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On Solving Non-homogeneous Ternary Higher Degree Diophantine Equation $w^2 + 2z^2 - 2wx - 4zx = x^{2s+1} - 3x^2, s > 0$

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ABSTRACT: Thenon-homogeneous ternary higherdegree Diophantine equation given by $w^2+2z^2-2wx-4zx = x^{2s+1} - 3x^2, s > 0$ is analyzed for its patterns of non-zero distinct integral solutions.

Key Words: Ternary higher degree equation, non-homogeneous equation, integral.

Contents

1 Introduction

A polynomial equation with variables that can only have integer values is known as a Diophantine equation. The word is named for Diophantus of Alexandria, a Hellenistic mathematician who explored such equations in great detail in the third century. The set of integers is used to find the solutions to these equations, which may involve one or more variables. The field saw resurgence in the 17th century with mathematicians like Pierre de Fermat, and its study continues to be a vibrant area of number theory today. Diophantine equations, while rooted in pure mathematics, have several practical applications in various fields that require finding integer solutions to problems. The Diophantine equation offers an unlimited field for research due to their variety [1-4]. In particular, one may refer [5-10] for quintic equationswith two, three and five unknowns. This communication concerns with an another interesting equation $w^2 + 2z^2 - 2wx - 4zx = x^{2s+1} - 3x^2$ representing non-homogeneous higher degree equation with three unknowns for determining its infinitely many non-zero integral points.utions are presented.

2 Method of Analysis

The given non-homogeneous ternary higher degree Diophantine equation is

$$w^2 + 2z^2 - 2wx - 4zx = x^{2s+1} - 3x^2, s > 0 \tag{2.1}$$

On completing the squares,(2.1) is written as

$$P^2 + 2Q^2 = x^{2s+1} \tag{2.2}$$

where

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$$\mathrm{P = w - x, Q = z - x} \tag{2.3}$$

We illustrate below the process of obtaining different sets of integer solutions to (2.1)

2.1 Set :

After some algebra, it is observed that (2.2) is satisfied by

$$\mathrm{P = m\left(m^2 + 2n^2\right)^s, Q = n\left(m^2 + 2n^2\right)^s} \tag{2.4}$$

and

$$x = m^2 + 2n^2 \tag{2.5}$$

From (2.4) and (2.3), we have

$$\mathrm{w = \left[m\left(m^2 + 2n^2\right)^{s-1} + 1\right]\left(m^2 + 2n^2\right), z = \left[n\left(m^2 + 2n^2\right)^{s-1} + 1\right]\left(m^2 + 2n^2\right)} \tag{2.6}$$

Thus,(2.5) and (2.6) represent the integer solutions to (2.1).

2.2 Set :

Assuming (2.5) in (2.2) and employing the method of factorization, one obtains

$$P + i\sqrt{2}Q = (m + i\sqrt{2}n)^{2s+1} \tag{2.7}$$

On equating the real and imaginary parts, we have

$$P = f(m,n), Q = g(m,n) \tag{2.8}$$

where

$$f(m,n) = \frac{(m + i\sqrt{2}n)^{2s+1} + (m - i\sqrt{2}n)^{2s+1}}{2}$$

,

$$g(m,n) = \frac{(m + i\sqrt{2}n)^{2s+1} - (m - i\sqrt{2}n)^{2s+1}}{i2\sqrt{2}}$$

Using (2.8) in (2.3), note that

$$\mathrm{z = m^2 + 2n^2 + g(m,n), w = m^2 + 2n^2 + f(m,n)} \tag{2.9}$$

Thus,(2.5) and (2.9) represent the integer solutions to (2.1).

2.3 Set:

Write (2.2) as

$$\mathrm{P^2 + 2Q^2 = x^{2\ s+1} * 1} \tag{2.10}$$

Consider 1 as

$$\mathrm{1 = \frac{(F(r,s) + i\sqrt{2}G(r,s))(F(r,s) - i\sqrt{2}G(r,s))}{[H(r,s)]^2}} \tag{2.11}$$

where

$$\mathrm{F(r,s) = 2r^2 - s^2, G(r,s) = 2rs, H(r,s) = 2r^2 + s^2}$$

Using (2.5) and (2.11) in (2.10) and employing the method of factorization,one has

$$\mathrm{P + i\sqrt{2}Q = \frac{(F(r,s) + i\sqrt{2}G(r,s))(m + i\sqrt{2}n)^{2\ s+1}}{H(r,s)}} \tag{2.12}$$

Equating the real and imaginary parts in (2.12) and replacing m by H(r, s)M and n by H(r, s)N, we get
$P = [H(r,s)]^{2\ s}[f(M,N)F(r,s) - 2\ g(M,N)G(r,s)]$

$$Q = [H(r,s)]^{2\ s}[f(M,N)G(r,s) + g(M,N)F(r,s)] \tag{2.13}$$

Also ,from (2.5) ,we have

$$x = [H(r,s)]^2 \left(M^2 + 2N^2\right) \tag{2.14}$$

Substituting (2.13) and (2.14) in (2.3) ,one obtains the corresponding values of z, w satisfying (2.1) .

2.4 Set :

The option

$$Q = kx^s \tag{2.15}$$

in (2.2) leads to

$$P^2 = x^{2\ s}\left(x - 2k^2\right)$$

which is satisfied by

$$x = \left(S^2 + 2\right) k^2 \tag{2.16}$$

and

$$P = S\left(S^2 + 2\right)^s k^{2\ s+1} \tag{2.17}$$

Using (16) in (15), we have

$$Q = \left(S^2 + 2\right)^s k^{2\ s+1} \tag{2.18}$$

Substituting (17) and (18) in (3), it is seen that

$$w = \left(S^2 + 2\right) k^2 \left[\ S\left(\ S^2 + 2\right)^{s-1} k^{2\ s-1} + 1\right]$$

$$z = \left(S^2 + 2\right) k^2 \left[\left(\ s^2 + 2\right)^{s-1} k^{2\ s-1} + 1\right] \tag{2.19}$$

Thus, (2.16) and (2.19) satisfy (2.1).

2.5 Set :

The option

$$P = x^s \tag{2.20}$$

in (2.2) leads to

$$2Q^2 = x^{2\ s}(x - 1)$$

which is satisfied by

$$x = \left(2k^2 + 1\right) \tag{2.21}$$

and

$$Q = k\left(2k^2 + 1\right)^s \tag{2.22}$$

Using (2.21) in (2.20), we have

$$P = \left(2k^2 + 1\right)^s \tag{2.23}$$

Substituting (2.21),(2.22) and (2.23) in (2.3), it is seen that

$$w = \left(2k^2 + 1\right)\left[\left(2k^2 + 1\right)^{s-1} + 1\right]$$

$$z = \left(2k^2 + 1\right)\left[k\left(2k^2 + 1\right)^{s-1} + 1\right] \tag{2.24}$$

Thus, (2.21) and (2.24) satisfy (2.1).

2.6 Set :

The choice

$$P = kQ \tag{2.25}$$

in (2.2) gives

$$\left(k^2 + 2\right)Q^2 = x^{2\,s+1}$$

which is satisfied by

$$Q = \left(k^2 + 2\right)^s \alpha^{(2\,s+1)\beta} \tag{2.26}$$

and

$$x = \left(k^2 + 2\right)\alpha^{2\beta} \tag{2.27}$$

From (2.25), it is seen that

$$P = k\left(k^2 + 2\right)^s \alpha^{(2\,s+1)\beta} \tag{2.28}$$

Using (2.26), (2.27) and (2.28) in (2.3), we have

$$w = k\left(k^2 + 2\right)^s \alpha^{(2\,s+1)\beta} + \left(k^2 + 2\right)\alpha^{2\beta} \tag{2.29}$$

$$z = \left(k^2 + 2\right)^s \alpha^{(2\,s+1)\beta} + \left(k^2 + 2\right)\alpha^{2\beta}$$

Thus, (2.1) is satisfied by (2.27) and (2.29).

2.7 Set :

The choice

$$Q = kP \tag{2.29}$$

in (2.2) gives

$$\left(2k^2 + 1\right)P^2 = x^{2\,s+1}$$

which is satisfied by

$$P = \left(2k^2 + 1\right)^s \alpha^{(2\,s+1)\beta} \tag{2.30}$$

and

$$x = \left(2k^2 + 1\right)\alpha^{2\beta} \tag{2.31}$$

From (2.29), it is seen that

$$\mathrm{Q} = \mathrm{k}\left(2\mathrm{k}^2+1\right)^{\mathrm{s}}\alpha^{(2\ \mathrm{s}+1)\beta} \tag{2.32}$$

Using (2.30), (2.31) and (2.32) in (2.3), we have

$$\mathrm{w} = \left(2\mathrm{k}^2+1\right)^{\mathrm{s}}\alpha^{(2\ \mathrm{s}+1)\beta} + \left(2\mathrm{k}^2+1\right)\alpha^{2\beta}$$

$$\mathrm{z} = \mathrm{k}\left(2\mathrm{k}^2+1\right)^{\mathrm{s}}\alpha^{(2\ \mathrm{s}+1)\beta} + \left(2\mathrm{k}^2+1\right)\alpha^{2\beta} \tag{2.33}$$

Thus, (2.1) is satisfied by (2.31) and (2.33).

3 Conclusion

In this paper, we have made an attempt to obtain all integer solutions to $w^2+2z^2-2wx-4zx = x^{2s+1}-3x^2, s>0$. To conclude, one may search for integer solutions to other choices of homogeneous or non-homogeneous higher degree equations with multiple variables.

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