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Structural Analysis of Pessimistic Multi-Granulation Nano Generalized Topological Spaces*

A.A. Abdallah , O. R. Sayed, E. El-Sanousy, Y. H. Ragheb Sayed , M. A. Bakhit , and Salahuddin †

ABSTRACT: This study establishes a fundamental connection between multi-granulation rough sets and topological spaces. By partitioning a universal set, we construct a novel pessimistic multi-granulation nanogeneric topological structure. We demonstrate that this topology is directly induced by the family of equivalence relations underlying a multi-granulation approximation space. A key finding is that the pessimistic multi-granulation lower and upper approximation operators are isomorphic to the nanogeneric interior and closure operators, respectively. Furthermore, we specify the conditions under which these operators are equivalent, formally bridging the concepts of approximation in rough set theory with topological interior and closure.

Key Words: The pessimistic multi-granulation nanogeneric topology, the pessimistic multi-granulation nanogeneric interior, the pessimistic multi-granulation nanogeneric closure.

Contents

1. Conceptual background

Rough set theory, introduced by Pawlak [9,10], is a useful mathematical tool for studying intelligent systems with incomplete or insufficient information. Thivagar [14] introduced nanotopology, a concept that incorporates lower approximation, upper approximation, and boundary regions.

The multi-granulation rough set developed by Qian et al. [11,12] differs from Pawlak's rough set. The multi-granulation rough set is defined by a family of equivalence relations, whereas Pawlak's rough set is defined by a single relation. The multi-granulation rough set comprises both the optimistic and pessimistic multi-granulation rough sets. In the lower approximation, the word "optimistic" denotes the idea that in multi-independent granular structures, at least one granular structure must satisfy the inclusion relation between the equivalence class and the undefinable set, whereas the word "pessimistic" denotes the idea that each granular structure must satisfy the inclusion relation between the equivalence class and the undefinable set. Several researchers investigated multi-granulation rough set models based on various types of relationships and came up with interesting ideas (see, for example, [1], [2], [3], [4], [5], [7], [8], [13]).
Inspired by the works of [1]- [15], our aim in this paper is to explore pessimistic multi-granulation nanogeneric topological spaces. We will investigate the connection between multi-granulation approximation operators and pessimistic multi-granulation nanogeneric closure (interior) operators.

For the purposes of this paper, consider $U = \{x_1, x_2, \cdots, x_n\}$ to be a nonempty and finite set of objects. 2^U represents all subsets of U, and $\mathcal{R}$ is a family of equivalence relations on U. The pair $(U, \mathcal{R})$

* Corresponding author. *Email address*: smohammad@jazanu.edu.sa
† Corresponding author.
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is described as a multi-granulation approximation space. For $x \in U$, $[x]_{R_i} = \{y \in U : (x,y) \in R_i\}$ is the equivalence class that includes x where $R_i \in \mathcal{R}$. Then $U/R_i = \{[x]_{R_i} : x \in U\}$ creates a partition of U.

Definition 1.1. [9,10] Assume $R_i \in \mathcal{R}$. The pair (U, R_i) is called a Pawlak approximation space. For any $X \subseteq U$, the lower and upper approximations of X on R_i are defined

$$L_{R_i}(X) = \{x \in U : [x]_{R_i} \subseteq X\},$$
$$U_{R_i}(X) = \{x \in U : [x]_{R_i} \cap X \neq \emptyset\}.$$

Definition 1.2. [11,12] Let $(U, \mathcal{R})$ be a multi-granulation approximation space and $R_1, R_2, R_3, \cdots, R_m \in \mathcal{R}$, where m is a natural number. For any $X \subseteq U$, the pessimistic multi-granulation lower and upper approximations of X on the family of equivalence relations $\{R_i : 1 \leq i \leq m\}$ are defined

$$PL_{\sum_{i=1}^{m} R_i}(X) = \{x \in U : [x]_{R_1} \subseteq X \text{ and } [x]_{R_2} \subseteq X \text{ and } \cdots \text{ and } [x]_{R_m} \subseteq X\},$$

$$\begin{aligned} PU_{\sum_{i=1}^{m} R_i}(X) &= \sim (PL_{\sum_{i=1}^{m} R_i}(\sim X)) \\ &= \{x \in U : [x]_{R_1} \cap X \neq \emptyset \text{ or } [x]_{R_2} \cap X \neq \emptyset \ldots \text{ or } [x]_{R_m} \cap X \neq \emptyset\}. \end{aligned}$$

Then $(PL_{\sum_{i=1}^{m} R_i}(X), PU_{\sum_{i=1}^{m} R_i}(X))$ is called the pessimistic multi-granulation rough set of X. The term "pessimistic" refers to the fact that $x \in PL_{\sum_{i=1}^{m} R_i}(X)$ if and only if for all equivalence relation R_i,

$$[x]_{R_i} \subseteq X, \forall x \in U \text{ and } 1 \leq i \leq m.$$

The pessimistic multi-granulation boundary region of X denoted by $PBR_{\sum_{i=1}^{m} R_i}(X)$ is defined as follows:

$$PBR_{\sum_{i=1}^{m} R_i}(X) = PU_{\sum_{i=1}^{m} R_i}(X) - PL_{\sum_{i=1}^{m} R_i}(X).$$

Proposition 1.3. [12] Let $(U, \mathcal{R})$ be a multi-granulation approximation space. For any $X, Y \subseteq U$, the following properties hold:

(1) $PL_{\sum_{i=1}^{m} R_i}(X) \subseteq X \subseteq PU_{\sum_{i=1}^{m} R_i}(X)$.

(2) $PL_{\sum_{i=1}^{m} R_i}(\emptyset) = PU_{\sum_{i=1}^{m} R_i}(\emptyset) = \emptyset$ and $PL_{\sum_{i=1}^{m} R_i}(U) = PU_{\sum_{i=1}^{m} R_i}(U) = U$.

(3) $PL_{\sum_{i=1}^{m} R_i}(\sim X) = \sim (PU_{\sum_{i=1}^{m} R_i}(X))$ and $PU_{\sum_{i=1}^{m} R_i}(\sim X) = \sim (PL_{\sum_{i=1}^{m} R_i}(X))$.

(4) $PL_{\sum_{i=1}^{m} R_i}(PL_{\sum_{i=1}^{m} R_i}(X)) = PU_{\sum_{i=1}^{m} R_i}(PL_{\sum_{i=1}^{m} R_i}(X)) = PL_{\sum_{i=1}^{m} R_i}(X)$.

(5) $PU_{\sum_{i=1}^{m} R_i}(PU_{\sum_{i=1}^{m} R_i}(X)) = PL_{\sum_{i=1}^{m} R_i}(PU_{\sum_{i=1}^{m} R_i}(X)) = PU_{\sum_{i=1}^{m} R_i}(X)$.

(6) $X \subseteq Y \Longrightarrow PL_{\sum_{i=1}^{m} R_i}(X) \subseteq PL_{\sum_{i=1}^{m} R_i}(Y)$.

(7) $X \subseteq Y \Longrightarrow PU_{\sum_{i=1}^{m} R_i}(X) \subseteq PU_{\sum_{i=1}^{m} R_i}(Y)$.

(8) $PL_{\sum_{i=1}^{m} R_i}(X \cap Y) \subseteq PL_{\sum_{i=1}^{m} R_i}(X) \cap PL_{\sum_{i=1}^{m} R_i}(Y)$ or
$PU_{\sum_{i=1}^{m} R_i}(X \cup Y) \supseteq PU_{\sum_{i=1}^{m} R_i}(X) \cup PU_{\sum_{i=1}^{m} R_i}(Y)$.

(9) $PL_{\sum_{i=1}^{m} R_i}(X \cup Y) \supseteq PL_{\sum_{i=1}^{m} R_i}(X) \cup PL_{\sum_{i=1}^{m} R_i}(Y)$ and
$PU_{\sum_{i=1}^{m} R_i}(X \cap Y) \subseteq PU_{\sum_{i=1}^{m} R_i}(X) \cap PU_{\sum_{i=1}^{m} R_i}(Y)$.

(10) $PL_{\sum_{i=1}^{m} R_i}(X) = \cap_i L_{R_i}(X)$ and $PU_{\sum_{i=1}^{m} R_i}(X) = \cup_i U_{R_i}(X)$ for $1 \le i \le m$.

2. Main results

Definition 2.1. Let $(U, \mathcal{R})$ be a multi-granulation approximation space, with $X \subseteq U$ and $\mathcal{T}_{\sum_{i=1}^{m} R_i}(X) = \{\emptyset, PL_{\sum_{i=1}^{m} R_i}(X), PU_{\sum_{i=1}^{m} R_i}(X), PBR_{\sum_{i=1}^{m} R_i}(X)\}$. Then $\mathcal{T}_{\sum_{i=1}^{m} R_i}(X)$ satisfies the following axioms:

(A1) $\emptyset \in \mathcal{T}_{\sum_{i=1}^{m} R_i}(X)$.

(A2) The union of elements in any subcollection of $\mathcal{T}_{\sum_{i=1}^{m} R_i}(X)$ is in $\mathcal{T}_{\sum_{i=1}^{m} R_i}(X)$.

That is, $\mathcal{T}_{\sum_{i=1}^{m} R_i}(X)$ forms a generic topology on U, called the pessimistic multi-granulation nanogeneric topology with respect to X. The pair $(U, \mathcal{T}_{\sum_{i=1}^{m} R_i}(X))$ is called the pessimistic multi-granulation nanogeneric topological space.

The elements of $\mathcal{T}_{\sum_{i=1}^{m} R_i}(X)$ are called the pessimistic multi-granulation nanogeneric open sets. $A \in \mathcal{T}_{\sum_{i=1}^{m} R_i}(X)$, $\sim A$ the complement of A, is known as the pessimistic multi-granulation nanogeneric closed set. $\sim \mathcal{T}_{\sum_{i=1}^{m} R_i}(X)$ is the family of complements for all pessimistic multi-granulation nanogeneric open sets.

Definition 2.2. Let $(U, \mathcal{T}_{\sum_{i=1}^{m} R_i}(X))$ be the pessimistic multi-granulation nanogeneric topological space with respect to X, where $X \subseteq U$. The pessimistic multi-granulation nanogeneric interior of Y is defined as:

$$PMN\ int(Y) = \cup\{A \in 2^U : A \subseteq Y, A \in \mathcal{T}_{\sum_{i=1}^{m} R_i}(X)\}.$$

The pessimistic multi-granulation nanogeneric closure of Y can be defined as:

$$PMN\ cl(Y) = \cap\{A \in 2^U : Y \subseteq A, A \in \sim \mathcal{T}_{\sum_{i=1}^{m} R_i}(X)\}.$$

Example 2.3. Let $U = \{a, b, c, d, e\}$, $P = U/R_1 = \{\{a\}, \{b, c\}, \{d, e\}\}$, $Q = U/R_2 = \{\{a, b\}, \{d\}, \{c\}, \{e\}\}$, $L = U/R_3 = \{\{a, d\}, \{b\}, \{c\}, \{e\}\}$, and $X = \{b, c\}$. Then,

$$PL_{P+Q+L}(X) = \{c\},$$

Also, we can find

$$PU_{P+Q+L}(X) = \{a, b, c\}, PBR_{P+Q+L}(X) = \{a, b\}.$$

Thus, the pessimistic multi-granulation nanogeneric topology on U with respect to X is

$$\mathcal{T}_{\sum_{i=1}^{m} R_i}(X) = \{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}.$$

Let $Y = \{a, b, c, d\}$, $Z = \{b, c, d, e\} \subseteq U$. Then, $PMN\ int(Z) = \{c\}$ and $PMN\ int(Y \cap Z) = \{c\}$. So

$$\{c\} = PMN\ int(Y \cap Z) \subseteq PMN\ int(Y) \cap PMN\ int(Z) = \{a, b, c\} \cap \{c\} = \{c\}.$$

Theorem 2.4. Let $(U, \mathcal{T}_{\sum_{i=1}^{m} R_i}(X))$ be the pessimistic multi-granulation nanogeneric topological space, with respect to $X \subseteq U$ and $Y, Z \subseteq U$. Then the following statements hold:

(I1): $PMN\ int(U) \subseteq U$ and $PMN\ int(\emptyset) = \emptyset$.
(I2): $PMN\ int(Y) \subseteq Y$.
(I3): If $Y \subseteq Z$ then $PMN\ int(Y) \subseteq PMN\ int(Z)$.
(I4): $PMN\ int(PMN\ int(Y)) = PMN\ int(Y)$.
(I5): $PMN\ int(Y \cap Z) \subseteq PMN\ int(Y) \cap PMN\ int(Z)$.

Theorem 2.5. Let $(U, \mathcal{T}_{\sum_{i=1}^{m} R_i}(X))$ be the pessimistic multi-granulation nanogeneric topological space with respect to $X \subseteq U$ and $Y, Z \subseteq U$. Then, the following statements hold:

(C1): $PMN\ cl(U) = U$ and $\emptyset \subseteq PMN\ cl(\emptyset)$.
(C2): $Y \subseteq PMN\ cl(Y)$.
(C3): If $Y \subseteq Z$ then $PMN\ cl(Y) \subseteq PMN\ cl(Z)$.
(C4): $PMN\ cl(PMN\ cl(Y)) = PMN\ cl(Y)$.
(C5): $PMN\ cl(Y) \cup PMN\ cl(Z) \subseteq PMN\ cl(Y \cup Z)$.

Theorem 2.6. Let $(U, \mathcal{T}_{\sum_{i=1}^{m} R_i}(X))$ be the pessimistic multi-granulation nanogeneric topological space with respect to X. For $Y \subseteq U$, define $(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{int}$ as follows:

$$(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{int} = \{Y \subseteq U : PMN\ int(Y) = Y\}.$$

Then $(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{int}$ is the pessimistic multi-granulation nanogeneric topology on U with respect to X.

Proof. We prove that $(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{int}$ satisfies **(A1)-(A2)**.

(A1): According to Theorem 2.4 **(I1)**,

$$\emptyset \in (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{int}.$$

(A2): Assume that $Y_j \in (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{int}$. Then,

$$PMN\ int(Y_j) = Y_j, \forall j \in \Gamma.$$

By **(I2)**, we have

$$PMN\ int(\cup_{j\in\Gamma} Y_j) \subseteq \cup_{j\in\Gamma} Y_j. \tag{2.1}$$

Conversely, $PMN\ int(Y_j) \subseteq \cup_{j\in\Gamma} PMN\ int(Y_j)$. By Theorem 2.4 **(I3)** and **(I4)**, we have

$$PMN\ int(Y_j) = PMN\ int(PMN\ int(Y_j)) \subseteq PMN\ int(\cup_{j\in\Gamma} PMN\ int(Y_j)).$$

Thus,

$$\cup_{j\in\Gamma} PMN\ int(Y_j) \subseteq PMN\ int(\cup_{j\in\Gamma} PMN\ int(Y_j)).$$

Since $PMN\ int(Y_j) = Y_j$, we have

$$\cup_{j\in\Gamma} Y_j \subseteq PMN\ int(\cup_{j\in\Gamma} Y_j). \tag{2.2}$$

Using equations (2.1) and (2.2), we get

$$PMN\ int(\cup_{j\in\Gamma} Y_j) = \cup_{j\in\Gamma} Y_j.$$

Therefore,

$$\cup_{j\in\Gamma} Y_j \in (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{int}.$$

Hence, $(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{int}$ represents the pessimistic multi-granulation nanogeneric topology on U with respect to X.

Theorem 2.7. Let $(U, \mathcal{T}_{\sum_{i=1}^{m} R_i}(X))$ be the pessimistic multi-granulation nanogeneric topological space with respect to X. For $Y \subseteq U$, define $(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{cl}$ as follows:

$$(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{cl} = \{Y \subseteq U : PMN\ cl(\sim Y) = \sim Y\}.$$

Then $(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{cl}$ is the pessimistic multi-granulation nanogeneric topology on U with respect to X.

Proof. We show that $(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{cl}$ satisfies **(A1)-(A2)**.

(A1): Based on Theorem 2.5 **(J1)**, we have

$$PMN\ cl(\sim \emptyset) = PMN\ cl(U) = U = \sim \emptyset.$$

Therefore,

$$\emptyset \in (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{cl}.$$

(A2): Consider that $Y_j \in (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{cl}$. For each $j \in \Gamma$,

$$PMN\ cl(\sim Y_j) = \sim Y_j.$$

By **(C2)**, we have

$$\sim (\cup_{j\in\Gamma} Y_j) \subseteq PMN\ cl(\sim (\cup_{j\in\Gamma} Y_j)). \tag{2.3}$$

Conversely, assume that $PMN\ cl(\sim (\cup_{j\in\Gamma} Y_j)) = PMN\ cl(\cap_{j\in\Gamma}(\sim Y_j)) \subseteq PMN\ cl(\sim Y_j)$. Therefore,

$$PMN\ cl(\sim (\cup_{j\in\Gamma} Y_j)) \subseteq \cap_{j\in\Gamma} PMN\ cl(\sim Y_j) = \cap_{j\in\Gamma}(\sim Y_j) = \sim (\cup_{j\in\Gamma} Y_j). \tag{2.4}$$

Equations (2.3) and (2.4) show that $PMN\ cl(\sim (\cup_{j\in\Gamma} Y_j)) = \sim (\cup_{j\in\Gamma} Y_j)$.
Therefore,

$$\cup_{j\in\Gamma} Y_j \in (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{cl}.$$

Consequently, $(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{cl}$ denotes the pessimistic multi-granulation nanogeneric topology on U with respect to X.

Theorem 2.8. Let $(U, \mathcal{T}_{\sum_{i=1}^{m} R_i}(X))$ be the pessimistic multi-granulation nanogeneric topological space with respect to X. For $Y \subseteq U$, define $(\mathcal{T}^*_{\sum_{i=1}^{m} R_i}(X))_{int}$ as follows:

$$(\mathcal{T}^*_{\sum_{i=1}^{m} R_i}(X))_{int} = \{PMN\ int(Y) : Y \subseteq U\}.$$

Then, $(\mathcal{T}^*_{\sum_{i=1}^{m} R_i}(X))_{int} = (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{int}$.

Proof. It therefore follows that

$$(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{int} = \{Y \subseteq U : PMN\ int(Y) = Y\} \subseteq (\mathcal{T}^*_{\sum_{i=1}^{m} R_i}(X))_{int}.$$

Conversely, for each $Y \subseteq U$, let

$$PMN\ int(Y) \in (\mathcal{T}^*_{\sum_{i=1}^{m} R_i}(X))_{int}.$$

Then, using Theorem 2.4 **(I4)**, we have

$$PMN\ int(PMN\ int(Y)) = PMN\ int(Y).$$

Thus, we get

$$PMN\ int(Y) \in (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{int}.$$

Hence,

$$(\mathcal{T}^*_{\sum_{i=1}^{m} R_i}(X))_{int} \subseteq (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{int}.$$

Therefore,

$$(\mathcal{T}^*_{\sum_{i=1}^{m} R_i}(X))_{int} = (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{int}.$$

Theorem 2.9. Let $(U, \mathcal{T}_{\sum_{i=1}^{m} R_i}(X))$ be the pessimistic multi-granulation nanogeneric topological space with respect to X. For $Y \subseteq U$, define $(\mathcal{T}^*_{\sum_{i=1}^{m} R_i}(X))_{cl}$ as follows:

$$(\mathcal{T}^*_{\sum_{i=1}^{m} R_i}(X))_{cl} = \{\sim (PMN\ cl(Y)) : Y \subseteq U\}.$$

Then, $(\mathcal{T}^*_{\sum_{i=1}^{m} R_i}(X))_{cl} = (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{cl}$.

Proof. Let $\sim (PMN\ cl(Y)) \in (\mathcal{T}^*_{\sum_{i=1}^{m} R_i}(X))_{cl}$ for each $Y \subseteq U$. Then,

$$\begin{aligned} PMN\ cl(\sim (\sim (PMN\ cl(Y)))) &= PMN\ cl(PMN\ cl(Y)) \\ &= PMN\ cl(Y) \\ &= \sim (\sim (PMN\ cl(Y))). \end{aligned}$$

Thus,

$$\sim (PMN\ cl(Y)) \in (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{cl}.$$

It suggests that

$$(\mathcal{T}^*_{\sum_{i=1}^{m} R_i}(X))_{cl} \subseteq (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{cl}.$$

Conversely, suppose that $Y \in (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{cl}$. Then, we get

$$Y = \sim (PMN\ cl(\sim Y)).$$

Consequently,

$$Y = \sim (PMN\ cl(\sim Y)) \in (\mathcal{T}^*_{\sum_{i=1}^{m} R_i}(X))_{cl}.$$

It implies that

$$(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{cl} \subseteq (\mathcal{T}^*_{\sum_{i=1}^{m} R_i}(X))_{cl}.$$

Hence,

$$(\mathcal{T}^*_{\sum_{i=1}^{m} R_i}(X))_{cl} = (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{cl}.$$

Theorem 2.10. Let $(U, \mathcal{R})$ be a multi-granulation approximation space, and $(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{\mathcal{R}} = \{X \subseteq U : PL_{\sum_{i=1}^{m} R_i}(X) = X\}$. Then $(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{\mathcal{R}}$ is the pessimistic multi-granulation nanogeneric topology with respect to X.

Proof. We show that $(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{\mathcal{R}}$ satisfies **(A1)** and **(A2)**.

(A1): According to Proposition 1.3 (2), we have

$$PL_{\sum_{i=1}^{m} R_i}(\emptyset) = \emptyset.$$

It suggests that

$$\emptyset \in (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{\mathcal{R}}.$$

(A2): Let $Y_j \in (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{\mathcal{R}}$, where $j \in \Gamma$. Then,

$$PL_{\sum_{i=1}^{m} R_i}(Y_j) = Y_j.$$

Using Proposition 1.3 (1), we get

$$PL_{\sum_{i=1}^{m} R_i}(\cup_j Y_j) \subseteq \cup_j Y_j \text{ for all } j \in \Gamma. \tag{2.5}$$

Conversely, using Proposition 1.3 (6), we obtain

$$Y_j = PL_{\sum_{i=1}^{m} R_i}(Y_j) \subseteq PL_{\sum_{i=1}^{m} R_i}(\cup_j Y_j).$$

It suggests that

$$\cup_j Y_j \subseteq PL_{\sum_{i=1}^{m} R_i}(\cup_j Y_j) \text{ for all } j \in \Gamma. \tag{2.6}$$

From equations (2.5) and (2.6), we obtain

$$\cup_j Y_j = PL_{\sum_{i=1}^{m} R_i} \cup_j Y_j.$$

Hence,

$$\cup_j Y_j \in (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{\mathcal{R}}.$$

Theorem 2.11. Let $(U, \mathcal{R})$ be a multi-granulation approximation space, and $(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{\mathcal{R}} = \{X \subseteq U : PU_{\sum_{i=1}^{m} R_i}(\sim X) = \sim X\}$. Then $(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{\mathcal{R}}$ is the pessimistic multi-granulation nanogeneric topology with respect to X.

Proof. Similar to prove Theorem 2.10.

Theorem 2.12. Let $(U, \mathcal{R})$ be the multi-granulation approximation space. Then for $X \subseteq U$,

$$(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{\mathcal{R}} = \{PL_{\sum_{i=1}^{m} R_i}(X) : X \subseteq U\}.$$

Proof. It is clear that

$$\begin{aligned}(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{\mathcal{R}} &= \{X \subseteq U : PL_{\sum_{i=1}^{m} R_i}(X) = X\} \\ &\subseteq \{PL_{\sum_{i=1}^{m} R_i}(X) : X \subseteq U\}.\end{aligned}$$

According to Proposition 1.3 (4), we have

$$PL_{\sum_{i=1}^{m} R_i}(PL_{\sum_{i=1}^{m} R_i}(X)) = PL_{\sum_{i=1}^{m} R_i}(X).$$

It implies that

$$PL_{\sum_{i=1}^{m} R_i}(X) \in (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{\mathcal{R}}.$$

Therefore,

$$\{PL_{\sum_{i=1}^{m} R_i}(X) : X \subseteq U\} \subseteq (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{\mathcal{R}}.$$

Hence,

$$(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{\mathcal{R}} = \{PL_{\sum_{i=1}^{m} R_i}(X) : X \subseteq U\}.$$

According to Theorem 2.12, a family of equivalence relations can generate the pessimistic multi-granulation nanogeneric topology, which is simply the family of all pessimistic multi-granulation lower approximations induced by the given family of equivalence relations.

Lemma 2.13. Let $(U, \mathcal{R})$ be a multi-granulation approximation space. Then for $X_j \subseteq U$,

$$PL_{\sum_{i=1}^{m} R_i}(\cup_j PL_{\sum_{i=1}^{m} R_i}(X_j)) = \cup_j PL_{\sum_{i=1}^{m} R_i}(X_j) \text{ with } j \in \Gamma.$$

Proof. Using Proposition 1.3 (1), we get

$$PL_{\sum_{i=1}^{m} R_i}(\cup_j PL_{\sum_{i=1}^{m} R_i}(X_j)) \subseteq \cup_j PL_{\sum_{i=1}^{m} R_i}(X_j). \tag{2.7}$$

On the other hand, since $PL_{\sum_{i=1}^{m} R_i}(X_j) \subseteq \cup_j(PL_{\sum_{i=1}^{m} R_i}(X_j))$. Then by Proposition 1.3 (6), we get

$$PL_{\sum_{i=1}^{m} R_i}(PL_{\sum_{i=1}^{m} R_i}(X_j)) \subseteq PL_{\sum_{i=1}^{m} R_i}(\cup_j PL_{\sum_{i=1}^{m} R_i}(X_j)).$$

According to Proposition 1.3 (4), we have

$$PL_{\sum_{i=1}^{m} R_i}(X_j) \subseteq PL_{\sum_{i=1}^{m} R_i}(\cup_j PL_{\sum_{i=1}^{m} R_i}(X_j)).$$

It implies that

$$\cup_j PL_{\sum_{i=1}^{m} R_i}(X_j) \subseteq PL_{\sum_{i=1}^{m} R_i}(\cup_j PL_{\sum_{i=1}^{m} R_i}(X_j)). \tag{2.8}$$

From equations (2.7) and (2.8), we have

$$PL_{\sum_{i=1}^{m} R_i}(\cup_j PL_{\sum_{i=1}^{m} R_i}(X_j)) = \cup_j PL_{\sum_{i=1}^{m} R_i}(X_j).$$

The following theorem investigates the relationship between the pessimistic multi-granulation nanogeneric closure (interior) operator and the pessimistic multi-granulation upper (lower) approximation.

Theorem 2.14. Let $(U, (\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{\mathcal{R}})$ be the pessimistic multi-granulation nanogeneric topological space induced by $\mathcal{R}$ with respect to $(U, \mathcal{R})$, i. e.

$$(\mathcal{T}_{\sum_{i=1}^{m} R_i}(X))_{\mathcal{R}} = \{PL_{\sum_{i=1}^{m} R_i}(X) : X \subseteq U\}.$$

Then, for $X \subseteq U$,

$$\begin{aligned} (1) \quad PL_{\sum_{i=1}^{m} R_i}(X) &= PMN\ int(X) \\ &= \cup\{PL_{\sum_{i=1}^{m} R_i}(Y) : PL_{\sum_{i=1}^{m} R_i}(Y) \subseteq X, Y \subseteq U\}. \end{aligned}$$

$$\begin{aligned} (2) \quad PU_{\sum_{i=1}^{m} R_i}(X) &= PMN\ cl(X) \\ &= \cap\{PU_{\sum_{i=1}^{m} R_i}(\sim Y) : X \subseteq PU_{\sum_{i=1}^{m} R_i}(\sim Y), Y \subseteq U\}. \end{aligned}$$

Proof. (1) By Proposition 1.3 (1), $PL_{\sum_{i=1}^{m} R_i}(X) \subseteq X$. So,

$$PL_{\sum_{i=1}^{m} R_i}(X) \subseteq \cup\{PL_{\sum_{i=1}^{m} R_i}(Y) : PL_{\sum_{i=1}^{m} R_i}(Y) \subseteq X, Y \subseteq U\}. \tag{2.9}$$

Conversely, assume that $\cup\{PL_{\sum_{i=1}^{m} R_i}(Y) : PL_{\sum_{i=1}^{m} R_i}(Y) \subseteq X, Y \subseteq U\} \subseteq X$. Therefore,

$$PL_{\sum_{i=1}^{m} R_i}\left(\cup\{PL_{\sum_{i=1}^{m} R_i}(Y) : PL_{\sum_{i=1}^{m} R_i}(Y) \subseteq X, Y \subseteq U\}\right) \subseteq PL_{\sum_{i=1}^{m} R_i}(X).$$

By Lemma 2.13, we have

$$\cup\{PL_{\sum_{i=1}^{m} R_i}(Y) : PL_{\sum_{i=1}^{m} R_i}(Y) \subseteq X, Y \subseteq U\} \subseteq PL_{\sum_{i=1}^{m} R_i}(X). \tag{2.10}$$

From (2.9) and (2.10), we can conclude

$$PL_{\sum_{i=1}^{m} R_i}(X) = \cup\{PL_{\sum_{i=1}^{m} R_i}(Y) : PL_{\sum_{i=1}^{m} R_i}(Y) \subseteq X, Y \subseteq U\}.$$

(2) Similar to prove (1).
The proof is complete.

3. Conclusion

This research has advanced the theoretical foundation of multi-granulation rough sets (MGRS) by establishing a formal bridge to topological structures. Building upon the seminal framework introduced by Qian et al. [11,12], we have proposed and rigorously defined the concept of pessimistic multi-granulation nanogeneric topological spaces.

Our investigation yielded several key findings:

Structural Foundation: We demonstrated that the family of equivalence relations, which forms the core of a multi-granulation approximation space, is sufficient to induce a unique pessimistic multi-granulation nanogeneric topological space. This establishes a direct structural link between the two paradigms.

Operator Equivalence: A central result of this work is the proven equivalence between the fundamental operators of these two structures. Specifically, we showed that the pessimistic multi-granulation lower approximation operator is functionally identical to the nanogeneric interior operator, and similarly, the upper approximation operator is identical to the nanogeneric closure operator.

Unified Framework: This equivalence is not merely a formal curiosity; it provides a powerful, unified lens through which to view MGRS. It allows us to reinterpret approximation problems—a cornerstone of rough set theory—in terms of well-established topological concepts like interior, closure, and boundary.

Implications and Future Work: The implications of this unification are significant. It enables the application of topological theorems and methodologies to problems in granular computing and knowledge discovery. Future research will focus on exploring the optimistic variant of these topological structures, investigating the properties of these topologies (e.g., separation axioms, compactness), and developing practical applications in domains such as feature selection, data mining, and uncertainty reasoning, where the pessimistic multi-granulation approach is most applicable.

4. Declaration statements

Availability of data and material: The data sets used and/or analysed during the current study are available from the corresponding author upon reasonable request.

Conflict of interests: No potential conflict of interest was reported by the author(s).

Consent for publication: Informed consent was obtained from all individual participants included in the study.

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Ethics and Consent: The authors stated that none of the research presented in this article involved either human subjects or animals.

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5. Author's Biography

Author Dr. Abd-Allah Ali Abd-Allah Nassr is currently an Assistant Professor in the Department of Mathematics at Jazan University, Jazan, Saudi Arabia. He earned his undergraduate, graduate, and master's degrees, as well as his PhD in Mathematics, all from the Department of Mathematics at Sohag University in Sohag, Egypt. Dr. Nassr has published several papers in the field of Fuzzy Topology and has held several managerial positions at Jazan University.

Author Dr. M. A. Bakhit serves as an Associate Professor in the Department of Mathematics at Jazan University, Saudi Arabia. Renowned for his prolific research contributions, Dr. Bakhit's scholarly work predominantly focuses on complex functional analysis, with particular emphasis on operator theory, harmonic mappings, and their diverse applications. He completed his undergraduate, graduate, and master's studies at Sohag University in Sohag, Egypt, and subsequently earned his PhD from Al-Azhar University, Assiut, Egypt. In recognition of his academic excellence, he was awarded the Prof. Dr. Bahaa El-Din Helmy Ismail Award for Best Master's Degree by the Egyptian Mathematical Society in 2010. Dr. Bakhit is dedicated to fostering collaborative research endeavors and mentoring emerging researchers, guiding them in leveraging advanced mathematical theories to address intricate real-world challenges. His academic portfolio encompasses nearly fifty research articles published in esteemed international journals and conference proceedings. In addition, Dr. Bakhit demonstrates steadfast commitment to academic leadership, contributing substantially to the governance of the Department of Mathematics and the academic operations of the College of Science at Jazan University through various administrative and advisory roles.

Author Dr. Salahuddin, a Professor in the Department of Mathematics at Jazan University, Saudi Arabia, is a researcher specialising in variational inequality theory. His work centres on fixed point theory, topological structures, and their diverse applications. Dr. Salahuddin's significant scholarly output includes over 300 papers published in reputable international journals and conferences, establishing him as a leading voice in his field. Beyond his research, Dr. Salahuddin is committed to academic leadership and the broader impact of mathematics. He has actively contributed to Jazan University's administration through various managerial roles. His passion lies in fostering collaborative research and exploring how advanced mathematical theories can address complex real-world challenges.

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A.A. Abdallah,
Department of Mathematics,
College of Science, Jazan University, Jazan-45142, P.O. Box 114,
Saudi Arabia.
E-mail address: anassr@jazanu.edu.sa

and

O. R. Sayed,
Department of Mathematics,
Faculty of Science, Assiut University, Assiut 71516,
EGYPT.
E-mail address: o_sayed@aun.edu.eg

and

E. El-Sanousy,
Department of Mathematics,
Department of Mathematics, Faculty of Science, Sohag University, Sohag, 82524,
EGYPT.
E-mail address: elsanowsy@yahoo.com

and

Y. H. Ragheb Sayed,
Department of Mathematics,
Faculty of Science, University of Hafr Al Batin, Hafar Al Batin 39524,
Saudi Arabia.
E-mail address: yhassanien@uhb.edu.sa

and

M. A. Bakhit,
Department of Mathematics,
College of Science, Jazan University, Jazan-45142, P.O. Box 114,
Saudi Arabia.
E-mail address: mabakhit@jazanu.edu.sa

and

Salahuddin,
Department of Mathematics,
College of Science, Jazan University, Jazan-45142, P.O. Box 114,
Saudi Arabia.
E-mail address: smohammad@jazanu.edu.sa