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Boundedness Analysis for a Coupled ϕ-Caputo Fractional System Using Measures of Noncompactness

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ABSTRACT: This paper investigates a system of two coupled nonlinear fractional differential equations with ϕ-Caputo derivatives of orders $q_1, q_2 \in (1, 2)$. The system includes proportional delays and nonlocal initial conditions to capture memory and hereditary effects. Solutions are studied in a Banach space using Kuratowski-type measures of noncompactness combined with Sadovskii's fixed point theorem. Existence and boundedness of solutions are established under suitable continuity, growth, and Lipschitz conditions. An integral reformulation and noncompactness techniques address challenges from nonlinearity and coupling. A numerical example illustrates the results, confirming the framework's applicability. Future work will consider stability, bifurcation, and extensions to multivariate or spatially distributed models.

Keywords: Solutions, boundedness, coupled, fractional, nonlocal.

Contents

1. Introduction

The theory of derivatives of non-integer order, as a natural generalization of classical differential calculus, has witnessed significant progress in recent decades owing to its ability to describe complex phenomena with memory and hereditary properties [18,21,17,23]. This ability to incorporate nonlocal temporal behavior makes fractional calculus an essential tool for modeling numerous real-world phenomena in physics, biology, finance, and engineering [13,26]. Among various proposed generalizations, the ϕ-Caputo fractional derivative introduced by Almeida [3] stands out for its exceptional flexibility. This derivative operator unifies several well-known fractional operators—including Caputo, Caputo-Hadamard, and more recent derivatives—through appropriate selection of the kernel function ϕ. The generalized formulation provides a unified framework that extends the scope of analytical and numerical methods for fractional differential equations [5,12] while enabling new approaches to problems with nonlocal initial or boundary conditions that depend globally on solution trajectories, particularly in abstract spaces.

Coupled systems of fractional differential equations play a fundamental role in modeling complex phenomena where multiple interdependent variables evolve simultaneously [1,2]. Such systems appear in diverse domains including population dynamics, electrical networks, viscoelastic processes, and economic modeling [11]. The study of these systems within Banach spaces permits analysis of abstract problems with broad applications while accounting for complex interactions and memory effects [4,24]. However, despite growing interest in fractional derivatives, the analysis of coupled systems involving ϕ-Caputo derivatives with nonlocal initial conditions remains underdeveloped. Most existing work focuses on special cases like classical Caputo or Hilfer derivatives [13], rarely addressing coupled systems with nonlocal initial conditions in a general abstract framework.

This paper investigates a coupled system of nonlinear fractional differential equations with ϕ-Caputo derivatives on a finite interval $[0, T]$:

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$$\begin{cases} D^{q_1,\phi_1}\eta(t) = f_1(t,\eta(t),\zeta(\varepsilon t)), \\ D^{q_2,\phi_2}\zeta(t) = f_2(t,\zeta(t),\eta(\varepsilon t)), \\ \eta'(0) = \zeta'(0) = 0, \\ \eta(0) + g_1(\eta) = \eta_0, \\ \zeta(0) + g_2(\zeta) = \zeta_0, \end{cases} \tag{1.1}$$

where $q_1, q_2 \in (1,2)$ are fractional orders, $\eta, \zeta : [0,T] \to \mathbb{R}$ are functions taking values in the real Banach space $\mathbb{R}$, $\phi_1, \phi_2 \in C^2([0,T])$ are strictly increasing functions with $\phi_i'(t) > 0$, and $\varepsilon \in (0,1)$ is a proportional delay parameter. The ϕ-Caputo fractional derivatives are defined as:

$$^{C}D^{q_i,\phi_i} f(t) = \frac{1}{\Gamma(2-q_i)} \int_0^t \phi_i'(s)(\phi_i(t)-\phi_i(s))^{1-q_i} \frac{d^2}{ds^2} f(s)\, ds, \quad i = 1,2.$$

The nonlinear functions $f_1, f_2 : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ model coupled delay terms, while the operators $g_1, g_2 : C([0,T],\mathbb{R}) \to \mathbb{R}$ represent nonlocal initial conditions dependent on global solution trajectories [11]. This general formulation accommodates various non-classical boundary conditions common in applications.

Under suitable continuity, growth, and Lipschitz conditions on f_i and g_i, we establish existence, uniqueness, positivity, and boundedness results for solutions to system (1.1). Our methodology combines advanced fixed point theory with noncompactness measure techniques to overcome challenges posed by the system's nonlinearity and coupled nature. Specifically, we employ an adapted version of Sadovskii's fixed point theorem suitable for noncompact operators. [19,10,9].

This study makes three key contributions: First, the unified ϕ-Caputo framework extends fractional differential operators, encompassing special cases in a coherent approach. Second, the Banach space setting provides flexibility and broad applicability across concrete models. Third, the novel combination of noncompactness measures with fixed point theorems offers a rigorous tool for analyzing coupled nonlinear fractional equations.

A concrete example with specific nonlinear functions and nonlocal conditions demonstrates our theoretical results and illustrates practical applicability. The paper is organized as follows: Section 2 presents preliminary definitions and tools, Section 3 presents the main results on existence and boundedness of solutions, Section 4 provides numerical implementation, and Section 5 discusses conclusions and future directions. This work fills a literature gap by establishing new results for general coupled systems while offering potential applications to concrete problems in applied sciences.

2. Preliminaries

In this section, we present the main notions from generalized fractional calculus relevant to our work, with a focus on the ϕ-Caputo type operators for integration and differentiation [3,5,12]. We also define the functional spaces employed throughout the paper and recall the analytical tools—such as those related to measures of noncompactness—needed for the study of the considered system [4,15,24].

Fractional Integrals and Derivatives of ϕ-Type

Let $K = [0,T] \subset \mathbb{R}$ be a fixed interval. We consider a function $\phi \in \mathcal{C}^n(K,\mathbb{R})$ that is strictly monotone increasing, meaning that $\phi'(t) > 0$ for every $t \in K$ [3].

Definition 2.1 *[3,21] Let $\xi > 0$. If μ is an integrable function on K, then the ϕ-Riemann–Liouville fractional integral of order ξ is defined by:*

$$I^{\xi,\phi}\mu(t) = \frac{1}{\Gamma(\xi)} \int_0^t \phi'(p) \left(\phi(t) - \phi(p)\right)^{\xi-1} \mu(p)\, dp.$$

Definition 2.2 *[3,21] Let $\xi > 0$ and set $n = \lfloor \xi \rfloor$. The ϕ-Riemann–Liouville fractional derivative is defined by:*

$$D^{\xi,\phi}\mu(t) = \left(\frac{1}{\phi'(t)}\frac{d}{dt}\right)^n \left[\frac{1}{\Gamma(n-\xi)}\int_0^t \phi'(p)\left(\phi(t)-\phi(p)\right)^{n-\xi-1}\mu(p)\,dp\right],$$

or equivalently,

$$D^{\xi,\phi}\mu = \left(\frac{1}{\phi'}\frac{d}{dt}\right)^n \left(I^{n-\xi,\phi}\mu\right).$$

Definition 2.3 *[3,5,12] Let $\xi > 0$ and define $n = \lfloor \xi \rfloor$. If $\mu \in \mathcal{C}^{n-1}(K,\mathbb{R})$ and $\phi \in \mathcal{C}^n(K)$ is such that $\phi'(t) > 0$ for all $t \in K$, the generalized Caputo fractional derivative with respect to ϕ is defined by*

$${}^C D^{\xi,\phi}\mu(t) = \frac{1}{\Gamma(n-\xi)}\int_0^t \phi'(p)\left(\phi(t)-\phi(p)\right)^{n-\xi-1}\mu_\phi^{[n]}(p)\,dp,$$

where

$$\mu_\phi^{[n]}(p) := \left(\frac{1}{\phi'(p)}\frac{d}{dp}\right)^n \mu(p).$$

Remark 2.1 *When $\phi(t) = t$, the operator ${}^C D^{\xi,\phi}$ reduces to the standard Caputo derivative [18]. For the particular choice $\phi(t) = \ln(t)$, the above definition reduces to the Caputo–Hadamard fractional derivative [18].*

Remark 2.2 *In the particular case $0 < \xi < 1$, one can write [3]*

$${}^C D^{\xi,\phi}\mu(t) = \frac{1}{\Gamma(1-\xi)}\left(\frac{1}{\phi'(t)}\frac{d}{dt}\right)\int_0^t \left(\phi(t)-\phi(p)\right)^{-\xi}\mu(p)\,dp.$$

Theorem 2.1 *[3] Let $\xi > 0$ and $\mu \in \mathcal{C}^{n-1}(K)$. The following two equalities are satisfied:*

1. ${}^C D^{\xi,\phi}\big(I^{\xi,\phi}\mu\big)(t) = \mu(t)$,
2. $I^{\xi,\phi}\big({}^C D^{\xi,\phi}\mu\big)(t) = \mu(t) - \displaystyle\sum_{i=0}^{n-1}\frac{\mu_\phi^{[i]}(0)}{i!}\big(\phi(t)-\phi(0)\big)^i$.

Theorem 2.2 *[21] Let $z_1 > z_2 > 0$ and $p \in \mathbb{N}$. Then:*

1. $I^{z_1,\phi}\big(\phi(t)-\phi(0)\big)^{z_2-1} = \dfrac{\Gamma(z_2)}{\Gamma(z_1+z_2)}\big(\phi(t)-\phi(0)\big)^{z_1+z_2-1}$,
2. $D^{z_1,\phi}\big(\phi(t)-\phi(0)\big)^{z_2-1} = \dfrac{\Gamma(z_2)}{\Gamma(z_2-z_1)}\big(\phi(t)-\phi(0)\big)^{z_2-z_1-1}$,
3. $D^{z_1,\phi}\big(\phi(t)-\phi(0)\big)^p = 0$ *for all* $p < n \in \mathbb{N}$.

Kuratowski Measure of Noncompactness and Associated Operators

Within the setting of nonlinear fractional differential equations subject to nonlocal conditions, the related integral operators are typically non-compact. To overcome this difficulty, the measure of noncompactness theory provides an effective tool for proving the existence of solutions via generalized fixed point results [4,15,20]. We now present the definition and fundamental properties of the Kuratowski measure of noncompactness.

Definition 2.4 *[15,4] Let M be a Banach space and let $\mathcal{B}_M$ denote the collection of all nonempty bounded subsets of M. The* Kuratowski measure of noncompactness *is the function $\rho : \mathcal{B}_M \to [0,+\infty)$ given by*

$$\rho(Y) := \inf\left\{ r > 0 \;\middle|\; \exists Y_1,\ldots,Y_n \subset M \text{ with } Y \subset \bigcup_{i=1}^n Y_i \text{ and } \operatorname{diam}(Y_i) \le r\ \forall i \right\},$$

where the diameter of a set $A \subset M$ is defined as $\operatorname{diam}(A) := \sup\{\|k_1 - k_2\| \,:\, k_1, k_2 \in A\}$.

Theorem 2.3 *[4] The Kuratowski measure of noncompactness ρ enjoys the following properties:*

1. *$\rho(F) = 0$ iff F is relatively compact in M.*
2. *For any scalar $a \in \mathbb{R}$, one has $\rho(aF) = |a|\,\rho(F)$.*
3. *For all $F_1, F_2 \in \mathcal{B}_M$, it holds that $\rho(F_1 + F_2) \le \rho(F_1) + \rho(F_2)$.*
4. *If $F_1 \subset F_2$, then $\rho(F_1) \le \rho(F_2)$.*
5. *For any $F_1, F_2 \in \mathcal{B}_M$, $\rho(F_1 \cup F_2) = \max\{\rho(F_1), \rho(F_2)\}$.*
6. *For every $F \in \mathcal{B}_M$, $\rho(F) = \rho(\overline{F}) = \rho(\mathrm{conv}(F))$, where $\overline{F}$ denotes the closure of F and $\mathrm{conv}(F)$ its convex hull.*

Definition 2.5 *[20] Let $\Theta : Y \subset M \to M$ be a bounded and continuous operator. We call Θ ρ-*Lipschitzian *if there exists a constant $\ell \ge 0$ such that*

$$\rho\big(\Theta(F^*)\big) \le \ell\,\rho(F^*), \quad \text{for every nonempty bounded subset } F^* \subset F.$$

*The number ℓ is referred to as the ρ-*Lipschitz constant *associated with Θ.*

Definition 2.6 *[20] A continuous function $\Theta : F \to M$ is said to be ρ-condensing whenever, for every bounded subset $F^* \subset F$ with $\rho(F^*) > 0$, we have strictly:*

$$\rho(\Theta(F^*)) < \rho(F^*).$$

We denote by $\mathcal{C}_\rho(F)$ the set of ρ-condensing functions from F into M.

Definition 2.7 *[7] Let $f : [d,e] \times D \to D$, where D is a metric space (or a Banach space). The function f is said to satisfy the* Carathéodory conditions *if the following hold:*

- *For each $\kappa \in D$, the function $\kappa \mapsto f(\kappa, u)$ is measurable on $[d,e]$;*
- *For almost every $\kappa \in [d,e]$, the function $u \mapsto f(\kappa, u)$ is continuous on D.*

Definition 2.8 *[9] A mapping $\mu : Y \to X$ is called Lipschitz if there exists a constant $\ell > 0$ such that*

$$\|\mu(y_1) - \mu(y_2)\| \le \ell \|y_1 - y_2\|, \quad \forall y_1, y_2 \in Y.$$

Lemma 2.1 *[4] Any Lipschitz mapping $\mu : F \to M$ with Lipschitz constant ℓ Moreover, the mapping Θ also satisfies the ρ-Lipschitz condition with the same constant ℓ.*

Lemma 2.2 *[4] Let $\Theta, \Delta : F \to M$ be two ρ-Lipschitz mappings with respective constants ℓ_1 and ℓ_2. Then:*

- *The mapping $\Theta + \Delta : F \to M$ is ρ-Lipschitz with constant $\ell_1 + \ell_2$.*
- *If Θ is compact, then $\ell_1 = 0$.*

Topological Degree Theorem for ρ-Condensing Mappings

The following results are adapted from the theory of topological degree for ρ-condensing operators [20, 25,24]

Theorem 2.4 *[20,25] Let M be a Banach space and $F \subset M$ a bounded open subset. Assume $\Theta : F \to M$ is a continuous ρ-condensing mapping with $\Theta \in \mathcal{C}_\rho(\overline{F})$.*

Consider the set of admissible triples defined by

$$\mathcal{T} := \left\{ (I - \Theta, F, j) \;\middle|\; j \in M \setminus \mathrm{int}\big((I - \Theta)(\partial F)\big) \right\}.$$

Then there exists a mapping, called the topological degree *corresponding to ρ, denoted by*

$$\deg : \mathcal{T} \longrightarrow \mathbb{Z},$$

which satisfies the following properties:

1. *For any* $s \in F$, $\deg(I, F, s) = 1$.

2. *If* F_1 *and* F_2 *are disjoint open subsets of* F *such that*

$$j \notin (I - \Theta)(\overline{F} \setminus (F_1 \cup F_2)),$$

then

$$\deg(I - \Theta, F, j) = \deg(I - \Theta, F_1, j) + \deg(I - \Theta, F_2, j).$$

3. *Let* $f : [0,1] \times \overline{F} \to M$ *be a continuous bounded mapping satisfying*

$$\rho\big(f([0,1] \times \Omega)\big) < \rho(\Omega), \quad \forall \Omega \subset \overline{F},\ \rho(\Omega) > 0.$$

If $j : [0,1] \to M$ *is a continuous function such that*

$$j(t) \neq z - f(\kappa, z), \quad \forall z \in \partial F, \quad \forall \kappa \in [0,1],$$

then the degree

$$\deg(I - f(\kappa, \cdot), F, j(t))$$

remains constant for all $\kappa \in [0,1]$.

4. *If* $\deg(I - \Theta, F, j) \neq 0$, *there exists* $z \in F$ *such that*

$$j = (I - \Theta)(z),$$

i.e., j *lies in the image* $(I - \Theta)(F)$.

5. *If* $F_1 \subset F$ *is an open subset and*

$$j \notin (I - \Theta)(\overline{F} \setminus F_1),$$

then

$$\deg(I - \Theta, F, j) = \deg(I - \Theta, F_1, j).$$

Theorem 2.5 *[20,25] Let* $\Theta : M \to M$ *be a continuous* ρ*-condensing operator. For each* $\delta \in [0,1]$*, define the set*

$$E_\delta := \{\, j \in M \mid j = \delta\, \Theta(j) \,\}.$$

If all the sets E_δ *are bounded, then there exists a closed ball* $B_q(0) \subset M$ *of radius* $q > 0$ *that contains* E_δ *for every* $\delta \in [0,1]$*. Moreover, for each* δ,

$$\deg(I - \delta\Theta, B_q(0), 0) = 1.$$

Consequently, the operator Θ *admits at least one fixed point; that is, there exists* $j \in B_q(0)$ *such that*

$$\Theta(j) = j.$$

Theorem 2.6 *[19] Let* M *be a Banach space and let* $B \subset M$ *be a nonempty, closed, bounded, and convex set. Assume that* $T : B \to B$ *is a continuous mapping that is* ρ*-contractive in the sense that there exists a measure of noncompactness* λ *on* M *satisfying*

$$\lambda(T(A)) \leq h\, \lambda(A) \quad \text{for every bounded subset } A \subset B,$$

with some constant $0 \leq h < 1$*. Then* T *has at least one fixed point in* B*.*

Armed with these tools, we are now prepared to reformulate system (1.1) in its integral form, thereby enabling the application of fixed point theory within a suitable functional framework.

3. Main Results

Functional space

The solutions (η, ζ) are sought in the functional space:

$$\mathcal{C} := C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R}), \tag{3.1}$$

endowed with the maximum norm:

$$\|(\eta,\zeta)\|_{\mathcal{C}} := \max\{\|\eta\|_\infty, \|\zeta\|_\infty\}. \tag{3.2}$$

Hypotheses on the conditions:

(H1) There exist positive constants L_{g_i} satisfying:

$$\|g_i(\eta) - g_i(\tilde{\eta})\| \leq L_{g_i}\|\eta - \tilde{\eta}\| \quad \text{for each } \eta, \tilde{\eta} \in \mathcal{C}, \quad i = 1,2. \tag{3.3}$$

(H2) There exist positive constants N_{g_i} and nonnegative constants M_{g_i} such that:

$$\|g_i(\eta)\| \leq N_{g_i}\|\eta\|^\alpha + M_{g_i}, \quad \alpha \in [0,1]. \tag{3.4}$$

(H3) The functions $f_i : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is measurable in the first argument t and continuous in the second and third arguments (η, ζ).

(H4) There exist essentially bounded functions $\lambda_{f_i} \in L^\infty([0,T],\mathbb{R}_+)$, for $i = 1,2$, such that

$$\|f_i(t,\eta,\zeta)\| \leq \lambda_{f_i}(t)\,\varphi_i\big(\|\eta\| + \|\zeta\|\big),$$

where $\varphi_i : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, nondecreasing functions.

(H5) There exist positive constants N_{f_i} and nonnegative constants M_{f_i} such that:

$$\|f_i(t,\eta(t),\zeta(\varepsilon t))\| \leq N_{f_i}\|(\eta,\zeta)\|^\beta + M_{f_i}, \quad \beta \in [0,1]. \tag{3.5}$$

Lemma 3.1 *Solving problem (1.1) is equivalent to finding functions that satisfy the following system of integral equations:*

$$\begin{cases} \eta(t) = \eta_0 - g_1(\eta) + \dfrac{1}{\Gamma(q_1)} \displaystyle\int_0^t \phi_1'(s)\big(\phi_1(t) - \phi_1(s)\big)^{q_1-1} f_1\big(s,\eta(s),\zeta(\varepsilon s)\big)\,ds, \\ \zeta(t) = \zeta_0 - g_2(\zeta) + \dfrac{1}{\Gamma(q_2)} \displaystyle\int_0^t \phi_2'(s)\big(\phi_2(t) - \phi_2(s)\big)^{q_2-1} f_2\big(s,\zeta(s),\eta(\varepsilon s)\big)\,ds. \end{cases} \tag{3.6}$$

Proof: Consider a function $f \in C^2([0,T],\mathbb{R})$. The ϕ-Caputo derivative of order $q_i \in (1,2)$ is given by:

$${}^C D_{0^+}^{q_i;\phi_i} f(t) = I_{0^+}^{2-q_i;\phi_i}\left(\frac{1}{\phi_i'(t)}\frac{d}{dt}\right)^2 f(t),$$

where $I_{0^+}^{\alpha;\phi_i}$ denotes the Riemann–Liouville fractional integral associated with ϕ_i. The general solution of ${}^C D_{0^+}^{q_i;\phi_i} f(t) = F(t)$ is:

$$f(t) = c_0 + c_1(\phi_i(t) - \phi_i(0)) + I_{0^+}^{q_i;\phi_i} F(t).$$

Applying this to the first equation of the system, using the initial conditions $\eta'(0) = 0$ and $\eta(0) + g_1(\eta) = \eta_0$, we obtain:

$$\eta(t) = \eta_0 - g_1(\eta) + \frac{1}{\Gamma(q_1)}\int_0^t \phi_1'(s)(\phi_1(t) - \phi_1(s))^{q_1-1} f_1(s,\eta(s),\zeta(\varepsilon s))\,ds,$$

with $c_1 = 0$ and $c_0 = \eta_0 - g_1(\eta)$. The same reasoning applies to the second equation.

Since the functions f_i are continuous and the functions ϕ_i are sufficiently regular, the delay term $\zeta(\varepsilon t)$ is well defined for all $t \in [0, T]$, with $\varepsilon \in (0, 1)$ and $\zeta \in C([0, T], \mathbb{R})$.

We introduce the operators on $\mathcal{C}$, namely the initial condition operator $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ and the integral operator $\mathcal{T} : \mathcal{C} \to \mathcal{C}$, defined respectively by:

$$\mathcal{A}(\eta, \zeta) := (\eta_0 - g_1(\eta),\ \zeta_0 - g_2(\zeta)), \tag{3.7}$$

$$\mathcal{T}(\eta, \zeta)(t) := \left(\frac{1}{\Gamma(q_1)} \int_0^t \phi_1'(s) f_1(s, \eta(s), \zeta(\varepsilon s))\, ds, \quad \frac{1}{\Gamma(q_2)} \int_0^t \phi_2'(s) f_2(s, \zeta(s), \eta(\varepsilon s))\, ds \right). \tag{3.8}$$

The fractional system can then be expressed in the compact operator form:

$$\mathcal{G}(\eta, \zeta) = \mathcal{A}(\eta, \zeta) + \mathcal{T}(\eta, \zeta).$$

The system is thus rewritten in the compact operator form:

$$\mathcal{G}(\eta, \zeta) = \begin{pmatrix} \eta_0 - g_1(\eta) + I^{q_1;\phi_1} f_1(t, \eta(t), \zeta(\varepsilon t)) \\ \zeta_0 - g_2(\zeta) + I^{q_2;\phi_2} f_2(t, \zeta(t), \eta(\varepsilon t)) \end{pmatrix}$$

□

Lemma 3.2 *The operator $\mathcal{A}$ possesses the ρ-Lipschitz property with constant $L_g := \max\{L_{g_1}, L_{g_2}\}$, and it additionally satisfies the inequality:*

$$\|\mathcal{A}(\eta, \zeta)\|_{\mathcal{C}} \le |\eta_0| + |\zeta_0| + N_g \|(\eta, \zeta)\|_{\mathcal{C}}^{\alpha} + M_g, \quad \forall (\eta, \zeta) \in \mathcal{C},$$

where $N_g := \max\{N_{g_1}, N_{g_2}\}$ and $M_g := \max\{M_{g_1}, M_{g_2}\}$.

Proof: Let $(\eta_1, \zeta_1), (\eta_2, \zeta_2) \in \mathcal{C}$. By the definition of $\mathcal{A}$ together with (H1), we obtain:

$$\begin{aligned} \|\mathcal{A}(\eta_1, \zeta_1) - \mathcal{A}(\eta_2, \zeta_2)\|_{\mathcal{C}} &= \max\{\|g_1(\eta_1) - g_1(\eta_2)\|,\ \|g_2(\zeta_1) - g_2(\zeta_2)\|\} \\ &\le \max\{L_{g_1} \|\eta_1 - \eta_2\|_\infty,\ L_{g_2} \|\zeta_1 - \zeta_2\|_\infty\} \\ &= L_g \|(\eta_1, \zeta_1) - (\eta_2, \zeta_2)\|_{\mathcal{C}}. \end{aligned}$$

Moreover, from (H2),

$$\|\mathcal{A}(\eta, \zeta)\|_{\mathcal{C}} = \max\{|\eta_0 - g_1(\eta)|,\ |\zeta_0 - g_2(\zeta)|\} \le |\eta_0| + |\zeta_0| + N_g \|(\eta, \zeta)\|_{\mathcal{C}}^{\alpha} + M_g.$$

□

Lemma 3.3 *Let $\mathcal{T}$ be the operator given by (3.8). Then $\mathcal{T}$ satisfies the following properties:*

1. *$\mathcal{T}$ is continuous.*
2. *For every $(\eta, \zeta) \in \mathcal{C}$,*

$$\|\mathcal{T}(\eta, \zeta)\|_{\mathcal{C}} \le \frac{\big(\phi(T) - \phi(0)\big)^{q^*}}{\Gamma(q^* + 1)} \left(N_f \|(\eta, \zeta)\|_{\mathcal{C}}^{\beta} + M_f \right), \tag{3.9}$$

where $q^* = \max\{q_1, q_2\}$ and $\phi = \max\{\phi_1, \phi_2\}$.

Proof: Let $(\eta_n, \zeta_n) \to (\eta, \zeta)$ be a sequence in $\mathcal{C}$. We want to show that:

$$\mathcal{T}(\eta_n, \zeta_n) \to \mathcal{T}(\eta, \zeta).$$

We denote the two components of the operator:

$$\mathcal{T}_1(\eta,\zeta)(t) = \frac{1}{\Gamma(q_1)} \int_0^t \phi_1'(s)\, f_1(s,\eta(s),\zeta(\varepsilon s))\, ds,$$

and

$$\mathcal{T}_2(\eta,\zeta)(t) = \frac{1}{\Gamma(q_2)} \int_0^t \phi_2'(s)\, f_2(s,\zeta(s),\eta(\varepsilon s))\, ds.$$

By continuity of the functions f_1 and f_2, and since $\eta_n \to \eta$ and $\zeta_n \to \zeta$ uniformly on $[0,T]$, we have pointwise convergence:

$$f_i(s,\eta_n(s),\zeta_n(\varepsilon s)) \to f_i(s,\eta(s),\zeta(\varepsilon s)), \quad \text{uniformly in } s \in [0,T],\ i = 1,2.$$

Moreover, thanks to the growth hypothesis on f_i, there exists an integrable majorant:

$$|f_i(s,\eta_n(s),\zeta_n(\varepsilon s))| \le N_{f_i}(\|\eta_n\|^\beta + \|\zeta_n\|^\beta) + M_{f_i},$$

Moreover, since this bound is uniform in n, we can thus apply the dominated convergence theorem to conclude that:

$$\mathcal{T}_i(\eta_n,\zeta_n)(t) \to \mathcal{T}_i(\eta,\zeta)(t), \quad \text{uniformly on } [0,T],\ i = 1,2.$$

Thus, we then obtain:

$$\mathcal{T}(\eta_n,\zeta_n) \to \mathcal{T}(\eta,\zeta) \quad \text{in } C([0,T],\mathbb{R})$$

thereby establishing the continuity of the operator $\mathcal{T}$.

For the second property, we define the norm:

$$\|\mathcal{T}(\eta,\zeta)\| := \max\{\|\mathcal{T}_1(\eta,\zeta)\|_\infty, \|\mathcal{T}_2(\eta,\zeta)\|_\infty\}.$$

By hypothesis on f_1, we have:

$$|f_1(s,\eta(s),\zeta(\varepsilon s))| \le N_{f_1}\|(\eta,\zeta)\|^\beta + M_{f_1}.$$

Then we get:

$$\begin{aligned}
|\mathcal{T}_1(\eta,\zeta)(t)| &\le \frac{1}{\Gamma(q_1)} \int_0^t \phi_1'(s)|f_1(s,\eta(s),\zeta(\varepsilon s))|\, ds \\
&\le \frac{N_{f_1}}{\Gamma(q_1)} \int_0^t \phi_1'(s)(|\eta(s)|^\beta + |\zeta(\varepsilon s)|^\beta)\, ds + \frac{M_{f_1}}{\Gamma(q_1)} \int_0^t \phi_1'(s)\, ds \\
&\le \frac{N_{f_1}\|(\eta,\zeta)\|^\beta}{\Gamma(q_1+1)}(\phi_1(t)-\phi_1(0))^{q_1} + \frac{M_{f_1}}{\Gamma(q_1+1)}(\phi_1(t)-\phi_1(0))^{q_1} \\
&\le \frac{(\phi_1(T)-\phi_1(0))^{q_1}}{\Gamma(q_1+1)} \left(N_{f_1}\|(\eta,\zeta)\|^\beta + M_{f_1}\right).
\end{aligned}$$

Thus we have:

$$\|\mathcal{T}_1(\eta,\zeta)\|_\infty \le \frac{(\phi_2(T)-\phi_2(0))^{q_1}}{\Gamma(q_1+1)} \left(N_{f_1}\|(\eta,\zeta)\|^\beta + M_{f_1}\right).$$

A similar reasoning applies to $\mathcal{T}_2$, yielding:

$$\|\mathcal{T}_2(\eta,\zeta)\|_\infty \le \frac{(\phi_2(T)-\phi_2(0))^{q_2}}{\Gamma(q_2+1)} \left(N_{f_2}\|(\eta,\zeta)\|^\beta + M_{f_2}\right).$$

Setting $q^* = \max\{q_1,q_2\}$ and $\phi = \max\{\phi_1,\phi_2\}$, and considering the supremum over $t \in [0,T]$, it follows that:

$$\|\mathcal{T}(\eta,\zeta)\|_{\mathcal{C}} \le \frac{(\phi(T)-\phi(0))^{q^*}}{\Gamma(q^*+1)} \left(N_f\|(\eta,\zeta)\|_{\mathcal{C}}^\beta + M_f\right).$$

□

Lemma 3.4 *The operator $\mathcal{T}$ is a compact mapping.*

Proof: To establish the **compactness** of the operator $\mathcal{T}$, we take a closed ball B_r of radius $r > 0$ and show that the image $\mathcal{T}(B_r)$ is **relatively compact** in $\mathbb{R}$. The proof is based on the *Arzelà–Ascoli theorem*, which requires checking that the set is both *uniformly bounded* and *equicontinuous*.

1. Boundedness.
Under the growth assumptions on the functions f_1 and f_2, there exist constants $N_{f_i}, M_{f_i} \geq 0$ and an exponent $\beta > 0$ such that:

$$|f_i(t, \eta, \zeta)| \leq N_{f_i}(|\eta|^\beta + |\zeta|^\beta) + M_{f_i}, \quad \text{for } i = 1, 2.$$

For every $(\eta, \zeta) \in B_r$, It can be seen that:

$$|f_i(t, \eta(t), \zeta(\varepsilon t))| \leq N_{f_i}(2r^\beta) + M_{f_i},$$

and therefore, for any $t \in [0, T]$:

$$|\mathcal{T}_i(\eta, \zeta)(t)| \leq \frac{N_{f_i}(2r^\beta) + M_{f_i}}{\Gamma(q_i)} (\phi_i(t) - \phi_i(0))^{q_i} \leq C_i,$$

with C_i being a constant that depends only on r, the data of the problem, and the functions ϕ_i. Therefore,

$$\|\mathcal{T}(\eta, \zeta)\| \leq C_1 + C_2,$$

which shows that the image $\mathcal{T}(B_r)$ is **uniformly bounded** in $\mathbb{R}$.

2. Equicontinuity.
Consider $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, and $(\eta, \zeta) \in B_r$. It follows that:

$$\begin{aligned} |\mathcal{T}_i(\eta, \zeta)(t_2) - \mathcal{T}_i(\eta, \zeta)(t_1)| &= \left| \frac{1}{\Gamma(q_i)} \int_{t_1}^{t_2} \phi_i'(s) f_i(s, \eta(s), \zeta(\varepsilon s))\, ds \right| \\ &\leq \frac{N_{f_i}(2r^\beta) + M_{f_i}}{\Gamma(q_i)} \big(\phi_i(t_2) - \phi_i(t_1)\big)^{q_i}. \end{aligned}$$

The continuity of ϕ_i ensures that this difference vanishes as $t_2 \to t_1$, **uniformly** for all $(\eta, \zeta) \in B_r$. Therefore, $\mathcal{T}(B_r)$ forms an **equicontinuous** set.

Moreover, since $\mathcal{T}(B_r)$ is both **bounded** and **equicontinuous** in a Banach space of continuous functions, the *Arzelà–Ascoli theorem* implies that it is **relatively compact**. This establishes that the operator $\mathcal{T}$ is **compact**. □

Corollary 3.1 *The operator $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is ρ-Lipschitz with constant equal to zero.*

Proof: By virtue of the compactness of the operator $\mathcal{T}$ and Lemma 2.2, it follows that $\mathcal{T}$ satisfies

$$\lambda(\mathcal{T}(B)) = 0 \leq 0 \cdot \lambda(B),$$

for any bounded set B. This corresponds to a ρ-Lipschitz constant $\rho = 0$, showing that $\mathcal{T}$ is a ρ-condensing operator with zero Lipschitz constant.

□

Theorem 3.1 *Under the assumptions* (H1)–(H5)*, the coupled fractional differential system of ϕ-Caputo type described in* (1) *admits at least one solution $(\eta, \zeta) \in \mathcal{C}$. In addition, the collection of solutions is bounded in the space $\mathcal{C}([0, T], \mathbb{R})^2$.*

Proof: Consider the space $\mathcal{C} = C([0,T],\mathbb{R})^2$, endowed with the norm:

$$\|(\eta,\zeta)\| := \max\{\|\eta\|_\infty, \|\zeta\|_\infty\}.$$

We consider the operators defined on $\mathcal{C}$:

- $\mathcal{A}(\eta,\zeta) = \big(\eta_0 - g_1(\eta),\ \zeta_0 - g_2(\zeta)\big)$,
- $\mathcal{T}(\eta,\zeta)(t) = \left(\frac{1}{\Gamma(q_1)}\int_0^t \phi_1'(s) f_1(s,\eta(s),\zeta(\varepsilon s))ds,\ \frac{1}{\Gamma(q_2)}\int_0^t \phi_2'(s) f_2(s,\zeta(s),\eta(\varepsilon s))ds\right)$,
- $\mathcal{G} := \mathcal{A} + \mathcal{T}$.

From assumptions (H1) and (H2), together with Lemma 3.2, the operator $\mathcal{A}$ is ρ-Lipschitzian with constant $L_g < 1$. Using Lemma 3.4, the operator $\mathcal{T}$ is compact and therefore ρ-Lipschitzian with zero constant (see Corollary 3.1). Then, applying the composition lemma for ρ-Lipschitzian operators (Lemma 2.2), we conclude that $\mathcal{G}$ is strictly ρ-condensing with constant $\rho = L_g < 1$.

Step 1: Construction of the homotopy set E_δ.
For $\delta \in [0,1]$, define:

$$E_\delta := \{(\eta,\zeta) \in \mathcal{C} : (\eta,\zeta) = \delta\,\mathcal{G}(\eta,\zeta)\}.$$

We show that E_δ is bounded in $\mathcal{C}$.
Let $(\eta,\zeta) \in E_\delta$ for some $\delta \in [0,1]$, then:

$$(\eta,\zeta) = \delta\mathcal{G}(\eta,\zeta) = \delta\,\mathcal{A}(\eta,\zeta) + \delta\,\mathcal{T}(\eta,\zeta).$$

Taking norms yields:

$$\|(\eta,\zeta)\| \le \delta\|\mathcal{A}(\eta,\zeta)\| + \delta\|\mathcal{T}(\eta,\zeta)\|.$$

By the growth assumptions (H2) on g_i and (H5) on f_i, there exist constants $N_g, M_g, N_f, M_f \ge 0$ and exponents $\alpha,\beta \in [0,1)$ such that:

$$\|\mathcal{A}(\eta,\zeta)\| \le |\eta_0| + |\zeta_0| + N_g\|(\eta,\zeta)\|^\alpha + M_g,$$
$$\|\mathcal{T}(\eta,\zeta)\| \le \frac{(\phi(T)-\phi(0))^{q^*}}{\Gamma(q^*+1)}\left(N_f\|(\eta,\zeta)\|^\beta + M_f\right),$$

where $q^* = \max(q_1,q_2)$ and $\phi = \max(\phi_1,\phi_2)$.
Hence,

$$\|(\eta,\zeta)\| \le \delta\left(|\eta_0| + |\zeta_0| + N_g\|(\eta,\zeta)\|^\alpha + M_g + \frac{(\phi(T)-\phi(0))^{q^*}}{\Gamma(q^*+1)}(N_f\|(\eta,\zeta)\|^\beta + M_f)\right). \tag{3.10}$$

Step 2: Proof by contradiction.
Argue by contradiction by considering a sequence $(\eta_n,\zeta_n) \in E_{\delta_n}$ for which

$$\|(\eta_n,\zeta_n)\| \to +\infty.$$

Dividing inequality (3.10) by $\|(\eta_n,\zeta_n)\|$ and taking limits yields:

$$1 \le \lim_{\xi\to\infty} \frac{|\eta_0| + |\zeta_0| + N_g\xi^\alpha + M_g + \frac{(\phi(T)-\phi(0))^{q^*}}{\Gamma(q^*+1)}(N_f\xi^\beta + M_f)}{\xi} = 0,$$

since $\alpha,\beta < 1$, which is a contradiction. Therefore, E_δ is bounded in $\mathcal{C}$ for all $\delta \in [0,1]$.

Step 3:Application of the fixed point result for ρ-condensing mappings.
As $\mathcal{G}$ is continuous, strictly ρ-condensing, and E_δ remains bounded for every $\delta \in [0,1]$, the application of

Sadovskii's fixed point theorem ensures the existence of a fixed point $(\eta, \zeta) \in \mathcal{C}$, which corresponds to a solution of the system.

Step 4: Boundedness of the solution set.
The inequality above also shows that any fixed point (η, ζ) of $\mathcal{G}$ satisfies:

$$\|(\eta, \zeta)\| \leq R,$$

for a certain constant $R > 0$, determined solely by the problem data, thereby guaranteeing that the set of solutions is uniformly bounded. □

Consequently, under the hypotheses (H1)–(H5), the fractional system (1.1) possesses at least one solution in the space $\mathcal{C}([0,T], \mathbb{R})^2$. This solution arises as a fixed point of the operator $\mathcal{G} = \mathcal{A} + \mathcal{T}$, and the set of all such solutions is uniformly bounded. This existence result lays the foundation for subsequent qualitative investigations (such as stability and bifurcation) and for the numerical simulation of the model.

Remark 3.1 *In particular, if we take $\alpha = \beta = 1$ in assumptions (H2) and (H5), the fixed point existence condition for the operator F simplifies to the following:*

$$N_g + N_f \frac{(\phi(T) - \phi(0))^{q^*}}{\Gamma(q^* + 1)} < 1.$$

This inequality ensures that $\mathcal{G}$ is a strict contraction on $\mathcal{C}$.

4. Example

Let us consider the following fractional system subject to nonlocal initial conditions:

$$\begin{cases} D^{\frac{7}{5}, e^t} \eta(t) = \dfrac{t^2}{50} \big(\eta(t) + \sin(t)\big) + \dfrac{1}{5} \sqrt{\pi} \cos\big(\zeta(\tfrac{t}{\sqrt{3}})\big), \quad t \in [0, 1], \\ D^{\frac{5}{3}, t^2+1} \zeta(t) = \dfrac{e^{1-t}}{20} \big(\zeta(t) + \arctan(t)\big) + \dfrac{\cos\big(\eta(\tfrac{t}{\sqrt{3}})\big)}{\pi + t^3}, \\ \eta'(0) = 0, \quad \zeta'(0) = 0, \\ \eta(0) = \displaystyle\sum_{j=1}^{10} \gamma_j \, |\eta(t_j)|, \quad \gamma_j > 0, \quad 0 < t_j < 1, \quad j = 1, \ldots, 10, \\ \zeta(0) = \displaystyle\sum_{j=1}^{10} \beta_j \, |\zeta(t_j)|, \quad \beta_j > 0, \quad 0 < t_j < 1, \quad j = 1, \ldots, 10. \end{cases} \tag{4.1}$$

The fractional orders are

$$q_1 = \frac{7}{5} = 1.4, \quad q_2 = \frac{5}{3} \approx 1.6667,$$

and the weight functions are given by

$$\phi_1(t) = e^t, \quad \phi_2(t) = t^2 + 1.$$

The functions ϕ_i are strictly increasing and differentiable on $[0, 1]$.
The fractional orders $q_1, q_2 \in (1, 2)$ suit the ϕ-Caputo derivatives of order greater than 1.

Verification of hypotheses (H1) and (H2)

Consider the nonlocal functions defined by:

$$g_1(\eta) = \sum_{j=1}^{10} \gamma_j |\eta(t_j)|, \quad g_2(\zeta) = \sum_{j=1}^{10} \beta_j |\zeta(t_j)|,$$

where $\gamma_j, \beta_j > 0$, $t_j \in (0,1)$, and

$$S_\gamma := \sum_{j=1}^{10} \gamma_j < 1, \quad S_\beta := \sum_{j=1}^{10} \beta_j < 1.$$

For all $\eta_1, \eta_2 \in C([0,1],\mathbb{R})$, we have:

$$\begin{aligned}|g_1(\eta_1) - g_1(\eta_2)| &= \left|\sum_{j=1}^{10} \gamma_j \big(|\eta_1(t_j)| - |\eta_2(t_j)|\big)\right| \\ &\leq \sum_{j=1}^{10} \gamma_j |\eta_1(t_j) - \eta_2(t_j)| \\ &\leq S_\gamma \|\eta_1 - \eta_2\|_\infty,\end{aligned}$$

which shows that g_1 is Lipschitz continuous with constant

$$L_{g_1} = S_\gamma < 1.$$

Moreover,

$$|g_1(\eta)| = \left|\sum_{j=1}^{10} \gamma_j |\eta(t_j)|\right| \leq \sum_{j=1}^{10} \gamma_j \|\eta\|_\infty = S_\gamma \|\eta\|_\infty,$$

giving controlled growth with

$$N_{g_1} = S_\gamma, \quad M_{g_1} = 0, \quad \alpha = 1.$$

Similarly, for g_2, we have

$$|g_2(\zeta_1) - g_2(\zeta_2)| \leq S_\beta \|\zeta_1 - \zeta_2\|_\infty, \quad L_{g_2} = S_\beta < 1,$$

and

$$|g_2(\zeta)| \leq S_\beta \|\zeta\|_\infty, \quad N_{g_2} = S_\beta, \quad M_{g_2} = 0, \quad \alpha = 1.$$

We now choose the sums

$$\sum_{j=1}^{10} \gamma_j =: S_\gamma = 0.18 < 1, \qquad \sum_{j=1}^{10} \beta_j =: S_\beta = 0.12 < 1,$$

- **(H3)-(H5)** — The right-hand side functions are locally Lipschitz in η, ζ. For example, for all $t \in [0,1]$, we have

$$\begin{aligned}|f_1(t,\eta(t),\zeta(\varepsilon t))| &= \left|\frac{t^2}{50}(\eta + \sin t) + \frac{1}{5}\sqrt{\pi}\cos\big(\zeta(\frac{t}{\sqrt{3}})\big)\right| \leq \frac{t^2}{50}|\eta| + \frac{t^2}{50}|\sin t| + \frac{1}{5}\sqrt{\pi}, \\ |f_2(t,\zeta(t),\eta(\varepsilon t))| &= \left|\frac{e^{1-t}}{20}(\zeta + \arctan t) + \frac{\cos\big(\eta(\frac{t}{\sqrt{3}})\big)}{\pi + t^3}\right| \leq \frac{e}{20}|\zeta| + \frac{e}{20}\frac{\pi}{4} + \frac{1}{\pi + t^3}.\end{aligned}$$

thus

$$\begin{aligned}|f_1(t,\eta,\zeta)| &\leq \frac{t^2}{50}\big(|\eta| + |\sin t|\big) + \frac{\sqrt{\pi}}{5} \leq \frac{t^2}{50}(|\eta| + 1) + \frac{\sqrt{\pi}}{5}. \\ |f_2(t,\zeta,\eta)| &\leq \frac{e^{1-t}}{20}\big(|\zeta| + |\arctan t|\big) + \frac{1}{\pi + t^3} \leq \frac{e}{20}|\zeta| + \frac{e}{20}\frac{\pi}{4} + \frac{1}{\pi + t^3}.\end{aligned}$$

For $t \in [0,1]$, we have uniform bounds

$$|f_1(t,\eta,\zeta)| \leq \frac{1}{50}|\eta| + \underbrace{\left(\frac{1}{50} + \frac{\sqrt{\pi}}{5}\right)}_{=: M_{f_1}}$$

and

$$|f_2(t,\zeta,\eta)| \le \frac{e}{20}|\zeta| + \underbrace{\Big(\frac{e\pi}{80} + \frac{1}{\pi}\Big)}_{=:M_{f_2}},$$

because $|\sin t| \le 1,\ e^{1-t} \le e,\ |\arctan t| \le \pi/4,\ |\cos| \le 1$ on $[0,1]$.

We thus choose $\beta = 1$ and

$$N_{f_1} = \frac{1}{50} = 0.02, \qquad M_{f_1} = \frac{1}{50} + \frac{\sqrt{\pi}}{5} \approx 0.37449077,$$
$$N_{f_2} = \frac{e}{20} \approx 0.13591409, \qquad M_{f_2} = \frac{e\pi}{80} + \frac{1}{\pi} \approx 0.42505656.$$

The auxiliary constants appearing in inequality (3.10) are:

$$A_1 = \frac{(\phi_1(1)-\phi_1(0))^{q_1}}{\Gamma(q_1+1)} = \frac{(e-1)^{1.4}}{\Gamma(2.4)} \approx 1.71771595,$$
$$A_2 = \frac{(\phi_2(1)-\phi_2(0))^{q_2}}{\Gamma(q_2+1)} = \frac{1^{\,1.6667}}{\Gamma(2.6667)} \approx 0.66463930.$$

We then verify

$$N_{f_1}A_1 \approx 0.02 \times 1.7177 \approx 0.03435 < 1, \qquad N_{f_2}A_2 \approx 0.135914 \times 0.66464 \approx 0.09033 < 1.$$

Thus (H5) is satisfied with $\alpha = 1$, $N_{f_1}, M_{f_1}, N_{f_2}, M_{f_2}$ as above.

From inequality (3.10), we obtain the bounds

$$\|\eta\|_\infty \le \frac{M_{f_1}A_1}{1-N_{f_1}A_1} \approx \frac{0.37449077 \times 1.71771595}{1-0.0343543} \approx 0.6662,$$
$$\|\zeta\|_\infty \le \frac{M_{f_2}A_2}{1-N_{f_2}A_2} \approx \frac{0.42505656 \times 0.66463930}{1-0.0903338} \approx 0.3106.$$

Conclusion. The numerical constants verify $N_{f_i}A_i < 1$ $(i = 1,2)$, so the hypotheses of Theorem (3.1) are fulfilled and the system (4.1) admits at least one solution $(\eta,\zeta) \in C([0,1])^2$. Moreover, the set of solutions is bounded by the above estimates:

$$\|\eta\|_\infty \lesssim 0.6662, \qquad \|\zeta\|_\infty \lesssim 0.3106.$$

The figures illustrate analytical and numerical results for the fractional system in (4.1) with nonlocal initial conditions. Figure 1 shows approximate solutions $\eta(t)$ and $\zeta(t)$ within the predicted bounds, while Figures 2 and 3 present 3D surfaces of f_1 and f_2, highlighting the dependence on delayed arguments and the nonlocal fractional nature of the model.

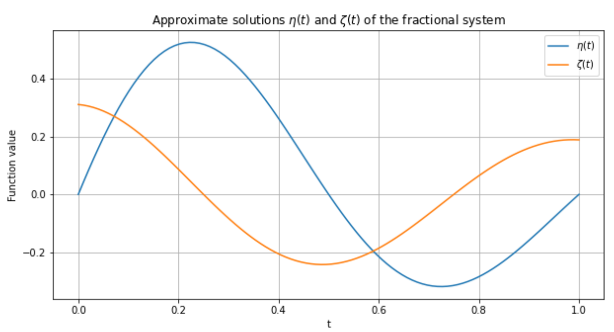

Figure 1: Numerical approximations of $\eta(t)$ (blue curve) and $\zeta(t)$ (orange curve) for the fractional ϕ-Caputo system on the interval $[0,1]$.

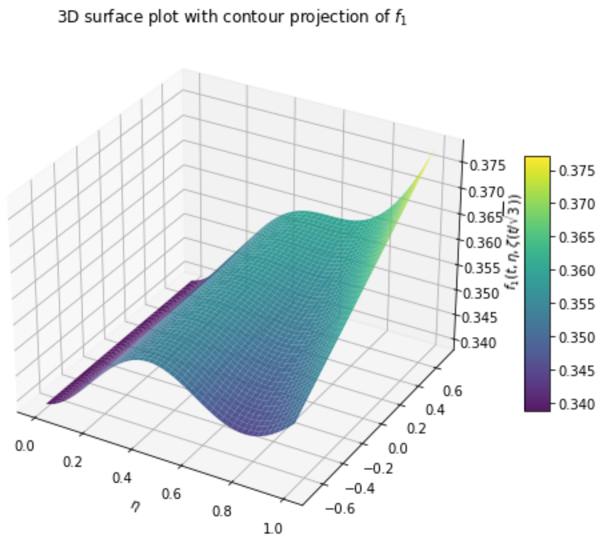

Figure 2: The graph of $f_1(t, \eta(t), \zeta(\epsilon t))$.

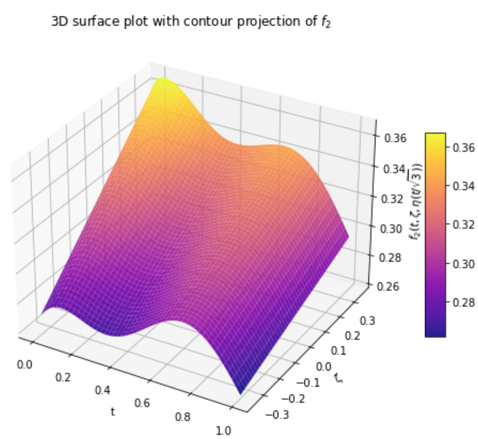

Figure 3: The graph of $f_2(t, \zeta(t), \eta(\epsilon t))$.

5. Conclusion

This study develops a rigorous analytical framework for coupled nonlinear fractional differential equations with ϕ-Caputo derivatives of orders in $(1, 2)$, incorporating proportional delays and nonlocal initial conditions. The approach leverages integral reformulations alongside advanced functional analysis techniques, such as measures of noncompactness and fixed point theory, to establish the existence and uniform boundedness of solutions in Banach spaces.

The developed framework effectively models systems exhibiting heterogeneous memory effects while accounting for both temporal delays and global dependencies in initial conditions. Numerical simulations confirm the validity of the theoretical results for complex dynamical systems where nonlocal interactions play a fundamental role.

Several promising directions for future research arise naturally from this work. These include the study of stability properties and bifurcation phenomena in relation to memory and delay parameters, extensions to higher-dimensional and spatially distributed systems with fractional operators, as well as the design of specialized computational methods tailored to such nonlocal fractional models.

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