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Fixed Point Theorem for F-Suzuki Contraction in M-Metric Space

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ABSTRACT: In this paper, we introduce fixed point theorem for F-Suzuki contraction principle in the setting of M-metric space. We give an example to exhibit the utility of our result. We also show that the existence of the solution of Fredholm integral equation by applying our fixed point result.

Key Words: Fixed point, F-Suzuki contraction, M-metric space, integral equation.

Contents

1. Introduction

Fixed point theory is one of the major research area in nonlinear functional analysis. The fundamental mathematical tool of fixed point technique is used to guarantee the existence of solutions that naturally occur in applications. It plays a significant part in a number of real-world issues and is used in many fields of current research interest, including nonlinear optimization problem, approximation theory, equilibrium problem, integral equations, complementarity problems, etc.

In 1922, the Polish mathematician Stefan Banach [18] established a remarkable fixed point result known as the Banach contraction principle. Due to the significance and clarity of Banach's contraction principle, many authors have produced a variety of intriguing extensions and generalizations of it. In 1962, M Edelstein [16] proved the new version of the Banach contraction principle. After that there are many contractions come in the field of Fixed point theory like Meir-Keeler contraction [1], F-contraction [2], Boyd and Wong nonlinear contraction [3] and several others. There are many authors who worked on different contractions in different metric spaces e.g. [6,7,10,11,12,13,14,15].

In 2008, the new result enlightened in the field of fixed point theory i.e., Suzuki [19] extended the Edelstein's results for compact metric space and F-contraction is a novel type of contraction i.e. In 2012, Wardowski [2] introduced for real-valued mapping F defined on positive real numbers and satisfying certain constraints and modified the Banach contraction principle. Subsequently, in 2013, Secelean [17] proved a condition for Wardowski's result [2].

After that several authors had worked on F-contraction mapping in different metric space i.e. In 2014, Kumam and Piri [5] applied weaker condition on self map on a complete metric space and extend the F-contraction as F-suzuki contraction. After that Aydi et al. [4] have done F-Suzuki contraction in b-metric space and obtained a fixed point theorem for it.

In this article, we will prove the fixed point theorem for F-Suzuki contraction in M-metric space.

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2. Preliminaries

In this section, we assemble some fundamental ideas, definitions, examples and auxiliary results.

In 2008, Suzuki [19] proved generalized versions of Edelstein's results in compact metric space as follows:

Definition 2.1 *[19] Let (ς, d) be a compact metric space and a mapping $\mathfrak{G} : \varsigma \to \varsigma$ is said to be Suzuki contraction. If for all $u, v \in \varsigma$ with $u \neq v$, such that*

$$\frac{1}{2} d(u, \mathfrak{G}u) < d(u, v) \Rightarrow d(\mathfrak{G}u, \mathfrak{G}v) < d(u, v).$$

Theorem 2.1 *[19] Let (ς, d) be a compact metric space and let $\mathfrak{G} : \varsigma \to \varsigma$ be a self mapping. Assume that for all $u, v \in \varsigma$ with $u \neq v$, such that*

$$\frac{1}{2} d(u, \mathfrak{G}u) < d(u, v) \Rightarrow d(\mathfrak{G}u, \mathfrak{G}v) < d(u, v).$$

Then, $\mathfrak{G}$ has a unique fixed point in ς.

In 2012, Wardowski [2] introduced F-contraction and the definition is as follows:

Definition 2.2 *[2] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:*
(F1) F is strictly increasing, i.e. for all $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta, F(\alpha) < F(\beta)$,
(F2) for each $\{\alpha_n\}_{n \in \mathbb{N}} \geq 0$ $\lim_{n\to\infty} \alpha_n = 0$ iff $\lim_{n\to\infty} F(\alpha_n) = -\infty$,
(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

Theorem 2.2 *[2] A mapping $\mathfrak{G} : \varsigma \to \varsigma$ is said to be an F-contraction if there exists $\tau > 0$ such that*

$$d(\mathfrak{G}u, \mathfrak{G}v) > 0 \Rightarrow \tau + F(d(\mathfrak{G}u, \mathfrak{G}v)) \leq d(u, v), \quad \text{for all } u, v \in \varsigma. \tag{2.1}$$

After that Secelean proved the following lemma.

Lemma 2.1 *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be an increasing mapping and $\{\alpha_n\}_{n=1}^{\infty} \geq 0$. Then, the following assertions hold:*
(a) If $\lim_{n\to\infty} F(\alpha_n) = -\infty$, then $\lim_{n\to\infty} \alpha_n = 0$.
(b) If $\inf F = -\infty$ and $\lim_{n\to\infty} \alpha_n = 0$, then $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

By proving Lemma (2.1), Secelean showed that the condition (F2) in Definition (2.1) can be replaced by a more simple condition,
(F2′) $\inf F = -\infty$,
(F2′′) there exists $\{\alpha_n\}_{n=1}^{\infty} \geq 0$, such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$,
(F3′) F is continuous on $(0, \infty)$.
We denote by $\mathfrak{F}$ the set of all functions satisfying this conditions (F1), (F2′) and (F3′).

Example 2.1 $F_1(\alpha) = \frac{-1}{\alpha} =$, $F_2(\alpha) = \frac{-1}{\alpha} + \alpha$, $F_3(\alpha) = \frac{1}{1-e^{\alpha}}$ and $F_4(\alpha) = \frac{1}{e^{\alpha}-e^{-\alpha}}$.
Then, F_1, F_2, F_3 and $F_4 \in \mathfrak{F}$.

After that, Piri et al. [5] introduced F-Suzuki contraction are as follows:

Definition 2.3 *Let (ς, d) be a metric space. A mapping $\mathfrak{G} : \varsigma \to \varsigma$ is said to be an F-Suzuki contraction if there exists $\tau > 0$ such that for all $u, v \in \varsigma$ with $\mathfrak{G}u \neq \mathfrak{G}v$*

$$\frac{1}{2} d(u, \mathfrak{G}u) < d(u, v) \Rightarrow \tau + F(d(\mathfrak{G}u, \mathfrak{G}v)) \leq d(u, v), \tag{2.2}$$

where $F \in \mathfrak{F}$.

In 2014, Asadi et al. introduced M-metric space are as follows:
Notation [8] The following notations are useful in the sequel:
(i) $m_{uv} := m(u,u) \vee m(v,v) = \min\{m(u,u), m(v,v)\}$,
(ii) $M_{uv} := m(u,u) \wedge m(v,v) = \max\{m(u,u), m(v,v)\}$.

Definition 2.4 *[8] If ς be a non-empty set. A function $m : \varsigma \times \varsigma \to \mathbb{R}^+$ is called a m-metric if it satisfying the following conditions:*
(1) $m(u,u) = m(v,v) = m(u,v) \iff u = v$,
(2) $m_{uv} \leq m(u,v)$,
(3) $m(u,v) = m(v,u)$,
(4) $(m(u,v) - m_{uv}) \leq (m(u,z) - m_{uz}) + (m(z,v) - m_{zv})$.
Then, the pair (ς, m) is called an M-metric space.

Definition 2.5 *[8] Let (ς, m) be an M-metric space. Then,*
(1) *A Sequence $\{u_n\}$ in ς converges to a point ς iff*

$$\lim_{n\to\infty}(m(u_n, u) - m_{u_n,u}) = 0. \tag{2.3}$$

(2) *A Sequence $\{u_n\}$ in ς is said to be m-Cauchy sequence iff*

$$\lim_{n,m\to\infty}(m(u_n, u_m) - m_{u_n,u_m}) \quad \textit{and} \quad \lim_{n,m\to\infty}(M(u_n, u_m) - m_{u_n,u_m}) \tag{2.4}$$

exist and finite.

(3) *An M-metric space is said to be m-complete if every m-Cauchy sequence $\{u_n\}$ converges to a point u such that*

$$\lim_{n\to\infty}(m(u_n, u) - m_{u_n,u}) = 0 \quad \textit{and} \quad \lim_{n\to\infty}(M(u_n, u) - m_{u_n,u}) = 0. \tag{2.5}$$

Lemma 2.2 *[8] If $u_n \to u$ as $n \to \infty$ in an M-metric space (ς, m). Then,*

$$\lim_{n\to\infty}(m(u_n, v) - m_{u_n,v}) = m(u,v) - m_{u,v}, \forall\ v \in \varsigma.$$

Lemma 2.3 *[8] If $u_n \to u$ and $v_n \to v$ as $n \to \infty$ in an M-metric space (ς, m). Then,*

$$\lim_{n\to\infty}(m(u_n, v_n) - m_{u_n,v_n}) = m(u,v) - m_{u,v}.$$

Lemma 2.4 *[8] If $u_n \to u$ and $v_n \to v$ as $n \to \infty$ in an M-metric space (ς, m). Then, $m(u,v) = m_{u,v}$. Further if $m(u,u) = m(v,v)$, then $u = v$.*

3. Main Results

In this section, we present the definition of F-Suzuki contraction in the framework of M-Metric space.

Definition 3.1 *Let (ς, m) be a M-metric space. A mapping $\mathfrak{G} : \varsigma \to \varsigma$ is said to be an F-Suzuki contraction if there exists $\tau > 0$ such that for all $u, v \in \varsigma$ with $\mathfrak{G}u \neq \mathfrak{G}v$*

$$\frac{1}{2}m(u, \mathfrak{G}u) < m(u,v) \Rightarrow \tau + F(m(\mathfrak{G}u, \mathfrak{G}v)) \leq m(u,v), \tag{3.1}$$

where $F \in \mathfrak{F}$.

Theorem 3.1 *Let (ς, m) be a complete M-metric space and $\mathfrak{G} : \varsigma \to \varsigma$ be an F-Suzuki contraction. Then, $\mathfrak{G}$ has a unique fixed point $u^* \in \varsigma$ and for every $u_0 \in \varsigma$, the sequence $\{\mathfrak{G}^n u_0\}_{n=1}^{\infty}$ converges to u^*.*

Proof: *Choose $u_0 \in \varsigma$ and define a sequence $\{u_n\}_{n=1}^{\infty}$ by*

$$u_1 = \mathfrak{G}u_0, \quad u_2 = \mathfrak{G}u_1 = \mathfrak{G}^2 u_0, \quad \cdots, \quad u_{n+1} = \mathfrak{G}u_n = \mathfrak{G}^{n+1}u_0, \quad \textit{for all } n \in \mathbb{N}. \tag{3.2}$$

If there exists $n \in \mathbb{N}$ such that $m(u_n, \mathfrak{G}u_n) = 0$, the proof is complete. So we assume that

$$0 < m(u_n, \mathfrak{G}u_n), \quad \text{for all } n \in \mathbb{N}.$$

Therefore,

$$\frac{1}{2}m(u_n, \mathfrak{G}u_n) - m_{u_n,\mathfrak{G}u_n} < m(u_n, \mathfrak{G}u_n) - m_{u_n,\mathfrak{G}u_n}, \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

For any $n \in \mathbb{N}$, we have

$$\tau + F(m(\mathfrak{G}u_n, \mathfrak{G}^2u_n) - m_{\mathfrak{G}u_n,\mathfrak{G}^2u_n}) \le F(m(u_n, \mathfrak{G}u_n) - m_{u_n,\mathfrak{G}u_n})$$

i.e.,

$$F(m(u_{n+1}, \mathfrak{G}u_{n+1}) - m_{\mathfrak{G}u_{n+1},\mathfrak{G}u_{n+1}}) \le F(m(u_n, \mathfrak{G}u_n) - m_{u_n,\mathfrak{G}u_n}) - \tau.$$

Repeating this process, we get

$$\begin{aligned} F(m(u_n, \mathfrak{G}u_n) - m_{u_n,\mathfrak{G}u_n}) &\le F(m(u_{n-1}, \mathfrak{G}u_{n-1}) - m_{\mathfrak{G}u_{n-1},\mathfrak{G}u_{n-1}}) - \tau \\ &\le F(m(u_{n-2}, \mathfrak{G}u_{n-2}) - m_{\mathfrak{G}u_{n-2},\mathfrak{G}u_{n-2}}) - 2\tau \\ &\le F(m(u_{n-3}, \mathfrak{G}u_{n-3}) - m_{\mathfrak{G}u_{n-3},\mathfrak{G}u_{n-3}}) - 3\tau \\ &\;\vdots \\ &\le F(m(u_0, \mathfrak{G}u_0) - m_{\mathfrak{G}u_0,\mathfrak{G}u_0}) - n\tau. \end{aligned}$$

$$F(m(u_n, \mathfrak{G}u_n) - m_{u_n,\mathfrak{G}u_n}) \le F(m(u_0, \mathfrak{G}u_0) - m_{\mathfrak{G}u_0,\mathfrak{G}u_0}) - n\tau. \tag{3.4}$$

From equation (3.4), we obtain $\lim_{n\to\infty} F(m(u_n, \mathfrak{G}u_n) - m_{u_n,\mathfrak{G}u_n}) = -\infty$ which together with (F2′) and Lemma (2.1) gives

$$\lim_{n\to\infty}(m(u_n, \mathfrak{G}u_n) - m_{u_n,\mathfrak{G}u_n}) = 0. \tag{3.5}$$

and

$$\lim_{n\to\infty}(M(u_n, \mathfrak{G}u_n) - m_{u_n,\mathfrak{G}u_n}) = 0. \tag{3.6}$$

Now, we claim that $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exist $\epsilon > 0$ and $\{r_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ of natural numbers such that

$$r(n) > s(n) > n, \quad \begin{cases} m(u_{r(n)}, u_{s(n)}) - m_{u_{r(n)},u_{s(n)}} \ge \epsilon, & \text{for all } n \in \mathbb{N}, \\ m(u_{r(n)-1}, u_{s(n)}) - m_{u_{r(n)-1,u_{s(n)}}} < \epsilon, & \text{for all } n \in \mathbb{N}. \end{cases} \tag{3.7}$$

So, we have

$$\begin{aligned} \epsilon &\le (m(u_{r(n)}, u_{s(n)}) - m_{u_{r(n)},u_{s(n)}}) \\ &\le (m(u_{r(n)}, u_{r(n)-1}) - m_{u_{r(n)},u_{r(n)-1}}) + (m(u_{r(n)-1}, u_{s(n)}) - m_{u_{r(n)-1},u_{s(n)}}) \\ &\le (m(u_{s(n)}, u_{s(n)-1}) - m_{u_{s(n)},u_{s(n)-1}}) + \epsilon. \end{aligned}$$

It follows from (3.6) and the above inequality that

$$\lim_{n\to\infty}(m(u_{r(n)}, u_{s(n)}) - m_{u_{r(n)},u_{s(n)}}) = \epsilon. \tag{3.8}$$

From (3.6) and (3.8), we can choose a positive integer $n \in \mathbb{N}$ such that

$$\frac{1}{2}(m(u_{r(n)}, \mathfrak{G}u_{r(n)}) - m_{u_{r(n)},\mathfrak{G}u_{r(n)}}) < \frac{1}{2}\epsilon < (m(u_{r(n)}, u_{s(n)}) - m_{u_{r(n)},u_{s(n)}}), \text{for all} \;\; n \;\; \ge \mathbb{N}.$$

So, from assumption of the Theorem, we get

$$\tau + F(m(\mathfrak{G}u_{r(n)}, \mathfrak{G}u_{s(n)}) - m_{\mathfrak{G}u_{r(n)},\mathfrak{G}u_{s(n)}}) \leq m(u_{r(n)}, u_{s(n)}) - m_{u_{r(n)},u_{s(n)}}, \text{for all } \ n \ \geq \mathbb{N}.$$

It follows from (3.2) that

$$\tau + F(m(u_{r(n)+1}, \mathfrak{G}u_{s(n)+1}) - m_{u_{r(n)+1},\mathfrak{G}u_{s(n)+1}}) \leq F(m(u_{r(n)}, u_{s(n)}) - m_{u_{r(n)},u_{s(n)}}), \text{for all } n \ \geq \mathbb{N}. \quad (3.9)$$

From $(F3)'$*, (3.8) and (3.9), we get* $\tau + F(\epsilon) \leq F(\epsilon)$*. This contradiction shows that* $\{u_n\}_{n=1}^{\infty}$ *is a Cauchy sequence. By completeness of* (X, m)*,* $\{u_n\}_{n=1}^{\infty}$ *converges to some point* $u^* \in \varsigma$*.*
Therefore,

$$\lim_{n\to\infty} (m(u_n, u^*) - m_{u_n,u^*}) = 0. \quad (3.10)$$

Now, we claim that

$$\frac{1}{2}(m(u_n, \mathfrak{G}u_n) - m_{u_n,\mathfrak{G}u_n}) < (m(u_n, u^*) - m_{u_n,u^*})$$

or

$$\frac{1}{2}(m(\mathfrak{G}u_n, \mathfrak{G}^2u_n) - m_{\mathfrak{G}u_n,\mathfrak{G}^2u_n}) \geq (m(\mathfrak{G}u_m, u^*) - m_{\mathfrak{G}u_m,u^*}), \text{for all } n \in \mathbb{N}. \quad (3.11)$$

Again, assume that there exists $m \in \mathbb{N}$ *such that*

$$\frac{1}{2}(m(x_m, \mathfrak{G}u_m) - m_{u_m,\mathfrak{G}u_m}) \geq (m(u_m, u^*) - m_{u_m,u^*})$$

and

$$\frac{1}{2}(m(\mathfrak{G}u_m, \mathfrak{G}^2u_m) - m_{\mathfrak{G}u_m,\mathfrak{G}^2u_m}) \geq (m(\mathfrak{G}u_m, u^*) - m_{\mathfrak{G}u_m,u^*}). \quad (3.12)$$

Therefore,

$$\begin{aligned} 2(m(u_m, u^*) - m_{u_m,u^*}) &\leq (m(u_m, \mathfrak{G}u_m) - m_{u_m,\mathfrak{G}u_m}) \\ &\leq (m(u_m, u^*) - m_{x_m,u^*}) + (m(u^*, \mathfrak{G}u_m) - m_{u^*,\mathfrak{G}u_m}) \end{aligned}$$

which implies that

$$(m(u_m, u^*) - m_{u_m,u^*}) \leq (m(u^*, \mathfrak{G}u_m) - m_{u^*,\mathfrak{G}u_m}). \quad (3.13)$$

It follows from (3.12) and (3.13) that

$$(m(u_m, u^*) - m_{u_m,u^*}) \leq (m(u^*, \mathfrak{G}u_m) - m_{u^*,\mathfrak{G}u_m}) \leq \frac{1}{2}(m(\mathfrak{G}u_m, \mathfrak{G}^2u_m) - m_{\mathfrak{G}u_m,\mathfrak{G}^2u_m}). \quad (3.14)$$

Since,

$$\frac{1}{2}(m(u_m, \mathfrak{G}u_m) - m_{u_m,\mathfrak{G}u_m}) < (m(u_m, \mathfrak{G}u_m) - m_{u_m,\mathfrak{G}u_m}),$$

By assumption of the theorem, we get

$$\tau + F((m(\mathfrak{G}u_m, \mathfrak{G}^2u_m) - m_{\mathfrak{G}u_m,\mathfrak{G}^2u_m})) \leq F(m(u_m, \mathfrak{G}u_m) - m_{u_m,\mathfrak{G}u_m}).$$

Since $\tau > 0$*, this implies that*

$$F((m(\mathfrak{G}u_m, \mathfrak{G}^2u_m) - m_{\mathfrak{G}u_m,\mathfrak{G}^2u_m})) < F(m(u_m, \mathfrak{G}u_m) - m_{u_m,\mathfrak{G}u_m}).$$

So from (F1), we get

$$(m(\mathfrak{G}u_m, \mathfrak{G}^2 u_m) - m_{\mathfrak{G}u_m, \mathfrak{G}^2 u_m}) < (m(u_m, \mathfrak{G}u_m) - m_{u_m, \mathfrak{G}u_m}). \tag{3.15}$$

It follows from (3.12), (3.14) *and* (3.15) *that*

$$\begin{aligned}(m(\mathfrak{G}u_m, \mathfrak{G}^2 u_m) - m_{\mathfrak{G}u_m, \mathfrak{G}^2 u_m}) &< F(m(u_m, \mathfrak{G}u_m) - m_{u_m, \mathfrak{G}u_m}) \\ &\leq (m(u_m, u^*) - m_{u_m, u^*}) + (m(u^*, \mathfrak{G}u_m) - m_{u^*, \mathfrak{G}u_m}) \\ &\leq \frac{1}{2}(m(\mathfrak{G}u_m, \mathfrak{G}^2 u_m) - m_{\mathfrak{G}u_m, \mathfrak{G}^2 u_m}) + \\ &\quad \frac{1}{2}(m(\mathfrak{G}u_m, \mathfrak{G}^2 u_m) - m_{\mathfrak{G}u_m, \mathfrak{G}^2 u_m}) \\ &= (m(\mathfrak{G}u_n, \mathfrak{G}^2 u_n) - m_{\mathfrak{G}u_n, \mathfrak{G}^2 u_n}).\end{aligned}$$

This is a contradiction. Hence (3.11) holds. So from (3.11) for every $n \in \mathbb{N}$, either

$$\tau + F(m(\mathfrak{G}u_n, \mathfrak{G}u^*) - m_{\mathfrak{G}u_n, \mathfrak{G}u^*}) \leq F(m(u_n, u^*) - m_{u_n, u^*})$$

or

$$\begin{aligned}\tau + F(m(\mathfrak{G}^2 u_n, \mathfrak{G}u^*) - m_{\mathfrak{G}^2 u_n, \mathfrak{G}u^*}) &\leq F(m(\mathfrak{G}u_n, u^*) - m_{\mathfrak{G}u_n, u^*}) \\ &= F(m(u_{n+1}, u^*) - m_{u_{n+1}, u^*})\end{aligned}$$

holds. In the first case, from (3.10), (F2′) and Lemma (2.1), we obtain

$$\lim_{n\to\infty} F(m(\mathfrak{G}u_n, \mathfrak{G}u^*) - m_{\mathfrak{G}u_n, \mathfrak{G}u^*}) = -\infty.$$

It follows from (F2′) and Lemma (2.1) that

$$\lim_{n\to\infty} (m(\mathfrak{G}u_n, \mathfrak{G}u^*) - m_{\mathfrak{G}u_n, \mathfrak{G}u^*}) = 0.$$

Therefore,

$$\begin{aligned}(m(u^*, \mathfrak{G}u^*) - m_{u^*, \mathfrak{G}u^*}) &= \lim_{n\to\infty} (m(u_{n+1}, \mathfrak{G}u^*) - m_{u_{n+1}, \mathfrak{G}u^*})) \\ &= \lim_{n\to\infty} (m(\mathfrak{G}u_n, \mathfrak{G}u^*) - m_{\mathfrak{G}u_n, \mathfrak{G}u^*}) = 0.\end{aligned}$$

Also, in the second case, from (3.10), (F2′) and Lemma (2.1), we obtain

$$\lim_{n\to\infty} F(m(\mathfrak{G}^2 u_n, \mathfrak{G}u^*) - m_{\mathfrak{G}^2 u_n, \mathfrak{G}u^*}) = -\infty.$$

It follows from (F2′) and Lemma (2.1) that

$$\lim_{n\to\infty} (m(\mathfrak{G}u_n, \mathfrak{G}u^*) - m_{\mathfrak{G}u_n, \mathfrak{G}u^*}) = 0.$$

Therefore,

$$\begin{aligned}(m(u^*, \mathfrak{G}u^*) - m_{u^*, \mathfrak{G}u^*}) &= \lim_{n\to\infty} (m(u_{n+2}, \mathfrak{G}u^*) - m_{u_{n+2}, \mathfrak{G}u^*}) \\ &= \lim_{n\to\infty} (m(\mathfrak{G}^2 u_n, \mathfrak{G}u^*) - m_{\mathfrak{G}^2 u_n, \mathfrak{G}u^*}) = 0.\end{aligned}$$

Hence, u^* is a fixed point of $\mathfrak{G}$. Now, let us show that $\mathfrak{G}$ has almost one fixed point. If $u^*, v^* \in \varsigma$ are two distinct fixed points of $\mathfrak{G}$. i.e., $\mathfrak{G}u^* = u^* \neq v^* = \mathfrak{G}v^*$.
Then,

$$(m(u^*, v^*) - m_{u^*, v^*}) > 0.$$

So, we have $0 = \frac{1}{2}(m(u^*, \mathfrak{G}u^*) - m_{u^*,\mathfrak{G}u^*}) < (m(u^*, v^*) - m_{u^*,v^*})$
and from the assumption of the theorem, we obtain

$$\begin{aligned} F(m(u^*, v^*) - m_{u^*,v^*}) &= F(m(\mathfrak{G}u^*, \mathfrak{G}v^*) - m_{\mathfrak{G}u^*,\mathfrak{G}v^*}) \\ &< \tau + F(m(\mathfrak{G}u^*, \mathfrak{G}v^*) - m_{\mathfrak{G}u^*,\mathfrak{G}v^*}) \\ &\leq F(m(u^*, v^*) - m_{u^*,v^*}). \end{aligned}$$

Which is a contradiction. Thus, the fixed point is unique. □

This is the sharply version by replacing F-Suzuki contraction condition by F-contraction condition.

Corollary 3.1 *Let $\mathfrak{G}$ be a self-mapping of a complete M-metric space ς into itself. Suppose $F \in \mathfrak{F}$ and there exists $\tau > 0$ such that*

$$\text{for all } u, v \in \varsigma, \quad [m(\mathfrak{G}u, \mathfrak{G}v) > 0 \Rightarrow \tau + F(m(\mathfrak{G}u, \mathfrak{G}v)) \leq F(m(u, v))]. \tag{3.16}$$

Then, $\mathfrak{G}$ has a unique fixed point $u^* \in \varsigma$ and for every $u_0 \in \varsigma$, the sequence $\{\mathfrak{G}^n u_0\}_{n=1}^{\infty}$ converges to u^*.

Example: Let $\varsigma = [0, \infty)$ and $m(u, v) = \frac{u+v}{2}$, for all $u, v \in \varsigma$. Then, (ς, m) is complete M-metric space. Define a mapping $\mathfrak{G} : \varsigma \to \varsigma$ such that $\mathfrak{G}(u) = \frac{u}{3}$, for all $u \in \varsigma$. Now, define the function $F : \mathbb{R}^+ \to \mathbb{R}$ by

$$F(r) = \frac{1}{(e^r - e^{-r})},$$

such that

$$\frac{1}{2}m(u, \mathfrak{G}u) < m(u, v) \Rightarrow \tau + F(m(\mathfrak{G}u, \mathfrak{G}v)) = \tau + F\left(\frac{u+v}{6}\right).$$

Let $\tau \leq 3$. Then

$$\tau + F\left(\frac{u+v}{6}\right) \leq 3 + \frac{1}{\left(e^{\left(\frac{u+v}{6}\right)} - e^{-\left(\frac{u+v}{6}\right)}\right)} = F(m(u, v)).$$

Thus, the contractive condition is satisfied for all $u, v \in \varsigma$. Hence, all hypotheses of the Theorem (3.1) are satisfied and $\mathfrak{G}$ has a unique fixed point $u = 0$.

4. Applications

In this section, we apply Theorem (3.1) to investigate the existence and uniqueness of solution of the Fredholm integral equation [9].
Let $\varsigma = C([a, b], \mathbb{R})$ be the set of continuous real valued functions defined on $[a, b]$.
Now, we consider the following integral equation:

$$u(t) = \int_a^b \zeta(t, q, u(t))dq, \quad \text{for } t, q \in [a, b] \tag{4.1}$$

where $\zeta \in C([a, b], \mathbb{R})$. Define $m : \varsigma \times \varsigma \to \mathbb{R}^+$ by

$$m(u(t), v(t)) = \sup_{t \in [a,b]} \frac{(|u(t) + v(t)|)}{2}, \quad \text{for all } u, v \in \varsigma. \tag{4.2}$$

Then, (ς, m) is an m-complete in M-metric space.

Theorem 4.1 *Suppose that there exist $\tau > 0$ and for all $u, v \in C([a, b], \mathbb{R})$*

$$|\zeta(t, q, u(t)) + \zeta(t, q, v(t))| \leq \frac{(|u(t) + v(t)|)}{(b - a)(1 + \tau(|u(t) + v(t)|))}, \quad \textit{for all } t, q \in [a, b].$$

Then, the integral equation (4.1) has a unique solution.

Proof: Define $\mathfrak{G}:\varsigma\to\varsigma$ by,

$$\mathfrak{G}(u(t))=\int_a^b \zeta(t,q,u(t))dq,\quad \text{for all}\quad t,q\in[a,b].$$

Observe that existence of a fixed point of the operator $\mathfrak{G}$ is equivalent to the existence of a solution of the integral equation (4.1).
Now, for all $u,v\in\varsigma$. We have

$$\begin{aligned} m(\mathfrak{G}u,\mathfrak{G}v) &= \left|\frac{\mathfrak{G}(u(t))+\mathfrak{G}(v(t))}{2}\right| \\ &= \left|\int_a^b\left(\frac{\zeta(t,q,u(t))+\zeta(t,q,v(t))}{2}\right)dq\right| \\ &\leq \int_a^b\left|\left(\frac{\zeta(t,q,u(t))+\zeta(t,q,v(t))}{2}\right)dq\right| \\ &\leq \frac{1}{(b-a)}\int_a^b\frac{|u(t)+v(t)|}{2(1+\tau|u(t)+v(t)|)}dq \\ &\leq \frac{1}{(b-a)}\int_a^b\frac{|u(t)+v(t)|}{2(1+\tau|u(t)+v(t)|)}dq. \end{aligned}\tag{4.3}$$

We know that

$$\begin{aligned} m(u,v) &= \frac{|u(t)+v(t)|}{2}, \\ 1+\tau m(u,v) &\leq 1+\tau|u+v|. \end{aligned}$$

From Equation (4.3)

$$\begin{aligned} m(\mathfrak{G}u,\mathfrak{G}v) &\leq \frac{1}{(b-a)}\left(\int_a^b\frac{(\frac{|u(t)+v(t)|}{2})}{(1+\tau m(u,v))}dq\right) \\ &\leq \frac{1}{(b-a)(1+\tau m(u,v))}\sup_{t\in[a,b]}\left(\frac{|u(t)+v(t)|}{2}\int_a^b dq\right) \\ &\leq \frac{1}{(b-a)}\frac{(m(u,v))}{(1+\tau m(u,v))}\left(\int_a^b dq\right) \\ &\leq \frac{m(u,v)}{(b-a)(1+\tau m(u,v))}(b-a) \\ &\leq \frac{m(u,v)}{(1+\tau m(u,v))}. \end{aligned}$$

Now,

$$\begin{aligned} m(\mathfrak{G}u,\mathfrak{G}v) &\leq \frac{m(u,v)}{1+\tau m(u,v)} \\ \frac{1}{m(\mathfrak{G}u,\mathfrak{G}v)} &\geq \frac{1+\tau m(u,v)}{m(u,v)} \\ -\frac{1}{m(\mathfrak{G}u,\mathfrak{G}v)} &\leq -\tau-\frac{1}{m(u,v)} \\ \tau-\frac{1}{m(\mathfrak{G}u,\mathfrak{G}v)} &\leq -\frac{1}{m(u,v)} \\ \tau+F(m(\mathfrak{G}u,\mathfrak{G}v)) &\leq F(m(u,v)). \end{aligned}$$

Thus, the Condition (3.1) is satisfied with $F(\alpha)=\frac{-1}{(\alpha)}$. Therefore, all the conditions of Theorem (3.1) are satisfied. Hence, the Fredholm integral equation (4.1) has a unique solution. □

5. Conclusion

As the M-metric is new generalization of p-metric. In this article, we established a fixed point theorem for F-Suzuki contraction in M-metric space and provide a corollary for it. We also take a suitable example and application to show the existence of the solution of Fredholm integral equation which supported our fixed point theorem.

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