k(u_n) - T_k((v_j)_\mu))^+$ and $W^{j}_{\mu,\beta} = (T_\beta(T_k(u) - T_k((v_j)_\mu))^+$.

By taking $\exp(D(|u_n|))W^{n,j}_{\mu,\beta}S_l(u_n)$ as a test function in (4.12), where $D(s) = \displaystyle\int_0^s \frac{d(r)}{\alpha}\,dr$, we obtain

$$I^1_{n,\mu,j,\beta,l} + I^2_{n,\mu,j,\beta,l} + I^3_{n,\mu,j,\beta,l} + I^4_{n,\mu,j,\beta,l} + I^5_{n,\mu,j,\beta,l} + I^6_{n,\mu,j,\beta,l} = I^7_{n,\mu,j,\beta,l} + I^8_{n,\mu,j,\beta,l}, \tag{4.29}$$

with

$$I^1_{n,\mu,j,\beta,l} = \int_{Q_T} \frac{\partial b_n(u_n)}{\partial t} \exp(D(|u_n|)) W^{n,j}_{\mu,\beta} S_l(u_n)\, dx\, dt,$$
$$I^2_{n,\mu,j,\beta,l} = \int_{Q_T} a_n(u_n, \nabla u_n) \cdot \nabla W^{n,j}_{\mu,\beta} \exp(D(|u_n|)) S_l(u_n)\, dx\, dt,$$
$$I^3_{n,\mu,j,\beta,l} = \int_{Q_T} a_n(u_n, \nabla u_n) \cdot \nabla u_n \exp(D(|u_n|)) W^{n,j}_{\mu,\beta} S'_l(u_n)\, dx\, dt,$$
$$I^4_{n,\mu,j,\beta,l} = \int_{Q_T} \phi_n(u_n) \cdot \nabla u_n \exp(D(|u_n|)) W^{n,j}_{\mu,\beta} S'_l(u_n)\, dx\, dt,$$
$$I^5_{n,\mu,j,\beta,l} = \int_{Q_T} \phi_n(u_n) \cdot \nabla W^{n,j}_{\mu,\beta} \exp(D(|u_n|)) S_l(u_n)\, dx\, dt,$$
$$I^6_{n,\mu,j,\beta,l} = \int_{Q_T} \phi_n(u_n) \cdot \nabla u_n \frac{d(|u_n|)}{\alpha} \exp(D(|u_n|)) W^{n,j}_{\mu,\beta} S_l(u_n)\, dx\, dt,$$
$$I^7_{n,\mu,j,\beta,l} = \int_{Q_T} f_n \exp(D(|u_n|)) W^{n,j}_{\mu,\beta} S_l(u_n)\, dx\, dt,$$
$$I^8_{n,\mu,j,\beta,l} = \int_{Q_T} h(x,t) \exp(D(|u_n|)) W^{n,j}_{\mu,\beta} S_l(u_n)\, dx\, dt.$$

Starting by the terms $I^7_{n,\mu,j,\beta,l}$ and $I^8_{n,\mu,j,\beta,l}$, we have $\exp(D(|u_n|)) W^{n,j}_{\mu,\beta} S_l(u_n) \rightharpoonup 0$ weak-$*$ in $L^\infty(Q_T)$ as $n, \mu, j \to \infty$ respectively, and $h \in L^1(Q_T)$ then

$$\lim_{j\to\infty} \lim_{\mu\to\infty} \lim_{n\to\infty} I^7_{n,\mu,j,\beta,l} = 0. \tag{4.30}$$

Similarly, we have $f_n \longrightarrow f$ strongly in $L^1(Q_T)$ as $n \to \infty$ then

$$\lim_{j\to\infty} \lim_{\mu\to\infty} \lim_{n\to\infty} I^8_{n,\mu,j,\beta,l} = 0. \tag{4.31}$$

For the term $I^3_{n,\mu,j,\beta,l}$, we have

$$\begin{aligned} I^3_{n,\mu,j,\beta,l} &= -\int_{\{l\le|u_n|\le l+1\}} a_n(u_n, \nabla u_n) \cdot \nabla u_n \exp(D(|u_n|)) W^{n,j}_{\mu,\beta} \operatorname{sign}(u_n)\, dx\, dt \\ &\le C \int_{\{l\le|u_n|\le l+1\}} a_n(u_n, \nabla u_n) \cdot \nabla u_n\, dx\, dt. \end{aligned}$$

By using (4.25), we get

$$I^3_{n,\mu,j,\beta,l} \le \varepsilon(n,l). \tag{4.32}$$

For the term $I^2_{n,\mu,\beta,j,l}$, we have

$$\begin{aligned} &I^2_{n,\mu,\beta,j,l} \\ &= \int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(v_j)\le\beta\}} a_n(T_k(u_n), \nabla T_k(u_n)) \cdot (\nabla T_k(u_n) - \nabla T_k(v_j)_\mu) S_l(u_n) \exp(D(|u_n|))\, dx\, dt \\ &\quad - \int_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(v_j)\le\beta\}} a_n(u_n, \nabla u_n) \cdot \nabla T_k(v_j)_\mu S_l(u_n) \exp(D(|u_n|))\, dx\, dt. \end{aligned}$$

In view of (4.23), there exists $\varpi_{k+\beta} \in (L_{\overline{\varphi}}(Q_T))^N$ that verifies $a_n(T_{k+\beta}(u_n), \nabla T_{k+\beta}(u_n)) \rightharpoonup \varpi_{k+\beta}$ weakly in $(L_{\overline{\varphi}}(Q_T))^N$, and since

$$\begin{aligned} &S_l(u_n) \exp(D(|u_n|)) \nabla T_k(v_j)_\mu \chi_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(v_j)_\mu\le\beta\}} \\ &\longrightarrow S_l(u) \exp(D(|u|)) \nabla T_k(v_j)_\mu \chi_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(v_j)_\mu\le\beta\}} \quad \text{strongly in } (E_\varphi(Q_T))^N \quad \text{as } n\to\infty. \end{aligned}$$

Then,

$$\lim_{n\to\infty}\int_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(v_j)_\mu\le\beta\}} a_n(u_n,\nabla u_n)S_l(u_n)\exp(D(|u_n|))\nabla T_k(v_j)_\mu\,dx\,dt$$
$$=\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(v_j)_\mu\le\beta\}} \varpi_{k+\beta}S_l(u)\exp(D(|u_n|))\nabla T_k(v_j)_\mu\,dx\,dt.$$

Hence,

$$\lim_{j\to\infty}\lim_{\mu\to\infty}\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(v_j)_\mu\le\beta\}} \varpi_{k+\beta}S_l(u)\exp(D(|u|))\nabla T_k(u)\,dx\,dt$$
$$=\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(u)\le\beta\}} \varpi_{k+\beta}S_l(u)\exp(D(|u|))\nabla T_k(u)\,dx\,dt$$
$$=0.$$

This implies from $\exp(D(|u_n|))\ge 1$ and $S_l(u_n)=1$ for $|u_n|\le k$ that

$$I^2_{n,\mu,j,\beta,l}=\int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(v_j)_\mu\le\beta\}} a_n(T_k(u_n),\nabla T_k(u_n))S_l(u_n)(\nabla T_k(u_n)-\nabla T_k(v_j)_\mu)\,dx\,dt+\varepsilon(n,\mu,j), \tag{4.33}$$

Concerning the term $I^5_{n,\mu,j,\beta,l}$, from (1.6) we obtain

$$|\phi_n(u_n)\exp(D(|u_n|))S_l(u_n)|\le C\overline{\varphi}_x^{-1}\varphi(x,\frac{\alpha_0}{\lambda}|b(l+1)|)\in L_{\bar\varphi}(Q_T),$$

and since $\bar\varphi$ satisfies the Δ_2-condition, then by applying Vitali's theorem we conclude

$$\phi_n(u_n)\exp(D(|u_n|))S_l(u_n)\to\phi(u)\exp(D(|u|))S_l(u)\quad\text{strongly in }\left(L_{\bar\varphi}(Q_T)\right)^N\quad\text{as }n\to\infty.$$

Moreover, we have $\nabla W^{n,j}_{\mu,\beta}\rightharpoonup 0$ weakly in $(L_\varphi(Q_T))^N$ as $n,\mu,j\to\infty$ then

$$I^5_{n,\mu,j,\beta,l}=\varepsilon(n,\mu,j) \tag{4.34}$$

Concerning the terms $I^4_{n,\mu,\beta,j,l}$ and $I^6_{n,\mu,\beta,j,l}$, for $n>l+1>k$, we have $\nabla u_n S_l'(u_n)=\nabla T_{l+1}(u_n)\operatorname{sign}(u_n)$ a.e. in Q_T. Since $\exp(D(|u_n|))W^{n,j}_{\mu,\beta}\rightharpoonup\exp(D(|u|))W^j_{\mu,\beta}$ weak-$*$ in $L^\infty(Q_T)$ as $n.\mu,j\to\infty$, and the sequence $(\phi_n(T_{l+1}(u_n)))_n$ converges strongly to $\phi(T_{l+1}(u))$ in $\left(E_{\overline{\varphi}}(Q_T)\right)^N$, thus from Vitali's theorem we deduce that

$$\phi_n(T_{l+1}(u_n))\exp(D(|u_n|))W^{n,j}_{\mu,\beta}\operatorname{sign}(u_n)\longrightarrow\phi(T_{l+1}(u))\exp(D(|u|))W^j_{\mu,\beta}\operatorname{sign}(u)\quad\text{strongly in }\left(E_{\overline{\varphi}}(Q_T)\right)^N.$$

In addition, we have $\nabla T_{l+1}(u_n)\rightharpoonup\nabla T_{l+1}(u)$ weakly in $(L_\varphi(Q_T))^N$ as $n\to+\infty$, then

$$I^4_{n,\mu,j,\beta,l}=\varepsilon(n,\mu,j). \tag{4.35}$$

Similarly, we obtain

$$I^6_{n,\mu,j,\beta,l}=\varepsilon(n,\mu,j). \tag{4.36}$$

Finally, for the term $I^1_{n,\mu,j,\beta,l}$, using the same argument followed in [15] we obtain

$$I^1_{n,\mu,j,\beta,l}=\varepsilon(n,\mu,j,\beta). \tag{4.37}$$

By combining (4.29) and (4.30)–(4.37) we conclude that

$$\int_{\{|T_k(u_n)-T_k(v_j)_\mu|\le\beta\}} a_n(T_k(u_n),\nabla T_k(u_n))S_l(u_n)(\nabla T_k(u_n)-\nabla T_k(v_j)_\mu)\,dx\,dt\le\varepsilon(n,\mu,j,\beta,l), \tag{4.38}$$

Let $r>0$, we set $Q^r=\{(x,t)\in Q_T:|\nabla T_k(u)|\le r\}$ and $Q^r_j=\{(x,t)\in Q_T:|\nabla T_k(v_j)|\le r\}$ and denoting by χ^r and χ^r_j the characteristic functions of Q^r and Q^r_j respectively. Also, we set

$$R_{n,k}=(a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u)).$$

For $0 < \delta < 1$, we have

$$0 \le \int_{Q^r} R_{n,k}^{\delta}\, dz\, dt = \int_{Q^r} R_{n,k}^{\delta} \chi_{\{|T_k(u_n)-T_k(v_j)_\mu| \le \beta\}}\, dx\, dt + \int_{Q^r} R_{n,k}^{\delta} \chi_{\{|T_k(u_n)-T_k(v_j)_\mu| > \beta\}}\, dx\, dt.$$

According to Hölder's inequality, we obtain

$$\begin{aligned}\int_{Q^r} R_{n,k}^{\delta} \chi_{\{|T_k(u_n)-T_k(v_j)_\mu| \le \beta\}}\, dx\, dt &\le \left(\int_{Q^r} R_{n,k} \chi_{\{|T_k(u_n)-T_k(v_j)_\mu| \le \beta\}}\, dx\, dt\right)^{\delta} \left(\int_{Q^r} dx\, dt\right)^{1-\delta} \\ &\le (\mathrm{meas}(Q_T))^{1-\delta} \left(\int_{Q^r} R_{n,k} \chi_{\{|T_k(u_n)-T_k(v_j)_\mu| \le \beta\}}\, dx\, dt\right)^{\delta},\end{aligned}$$

and

$$\int_{Q^r} R_{n,k}^{\delta} \chi_{\{|T_k(u_n)-T_k(v_j)_\mu| > \beta\}}\, dx\, dt \le \left(\int_{Q^r} R_{n,k}\, dx\, dt\right)^{\delta} \left(\int_{\{|T_k(u_n)-T_k(v_j)_\mu| > \beta\}} dx\, dt\right)^{1-\delta}.$$

Since $a(T_k(u_n), \nabla T_k(u_n))$ is bounded in $(L_{\overline{\varphi}}(Q_T))^N$, thus

$$\int_{Q^r} R_{n,k}^{\delta} \chi_{\{|T_k(u_n)-T_k(v_j)_\mu| > \beta\}}\, dx\, dt \le C_2 \mathrm{meas}\{(x,t) \in Q_T : |T_k(u_n) - T_k(v_j)_\mu| > \beta\}^{1-\delta}.$$

Hence,

$$\begin{aligned}\int_{Q^r} R_{n,k}^{\delta}\, dx\, dt \le C_1 &\left(\int_{Q^r} R_{n,k} \chi_{\{|T_k(u_n)-T_k(v_j)_\mu| \le \beta\}}\, dx\, dt\right)^{\delta} \\ &+ C_2 \mathrm{meas}\{|T_k(u_n) - T_k(v_j)_\mu| > \beta\}^{1-\delta}.\end{aligned}$$

Additionally,

$$\begin{aligned}&\int_{Q^r} R_{n,k} \chi_{\{|T_k(u_n)-T_k(v_j)_\mu| \le \beta\}}\, dx\, dt \\ &\le \int_{\{|T_k(u_n)-T_k(v_j)_\mu| \le \beta\}} (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)\chi^r)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)\chi^r)\, dx\, dt.\end{aligned}$$

Let $\tau > 0$, by taking $r > \tau$, we obtain

$$\begin{aligned}0 &\le \int_{Q^\tau \cap \{|T_k(u_n)-T_k(v_j)_\mu| \le \beta\}} (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))) \cdot (\nabla T_k(u_n) - \nabla T_k(u))\, dx\, dt \\ &= \int_{Q^\tau \cap \{|T_k(u_n)-T_k(v_j)_\mu| \le \beta\}} (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)\chi^\tau)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)\chi^\tau)\, dx\, dt \\ &= \int_{\{|T_k(u_n)-T_k(v_j)_\mu| \le \beta\}} (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(v_j)\chi_j^\tau)) \cdot (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^\tau)\, dx\, dt \\ &\quad + \int_{\{|T_k(u_n)-T_k(v_j)_\mu| \le \beta\}} a(T_k(u_n), \nabla T_k(u_n)) \cdot (\nabla T_k(v_j)\chi_j^\tau - \nabla T_k(u)\chi^\tau)\, dx\, dt \\ &\quad + \int_{\{|T_k(u_n)-T_k(v_j)_\mu| \le \beta\}} (a(T_k(u_n), \nabla T_k(v_j)\chi_j^\tau) - a(T_k(u_n), \nabla T_k(u)\chi^\tau)) \cdot \nabla T_k(u_n)\, dx\, dt \\ &\quad - \int_{\{|T_k(u_n)-T_k(v_j)_\mu| \le \beta\}} a(T_k(u_n), \nabla T_k(v_j)\chi_j^\tau) \cdot \nabla T_k(v_j)\chi_j^\tau\, dx\, dt \\ &\quad + \int_{\{|T_k(u_n)-T_k(v_j)_\mu| \le \beta\}} a(T_k(u_n), \nabla T_k(u)\chi^\tau) \cdot \nabla T_k(u)\chi^\tau\, dx\, dt \\ &= J_1 + J_2 + J_3 + J_4 + J_5.\end{aligned}$$

Now, we passing n, j, μ, and τ to infinity in each term of the previous inequality.
For the first term J_1, we have

$$\begin{aligned} J_1 =& \int_{\{|T_k(u_n)-T_k(v_j)_\mu|\leq\beta\}} a(T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(v_j)_\mu)\,dx\,dt \\ &-\int_{\{|T_k(u_n)-T_k(v_j)_\mu|\leq\beta\}} a(T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(v_j)\chi_j^\tau-\nabla T_k(v_j)_\mu)\,dx\,dt \\ &-\int_{\{|T_k(u_n)-T_k(v_j)_\mu|\leq\beta\}} a(T_k(u_n),\nabla T_k(v_j)\chi_j^\tau)\cdot(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^\tau)\,dx\,dt. \end{aligned}$$

Thanks to (4.38), we get

$$\int_{\{|T_k(u_n)-T_k(v_j)|\leq\beta\}} a(T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(v_j)_\mu)\,dx\,dt\leq\varepsilon(n,\mu,j,l)$$

Since $a(T_k(u_n),\nabla T_k(u_n))$ is bounded in $(L_{\overline{\varphi}}(Q_T))^N$, then there exists a measurable function $\varpi_k\in(L_{\overline{\varphi}}(Q_T))^N$ satisfying

$$a(T_k(u_n),\nabla T_k(u_n))\rightharpoonup\varpi_k\quad\text{ weakly in }((L_{\overline{\varphi}}(Q_T))^N\quad\text{for}\quad\sigma(\Pi L_{\overline{\varphi}},\Pi E_\varphi).$$

Moreover, we have

$$(\nabla T_k(v_j)\chi_j^r-\nabla T_k(v_j)_\mu)\chi_{\{|T_k(u_n)-T_k(v_j)_\mu|\leq\beta\}}\longrightarrow(\nabla T_k(v_j)\chi_j^r-\nabla T_k(v_j)_\mu)\chi_{\{|T_k(u)-T_k(v_j)_\mu|\leq\beta\}},$$

strongly in $(E_\varphi(Q_T))^N$ as $n\to\infty$. Then,

$$\begin{aligned} &\lim_{n\to\infty}\int_{\{|T_k(u_n)-T_k(v_j)_\mu|\leq\beta\}} a(T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(v_j)\chi_j^\tau-\nabla T_k(v_j)_\mu)\,dx\,dt, \\ &=\int_{\{|T_k(u)-T_k(v_j)|\leq\beta\}}\varpi_k(\nabla T_k(v_j)\chi_j^r-\nabla T_k(v_j)_\mu)\,dx\,dt, \end{aligned}$$

On the other hand, we have

$$a(T_k(u_n),\nabla T_k(v_j))\chi_{\{|T_k(u_n)-T_k(v_j)_\mu|\leq\beta\}}\longrightarrow a(T_k(u),\nabla T_k(v_j))\chi_{\{|T_k(u)-T_k(v_j)_\mu|\leq\beta\}}$$

strongly in $(E_{\overline{\varphi}}(Q_T))^N$, and since $(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^r)\rightharpoonup(\nabla T_k(u)-\nabla T_k(v_j)\chi_j^r)$, weakly in $(L_\varphi(Q_T))^N$ for the weak topology $\sigma(\Pi L_{\overline{\varphi}},\Pi E_\varphi)$.
Consequently, by passing j and μ to infinity and applying the Lebesgue's theorem, we conclude that

$$J_1\leq\varepsilon(n,\mu,j,l,r).$$

Similarly, for the term J_2, we have

$$\lim_{j\to\infty}\lim_{\mu\to\infty}\lim_{n\to\infty}J_2=\lim_{j\to\infty}\lim_{\mu\to\infty}\int_{\{|T_k(u)-T_k(v_j)_\mu|\leq\beta\}}\varpi_k(\nabla T_k(v_j)\chi_j^r-\nabla T_k(u)\chi^r)\,dx\,dt=0.$$

Similarly, we obtain

$$\lim_{j\to\infty}\lim_{\mu\to\infty}\lim_{n\to\infty}J_3=\int_{Q_T}a(T_k(u),\nabla T_k(u)\chi^\tau)\cdot\nabla T_k(u)(1-\chi^\tau)\,dx\,dt=0,$$

$$\lim_{j\to\infty}\lim_{\mu\to\infty}\lim_{n\to\infty}J_4=-\int_{Q_T}a(T_k(u),\nabla T_k(u)\chi^\tau)\cdot\nabla T_k(u)\chi^\tau\,dx\,dt,$$

$$\lim_{j\to\infty}\lim_{\mu\to\infty}\lim_{n\to\infty}J_5=\int_{Q_T}a(T_k(u),\nabla T_k(u)\chi^\tau)\cdot\nabla T_k(u)\chi^\tau\,dx\,dt.$$

Finally, we deduce that

$$\int_{Q^r} R_{n,k}\,dx\,dt \le C_1(\epsilon(n,\mu,j,l,\beta))^{\delta} + C_2(\varepsilon(n,\mu))^{1-\delta}.$$

Letting β tends to infinity, we obtain

$$\int_{Q^\tau} (a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n) - \nabla T_k(u))\,dx\,dt = \varepsilon(n). \tag{4.39}$$

Thus, in view of Lemma 3.7 we conclude that $\nabla u_n \longrightarrow \nabla u$ a.e. in Q^τ, and since τ is arbitrary,

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } Q_T, \tag{4.40}$$

Furthermore, by using (4.19), (4.22),(4.23),(4.40) and Fatou's lemma, we get

$$\int_{\{l\le|u|\le l+1\}} a(u,\nabla u)\cdot\nabla u\,dx\,dt \le \liminf_{n\to+\infty}\int_{\{l\le|u_n|\le l+1\}} a(u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt$$
$$\longrightarrow 0 \quad \text{as} \quad l\to\infty$$

Then, the condition (4.3) is established.

Step 6 : Equi-integrability of the non-linearities

In this step, we will show that

$$H_n(u_n,\nabla u_n) \to H(u,\nabla u) \quad \text{strongly in } \; L^1(Q_T).$$

From (4.19) and (4.40), we have

$$H_n(u_n,\nabla u_n) \to H(u,\nabla u) \quad \text{a.e. in } Q_T,$$

by using (1.5) and (1.7), we obtain

$$\int_K |H_n(u_n,\nabla u_n)|\,dx\,dt = \int_{K\cap\{|u_n|\le l\}} |H_n(u_n,\nabla u_n)|\,dx\,dt + \int_{K\cap\{|u_n|>l\}} |H_n(u_n,\nabla u_n)|\,dx\,dt$$
$$\le \int_K h(x,t)\,dx\,dt + \frac{\|d(\cdot)\|_{L^\infty(\mathbb{R}^+)}}{\alpha}\int_K a_n(T_l(u_n),\nabla T_l(u_n))\cdot\nabla T_l(u_n)\,dx\,dt$$
$$+ \int_{\{|u_n|>l\}} |H_n(u_n,\nabla u_n)|\,dx\,dt,$$

where K be a measurable subset of Q_T and $l>0$.
Since $h\in L^1(Q_T)$ and by using (4.26) we have

$$\lim_{l\to\infty}\limsup_{n\to+\infty}\int_{\{|u_n|>l\}} |H_n(u_n,\nabla u_n)|\,dx\,dt = 0,$$

According to Lemma 3.7 the sequence

$$(a_n(T_l(u_n),\nabla T_l(u_n))\cdot\nabla T_l(u_n))_n \quad \text{is equi-integrable in } Q_T$$

Consequently

$$\lim_{|K|\to 0}\sup_n \int_K |H_n(u_n,\nabla u_n)|\,dx\,dt = 0.$$

This proves that $H_n(u_n,\nabla u_n)$ is equi-integrable.
Therefore, Vitali's theorem allows us to get $H(u,\nabla u)\in L^1(Q_T)$, and

$$H_n(u_n,\nabla u_n) \to H(u,\nabla u) \quad \text{strongly in } \; L^1(Q_T). \tag{4.41}$$

Step 7 : Passage to the limit

Let $S \in W^{2,\infty}(\mathbb{R})$ which is piecewise C^1-function such that $\mathrm{supp}(S') \subset [-k,k]$. Multiplying the approximate problem (4.12) by $S'(u_n)$, we get

$$\begin{cases} \dfrac{\partial B_S(u_n)}{\partial t} - \mathrm{div}\left(S'(u_n)(a_n(u_n,\nabla u_n))\right) + S''(u_n)a_n(u_n,\nabla u_n)\cdot\nabla u_n \\ + H_n(u_n,\nabla u_n)S'(u_n) = fS'(u_n) + \mathrm{div}\left(S'(u)\phi_n(u_n)\right) - S''(u_n)\phi_n(u_n)\cdot\nabla u_n \qquad \text{in } \mathcal{D}'(Q_T), \end{cases} \tag{4.42}$$

where $B_S(w) = \displaystyle\int_0^w b'(r)S'(r)dr$.

Now, we will pass to the limit as $n \to +\infty$ of each term of (4.42).

- *Limit of* $\dfrac{\partial B_S(u_n)}{\partial t}$: since S is bounded, and $B_S(u_n)$ converges to $B_S(u)$ a.e. in Q_T and weakly-* in $L^\infty(Q_T)$, then $\dfrac{\partial B_S(u_n)}{\partial t}$ converges to $\dfrac{\partial B_S(u)}{\partial t}$ in $\mathcal{D}'(Q_T)$ as n tends to $+\infty$.
- *Limit of* $S'(u_n)a(u_n,\nabla u_n)$: since $\mathrm{supp}(S') \subset [-k,k]$ we have
$$S'(u_n)a(u_n,\nabla u_n) = S'(u_n)a(T_k(u_n),\nabla T_k(u_n)) \quad \text{a.e. in } Q_T.$$
The pointwise convergence of u_n to u as n tends to $+\infty$, the bounded character of S', and by Lemma 3.7, we conclude $a(T_k(u_n),\nabla T_k(u_n))$ converges to $a(T_k(u),\nabla T_k(u))$ weakly in $(L_{\overline{\varphi}}(Q_T))^N$ allows us to obtain $S'(u_n)A(T_k(u_n),\nabla T_k(u_n))$ converges to $S'(u)a(T_k(u),\nabla T_k(u))$ weakly for $\sigma(\Pi L_{\overline{\varphi}},\Pi E_\varphi)$, and $S'(u)a(T_k(u),\nabla T_k(u)) = S'(u)a(u,\nabla u)$ a.e. in Q_T.
- *Limit of* $S''(u_n)a(u_n,\nabla u_n)\cdot\nabla u_n$: since $\mathrm{supp}(S') \subset [-k,k]$, we get
$$S''(u_n)a(u_n,\nabla u_n)\cdot\nabla u_n = S''(u_n)a(T_k(u_n),\nabla T_k(u_n))\cdot\nabla u_n \quad \text{a.e. in } Q_T.$$
The pointwise convergence of $S''(u_n)$ to $S''(u)$ as n tends to $+\infty$, the bounded character of S'' and by Lemma 3.7, we conclude
$$S''(u_n)a(T_k(u_n),\nabla T_k(u_n))\cdot\nabla u_n \rightharpoonup S''(u)a(T_k(u),\nabla T_k(u))\cdot\nabla u \quad \text{weakly in } L^1(Q_T)$$
as $n \longrightarrow +\infty$, and
$$S''(u)a(T_k(u),\nabla T_k(u))\cdot\nabla u = S''(u)a(u,\nabla u)\cdot\nabla u \quad \text{a.e. in } Q_T.$$
- *Limit of* $S'(u_n)\phi(u_n)$: since $\mathrm{supp}(S') \subset [-k,k]$ we have
$$S'(u_n)\phi(u_n) = S'(u_n)\phi(T_k(u_n)) \quad \text{a.e. in } Q_T.$$
In a similar way, we obtain
$$S'(u_n)\phi(u_n) \rightharpoonup S'(u)\phi(u) \quad \text{weakly for } \sigma(\Pi L_{\overline{\varphi}},\Pi E_\varphi).$$
- *Limit of* $S''(u_n)\phi(u_n)\cdot\nabla u_n$: also we have
$$S''(u_n)\phi(u_n)\cdot\nabla u_n = S''(u_n)\phi(T_k(u_n))\cdot\nabla T_k(u_n).$$
Using the weakly convergence of truncation, it is possible to prove that,
$$S''(u_n)\phi(u_n)\cdot\nabla u_n \longrightarrow S''(u)\phi(u)\cdot\nabla u \text{ strongly in } L^1(Q_T).$$
- *Limit of* $H(u_n,\nabla u_n)S'(u_n)$: we have $u_n \longrightarrow u$ a.e. in Q_T, S' is piecewise C^1. It is enough to use (4.41) to get that $H(u_n,\nabla u_n)S'(u_n) \longrightarrow H(u,\nabla u)S'(u)$ strongly in $L^1(Q_T)$.

– *Limit of* $f_n S'(u_n)$: we have $u_n \longrightarrow u$ a.e. in Q_T, S' is piecewise C^1. It is enough to use (4.10) to get that $f_n S'(u_n) \longrightarrow f S'(u)$ strongly in $L^1(Q_T)$.

As a consequence of the above convergence result, we are in a position to pass to the limit as n tends to $+\infty$ in equation (4.42) and to conclude that u satisfies (4.4).
It remains to show that $B_S(u_n)$ satisfies the initial condition (4.5), remark that S being bounded, $B_S(u_n)$ is bounded in $L^\infty(Q_T)$. The equation (4.42) allows to show that $\dfrac{\partial B_S(u_n)}{\partial t}$ is bounded in $W^{-1,x}L_{\overline{\varphi}}(Q_T) + L^1(Q_T)$. By Lemma 3.6 implies that $B_S(u_n)$ lies in a compact set of $C^0([0,T]; L^\infty(\Omega))$. It follows that, on one hand, $B_S(u_n)(t=0))$ converges to $B_S(u)(t=0))$ strongly in $L^1(Q_T)$. On the other hand, the smoothness of S imply that $B_S(u)(t=0)) = B_S(u_0)$ in Ω. This complete the existence result. □

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