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Solving a System of Nonlinear Fractional Differential Equations via Novel Best Proximity Pair Results in Regular Semimetric Space

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ABSTRACT: This paper is devoted to examining the existence of optimal solutions for a coupled system of differential equations characterized by right sided-Hilfer fractional derivatives under initial conditions as form:

$$\begin{cases} {}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi}\mu_1(\kappa) = \lambda_1(\kappa, \mu_2(\kappa)), \\ {}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi}\mu_2(\kappa) = \lambda_2(\kappa, \mu_1(\kappa)), \end{cases}$$

for $b < \kappa \leq v$. To this end, we develop a series of best proximity pair theorems for a new category of proximal contractions, referred to as the α-generalized Geraghty proximal interpolative contraction pair, formulated within the framework of a regular semimetric space $(\mathfrak{Q}, \rho, \Phi)$.

Key Words: Common best proximity point, Hardy-Rogers contraction, ψ-Hilfer fractional derivative, Proximal Geraghty contraction, interpolation, regular semimetric space.

Contents

1. Introduction

The theory of the fixed point (FP) is extremely important and plays an indisputable role in the progress and development of various branches of mathematical science. In essence, numerous scientific problems can be expressed and modeled as a FP equation, that is taking the form of equation $Tu = u$, where T is an application defined over a metric space (MS), a normal linear space, or another suitable space.

In 1931, Wilson studied the theory of semimetric spaces (SMSs) [40] and then *McAuley* in 1956, *Burke* in 1972, *Galvin et al.* in 1984 obtained scientific results in the space [28,10,16]. In 2014, *Bessenyei et al.* [9] introduced the definition of a triangular function for a SMS, $(\mathfrak{Q}, \rho)$ in other words they proposed the notion of regularity of a SMS. Let us consider non-empty subsets A_1 and A_2 within a SMS, with $T : A_1 \to A_2$ denoting a non-self mapping. In certain scenarios, the aforementioned equation may not possess a solution. In such cases, we can aim to find a point u that minimizes the error $\delta(u, Tu)$, where δ denote a metric, leading to the proximity of u and Tu. Consequently, the best proximity point (BPP) results ensures an optimal solution for globally minimizing $\delta(u, Tu)$, whenever the condition $\delta(u, Tu) = \mathrm{dist}(A_1, A_2)$ holds with an approximate solution u. These approximate solutions are referred to as BPPs of T, providing an approximate solution to the equation $Tu = u$. In the context of a Hausdorff locally convex topological vector space $\mathfrak{E}$, consider a nonempty convex compact subset A, and a continuous non-self mapping $T : A \to \mathfrak{E}$. An influential result in the form of a BPP theorem, initially presented by *Fan* [15], guarantees the existence of a point $u \in A$ satisfying $\delta(u, Tu) = \delta(Tu, A)$. This innovative result by Fan subsequently served as a basis for further improvements and generalizations made by various authors, such as *Prolla* [34], *Reich* [35], *Sehgal* and *Singh* [37], who explored different directions. Moreover, several other studies have also contributed to the establishment of the existence of BPP theorems, as evidenced by the works referenced in [8,13,33,26,17,38].

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A BPP result for generalized proximal C contractions in MS equipped with partial orders has been elicited in [30,31] by *Mongkolkeha et al.*. Generalized α-ϕ-Geraghty contraction type mapping has been introduced by *Karapinar* ([22]), *Hamzehnejadi and Lashkaripour* ([18]) initiated generalized α-ϕ-Geraghty proximal contractions. They have investigated BPPs for these types of mappings. *Poom Kumam* in [25], has establish some common BPPs results for generalized α-ϕ-Geraghty contraction mappings. Additionally, BPP for this class of contraction have been discussed in [4,20,21,14,27,32,36]. The Hardy-Rogers (HR) interpolative contractions was introduced and examined by *Karapinar et al.* [23].

Theorem 1.1 **([23])** *Let* $(\mathfrak{Q},\delta)$ *be a* MS. *If* $S:\mathfrak{Q}\to\mathfrak{Q}$ *is an interpolative* HR *type contraction i.e., there exist* $0\le\lambda<1$ *and* $\sigma,\theta,\eta\in(0,1)$ *with* $\sigma+\theta+\eta<1$, *s.t.*

$$\delta(S\mu_1,S\mu_2)\leqslant\lambda[\delta(\mu_1,\mu_2)]^{\sigma}[\delta(\mu_1,S\mu_2)]^{\theta}[\delta(\mu_2,S\mu_2)]^{\eta}\left[\frac{\delta(\mu_1,S\mu_2)+\delta(\mu_2,S\mu_1)}{2}\right]^{1-\sigma-\theta-\eta}, \tag{1.1}$$

for all $\mu_1,\mu_2\in\mathfrak{Q}\setminus\mathrm{Fix}(S)$, *then* S *possesses a* FP *of* $\mathfrak{Q}$.

Fractional calculus, which generalizes integer-order integration and differentiation to arbitrary orders, has emerged as a rapidly expanding field of research, driven by the significant outcomes achieved through the application of fractional operators in modelling various phenomena [12,19]. To enhance the understanding of certain real-world problems, recent studies have introduced novel fractional operators, discussed in [1,3,11] are particularly noteworthy.

Simultaneously, the existence and uniqueness of solutions to differential and integral equations involving fractional operators have been well investigated using fixed point theorems. For a deep study, one refers to [2,5,6,7,24,29].

The content is organized as takes after: Section 1, recalls the foundational results that will be utilized in the subsequent sections. We introduce an innovation class of contraction providing improved results at proximity points, in Section 2. Each contraction in this class is termed a Geragthy proximal interpolative G_{pi} of type HR (see Definition 2.4). We provide the necessary conditions to establish the common best proximity points (CBPPs) for this mapping in complete regular SMS. To illustrate our results, we provide an example. The findings of this study extend and generalize the results of *Sadiq Basha* in [8], *Karapinar et al.* in [23] and other related works from the literature. In Section 3, we apply the result obtained from the corollary 2.1 to investigate the existence of optimal solutions of a coupled system of fractional differential equations ($\mathbb{FDE}$s) under initial values involving a fractional ψ-Hilfer derivative.

2. Theorems, lemmas, and other proclamations

Assume that $\mathfrak{Q}\neq\varnothing$ be an arbitrary set. A function $\rho:\mathfrak{Q}^2\to\mathbb{R}^{\geq 0}$ is called semimetric (SM) on $\mathfrak{Q}$ whenever $\rho(g_1,g_2)=\rho(g_2,g_1)$, $\rho(g_1,g_2)=0\iff g_1=g_2$ for each $g_1,g_2\in\mathfrak{Q}$ and a pair $(\mathfrak{Q},\rho)$, is called a SMS if ρ is a SM on $\mathfrak{Q}$. A SM ρ is metric if the triangle inequality $\rho(\mu_1,\mu_2)\leqslant\rho(\mu_1,\mu_3)+\rho(\mu_3,\mu_2)$ holds, for each $\mu_i\in\mathfrak{Q}$, $i=1,2,3$. In the context of a SMS $(\mathfrak{Q},\rho)$, the distance between nonempty subsets A_1 and A_2 can established as follows

$$\mathrm{dist}(A_1,A_2):=\inf\Big\{\rho(\mu_1,\mu_2):\ \mu_1\in A_1,\mu_2\in A_2\Big\}. \tag{2.1}$$

If A_1 consists of a single point μ, then we denote the distance between μ and A_2 as $\mathrm{dist}(\mu,A_2)$ rather than $\mathrm{dist}(\{\mu_1\},A_2)$. Consider a SMS $(\mathfrak{Q},\rho)$ and nonempty subsets E_1 and E_2 of $\mathfrak{Q}$. We can define the following sets

$$E_{1_0}:=\Big\{\mu\in E_1:\ \rho(\mu,\upsilon)=\mathrm{dist}(E_1,E_2)\text{ for some }\upsilon\in E_2\Big\}, \tag{2.2}$$

$$E_{2_0}:=\Big\{\upsilon\in E_2:\ \rho(\mu,\upsilon)=\mathrm{dist}(E_1,E_2)\text{ for some }\mu\in E_2\Big\}. \tag{2.3}$$

A pair $(\mu_0,\upsilon_0)\in E_{1_0}\times E_{2_0}$ for which $\rho(\mu_0,\upsilon_0)=\mathrm{dist}(E_1,E_2)$ referred to as a *"best proximity pair"* for E_1 and E_2.

Definition 2.1 *Let* $E_i\neq\varnothing$, $i=1,2$, *be subset of* SMS $(\mathfrak{Q},\rho)$.

- *An element* $a^* \in A$ *is defined as a* BPP *of the mapping* $T : E_1 \to E_2$ *if* $\rho(a^*, Ta^*) = \mathrm{dist}(E_1, E_2)$*;*
- CBPP *of non-self mappings* $S, T : E_1 \to E_2$ *are points* $\upsilon \in \mathfrak{Q}$ *s.t.*

$$\rho(\upsilon, S\upsilon) = \rho(\upsilon, T\upsilon) = \rho(E_1, E_2). \tag{2.4}$$

In the case where $\mathrm{dist}(E_1, E_2) = 0$*, then a common* FP *coincides with a* CBPP *of* S *and* T*.*

Bessenyei et al. [9] introduced the following definition of a *triangular function* for a SMS $(\mathfrak{Q}, \rho)$,

Definition 2.2 ([9]) *Consider a* SMS $(\mathfrak{Q}, \rho)$*. A* triangle function $\Phi : \mathbb{R}^2_{\geq 0} \to \overline{\mathbb{R}_{>0}}$ *for* ρ*, is symmetric and monotone increasing in both of its arguments, with properties* $\Phi(0, 0) = 0$ *and*

$$\rho(g_1, g_2) \leqslant \Phi(\rho(g_1, g_3), \rho(g_2, g_3)), \qquad \mu_i \in \mathfrak{Q},\ i = 1, 2, 3. \tag{2.5}$$

In SMSs, we say the triangular function Φ is *regular* whenever Φ is continuous at the origin. Let $\tilde{\lambda}_1, \tilde{\lambda}_2 \in \mathbb{R}_+$,

- $\Phi(\tilde{\lambda}_1, \tilde{\lambda}_2) = c(\tilde{\lambda}_1 + \tilde{\lambda}_2)$ (c-relaxed triangular inequality, $c \geqslant 1$);
- $\Phi(\tilde{\lambda}_1, \tilde{\lambda}_2) = c \max\{\tilde{\lambda}_1, \tilde{\lambda}_2\}$ (c-infrared inequality, $c \geqslant 1$);
- $\Phi(\tilde{\lambda}_1, \tilde{\lambda}_2) = (\tilde{\lambda}_{1_p} + \tilde{\lambda}_{2_p})^{\frac{1}{p}}$ (triangular inequality of order p, where $p > 0$).

The topology of a regular SMS is Hausdorff and a convergent sequence in a regular SMS has a unique limit, possesses the Cauchy property. Moreover, a SMS $(\mathfrak{Q}, \rho)$ is regular iff $\lim\limits_{r \to +\infty} \sup\limits_{p \in \mathfrak{Q}} diam(\mathcal{B}_r(p)) = 0$, where $\mathcal{B}_r(g) = \{y \in \mathfrak{Q} : \rho(y, p) < r\}$, $g \in \mathfrak{Q}$ and $r > 0$ [9]. We give the same definitions of the two articles [22](1^*) and [34](2^*) in SMSs.

Definition 2.3 *Let* $(\mathfrak{Q}, \rho)$ *be a* SMS*,* $\alpha : \mathfrak{Q}^2 \to \mathbb{R}^{\geq 0}$ *be a function.*

(1^*) *The sequence* $(u_n)_{n \geqslant 0}$ *is said to be* α*-regular on* $\mathfrak{Q}$ *if* $\alpha(u_n, u_{n+1}) \geqslant 1$, $n \in \mathbb{N}$ *and* $\lim\limits_{n \to +\infty} u_n = u \in \mathfrak{Q}$*, then there exists a subsequence* $(u_{n_k})_{k \geqslant 0}$ *of* $(u_n)_{n \geqslant 0}$ *s.t. for each* $k \in \mathbb{N}$*,* $\alpha(u_{n_k}, u) \geqslant 1$*;*

(2^*) α *is said to be triangular if* $\alpha(\mu_1, \mu_2) \geqslant 1$ *and* $\alpha(\mu_2, \mu_3) \geqslant 1$ *then* $\alpha(\mu_1, \mu_3) \geqslant 1$*;*

(3^*) *a pair* (V, S) *of non-self mappings from* E_1 *to* E_2 *is said* α*-proximal admissible if condition,*

$$\begin{cases} \alpha(\mu_1, \mu_2) \geqslant 1, \\ \rho(\mathfrak{t}, V\mu_1) = \mathrm{dist}(E_1, E_2), \\ \rho(\mathfrak{s}, S\mu_2) = \mathrm{dist}(E_1, E_2), \end{cases} \implies \min\Big\{\alpha(\mathfrak{t}, \mathfrak{s}), \alpha(\mathfrak{s}, \mathfrak{t})\Big\} \geqslant 1, \tag{2.6}$$

$\forall\, \mu_1, \mu_2, \mathfrak{t}, \mathfrak{s} \in E_1$*, holds,*

Theorem 2.1 ([4]) *Consider nonempty subsets* E_1 *and* E_2 *of a complete* MS $(\mathfrak{Q}, \rho)$*. suppose that* E_{1_0} *is nonempty and* E_1 *is closed set. Let* V *and* S *be a pair of* α*-*ψ*-proximal contraction pair of the first kind from* E_1 *to* E_2 *s.t.* (i) $V(E_{1_0}) \subset E_{1_0}$*,* $S(E_{1_0}) \subset E_{1_0}$*;* (ii) (V, S) *is an* α*-proximal admissible pair;* (iii) $\exists\, (\mu_0, \mu_1) \in E_{1_0}^{\,2}$ *s.t.*

$$\min\Big\{\alpha(\mu_0, \mu_1), \alpha(\mu_1, \mu_0)\Big\} \geqslant 1, \qquad \rho(\mu_1, S\mu_0) = \mathrm{dist}(E_1, E_2); \tag{2.7}$$

(iv) V *and* S *are continuous. Then,* $\exists\, \mu^* \in E_1$ *s.t.* $\rho(\mu^*, V\mu^*) = \rho(\mu^*, S\mu^*) = \mathrm{dist}(E_1, E_2)$*.*

Consider a SMS $(\mathfrak{Q}, \rho)$. We assume that $\alpha : \mathfrak{Q}^2 \to \mathbb{R}^{\geq 0}$ is triangular and consider the values of the triangular function Φ in $\mathbb{R}_{>0}$. Denote the family of functions $\psi : \mathbb{R}^{\geq 0} \to [0, 1)$ by Ψ, satisfying the property as

$$\limsup_{n \to +\infty} \psi(t_n) = 1 \implies \exists (t_{n_k})_{k \geqslant 0} \subset (t_n)_{n \geqslant 0} : \lim_{k \to +\infty} t_{n_k} = 0. \tag{2.8}$$

In what follows, we introduce a new class of proximal contractions, the so-called α-G_{pi} contraction type HR.

Definition 2.4 *Let $V, S : E_1 \to E_2$ be two mappings. The couple (V, S) is called a α-generalized G_{pi} contraction pair type* HR, *if $\forall\, \mathfrak{z}, \mathfrak{s}, \breve{\mu}_1, \breve{\mu}_2 \in E_1$, the conditions*

$$\alpha(\breve{\mu}_1, \breve{\mu}_2) \geqslant 1, \quad \rho(\mathfrak{z}, V\breve{\mu}_1) = \rho(E_1, E_2), \quad \rho(\mathfrak{s}, S\breve{\mu}_2) = \rho(E_1, E_2), \tag{2.9}$$

imply that

$$\begin{aligned}\alpha(\breve{\mu}_1, \breve{\mu}_2)\Phi\left(\rho(\mathfrak{z}, \mathfrak{s}), \rho(\mathfrak{z}, \mathfrak{s})\right) \leqslant \psi(\rho(\breve{\mu}_1, \breve{\mu}_2)) \left(\rho(\breve{\mu}_1, \breve{\mu}_2)\right)^{\eta} \left(\rho(\breve{\mu}_1, \mathfrak{z})\right)^{\theta} \\ \cdot \left(\rho(\breve{\mu}_2, \mathfrak{s})\right)^{\theta} \left[\rho(\breve{\mu}_1, \mathfrak{s}) + \rho(\breve{\mu}_2, \mathfrak{z})\right]^{1-\eta-2\theta},\end{aligned} \tag{2.10}$$

where $0 < \theta, \eta < 1$ with $\eta + 2\theta < 1$.

• If Φ is c-relaxed inequality triangular, then Eq. (2.10) becomes

$$\begin{aligned}\alpha(\breve{\mu}_1, \breve{\mu}_2)\, 2c\, \rho(\mathfrak{z}, \mathfrak{s}) \leqslant \psi(\rho(\breve{\mu}_1, \breve{\mu}_2))(\rho(\breve{\mu}_1, \breve{\mu}_2))^{\eta}(\rho(\breve{\mu}_1, \mathfrak{z}))^{\theta} \\ \cdot \left(\rho(\breve{\mu}_2, \mathfrak{s})\right)^{\theta} \left[\rho(\breve{\mu}_1, \mathfrak{s}) + \rho(\breve{\mu}_2, \mathfrak{z})\right]^{1-\eta-2\theta}.\end{aligned} \tag{2.11}$$

We obtain

$$\begin{aligned}\alpha(\breve{\mu}_1, \breve{\mu}_2)\, \rho(\mathfrak{z}, \mathfrak{s}) \leqslant \psi(\rho(\breve{\mu}_1, \breve{\mu}_2))(\rho(\breve{\mu}_1, \breve{\mu}_2))^{\eta}(\rho(\breve{\mu}_1, \mathfrak{z}))^{\theta} \\ \cdot \left(\rho(\breve{\mu}_2, \mathfrak{s})\right)^{\theta} \left[\tfrac{\rho(\breve{\mu}_1, \mathfrak{s}) + \rho(\breve{\mu}_2, \mathfrak{z})}{2c}\right]^{1-\eta-2\theta}.\end{aligned} \tag{2.12}$$

• If Φ is c-infrared inequality triangular, then Eq. (2.10) becomes

$$\begin{aligned}\alpha(\mu_1, \mu_2)\, \rho(\mathfrak{t}, \mathfrak{s}) \leqslant \psi(\rho(\mu_1, \mu_2))(\rho(\mu_1, \mu_2))^{\eta}(\rho(\mu_1, \mathfrak{t}))^{\theta} \\ \cdot \left(\rho(\mu_2, \mathfrak{s})\right)^{\theta} \left[\rho(\mu_1, \mathfrak{s}) + \rho(\mu_2, \mathfrak{t})\right]^{1-\eta-2\theta}.\end{aligned} \tag{2.13}$$

In the sequel, we denote $(\mathfrak{Q}, \rho, \Phi)$, the regular SMS $(\mathfrak{Q}, \rho)$ provided with a triangular function Φ continuous at the point $(0, 0)$ s.t. $\forall \mathfrak{z} \in \mathbb{R}_{>0}$, $\Phi(\mathfrak{z}, \mathfrak{z}) \geqslant \mathfrak{z}$.

Theorem 2.2 *Consider $\varnothing \neq E_i \subseteq \mathfrak{Q}$, $i = 1, 2$, in a regular* SMS $(\mathfrak{Q}, \rho, \Phi)$ *and suppose that $(\mathfrak{Q}, \rho)$ is complete and E_{1_0} is a closed set. Let $V, S : E_1 \to E_2$ fulfilling The subsequent requirements*

(i) *(V, S) is a α-generalized G_{pi} contraction type* HR *pair;*

(ii) *any sequence $(u_n)_{n\geqslant 0}$ of E_1 is α-regular and (V, S) is α-proximal admissible;*

(iii) *$V(E_{1_0}) \subset E_{2_0}$ and $S(E_{1_0}) \subset E_{2_0}$;*

(iv) *there exists $(\kappa_0, \kappa_1) \in E_{1_0}^2$ where*

$$\min\Big\{\alpha(\kappa_0, \kappa_1), \alpha(\kappa_1, \kappa_0)\Big\} \geqslant 1, \qquad \rho(\kappa_1, V\mathfrak{t}_0) = \mathrm{dist}(E_1, E_2). \tag{2.14}$$

Then, there exists a point $\mu^* \in E_1$ s.t. $\rho(\mu^*, V\mu^*) = \rho(\mu^*, S\mu^*) = \rho(E_1, E_2)$.

Proof: By (iv), E_{1_0} is a nonempty set and there exist $\kappa_0, \kappa_1 \in E_{1_0}$ s.t.

$$\min\Big\{\alpha(\kappa_0, \kappa_1), \alpha(\kappa_1, \kappa_0)\Big\} \geqslant 1, \qquad \rho(\kappa_1, V\kappa_0) = \mathrm{dist}(E_1, E_2). \tag{2.15}$$

According to the hypothesis $S(E_{1_0}) \subset E_{2_0}$, we can conclude the existence of $\kappa_2 \in E_{1_0}$ s.t. $\rho(\kappa_2, S\kappa_1) = \mathrm{dist}(E_1, E_2)$. Since (V, S) is a pair of α-proximal admissible mappings and

$$\begin{cases}\min\Big\{\alpha(\kappa_0, \kappa_1), \alpha(\kappa_1, \kappa_0)\Big\} \geqslant 1, \\ \rho(\kappa_1, V\kappa_0) = \mathrm{dist}(E_1, E_2), \\ \rho(\kappa_2, S\kappa_1) = \mathrm{dist}(E_1, E_2),\end{cases} \quad \Rightarrow \min\Big\{\alpha(\kappa_1, \kappa_2), \alpha(\kappa_2, \kappa_1)\Big\} \geqslant 1. \tag{2.16}$$

Similarly, we assert the existence of $\kappa_3 \in E_{1_0}$ where $\rho(\kappa_3, V\kappa_2) = \mathrm{dist}(E_1, E_2)$. Since (V, S) is a pair of α-proximal admissible mappings and

$$\begin{cases} \min\Big\{\alpha(\kappa_1,\kappa_2),\alpha(\kappa_2,\kappa_1)\Big\} \geqslant 1, \\ \rho(\kappa_3, V\kappa_2) = \mathrm{dist}(E_1,E_2), \\ \rho(\kappa_2, S\kappa_1) = \mathrm{dist}(E_1,E_2), \end{cases} \Rightarrow \min\Big\{\alpha(\kappa_2,\kappa_3),\alpha(\kappa_3,\kappa_2)\Big\} \geqslant 1. \tag{2.17}$$

Continuing this process, we construct a sequence $(\tilde\sigma_n)_{n\geqslant 0}$ in E_{1_0} s.t.

$$\begin{cases} \min\Big\{\alpha(\tilde\sigma_n,\tilde\sigma_{n+1}),\alpha(\tilde\sigma_{n+1},\tilde\sigma_n)\Big\} \geqslant 1, \\ \rho(\tilde\sigma_{2n+1}, V\tilde\sigma_{2n}) = \mathrm{dist}(E_1,E_2), \\ \rho(\tilde\sigma_{2n+2}, S\tilde\sigma_{2n+1}) = \mathrm{dist}(E_1,E_2), \qquad \forall n \in \mathbb{N}, \end{cases} \tag{2.18}$$

(V, S) is a α-generalized G_{pi} contraction type HR pair and $\alpha(\tilde\sigma_n, \tilde\sigma_{n+1}) \geqslant 1$, then $\forall\, n \in \mathbb{N}$,

$$\begin{aligned} &\Phi(\rho(\tilde\sigma_{2n+1},\tilde\sigma_{2n+2}),\rho(\tilde\sigma_{2n+1},\tilde\sigma_{2n+2})) \\ &\quad \leqslant (\rho(\tilde\sigma_{2n},\tilde\sigma_{2n+1}))^{\eta+\theta}(\rho(\tilde\sigma_{2n+1},\tilde\sigma_{2n+2}))^{\theta}\left(\rho(\tilde\sigma_{2n},\tilde\sigma_{2n+2})\right)^{1-\eta-2\theta} \\ &\quad \leqslant (\rho(\tilde\sigma_{2n},\tilde\sigma_{2n+1}))^{\eta+\theta}(\rho(\tilde\sigma\tilde\sigma_{2n+1},\tilde\sigma_{2n+2}))^{\theta} \\ &\qquad \cdot\left[\Phi(\rho(\tilde\sigma_{2n},\tilde\sigma_{2n+1}),\rho(\tilde\sigma_{2n+1},\tilde\sigma_{2n+2}))\right]^{1-\eta-2\theta}. \end{aligned} \tag{2.19}$$

Also, one has

$$\begin{cases} \alpha(\tilde\sigma_n,\tilde\sigma_{n+1}) \geqslant 1, \\ \rho(\tilde\sigma_{2n+2}, S\tilde\sigma_{2n+1}) = \mathrm{dist}(E_1,E_2), \\ \rho(\tilde\sigma_{2n+3}, V\tilde\sigma_{2n+2}) = \mathrm{dist}(E_1,E_2), \qquad \forall n \in \mathbb{N}. \end{cases} \tag{2.20}$$

Hence,

$$\begin{aligned} &\Phi(\rho(\tilde\sigma_{2n+2},\tilde\sigma_{2n+3}),\rho(\tilde\sigma_{2n+2},\tilde\sigma_{2n+3})) \\ &\quad \leqslant (\rho(\tilde\sigma_{2n+1},\tilde\sigma_{2n+2}))^{\eta+\theta}(\rho(\tilde\sigma_{2n+2},\tilde\sigma_{2n+3}))^{\theta}\left(\rho(\tilde\sigma_{2n+1},\tilde\sigma_{2n+3})\right)^{1-\eta-2\theta} \\ &\quad \leqslant (\rho(\tilde\sigma_{2n+1},\tilde\sigma_{2n+2}))^{\eta+\theta}(\rho(\tilde\sigma_{2n+2},\tilde\sigma_{2n+3}))^{\theta} \\ &\qquad \cdot\left[\Phi(\rho(\tilde\sigma_{2n+1},\tilde\sigma_{2n+2}),\rho(\tilde\sigma_{2n+2},\tilde\sigma_{2n+3}))\right]^{1-\eta-2\theta}. \end{aligned} \tag{2.21}$$

Thus, for each integer $n \geqslant 0$,

$$\begin{aligned} &\Phi(\rho(\tilde\sigma_{n+1},\tilde\sigma_{n+2}),\rho(\tilde\sigma_{n+1},\tilde\sigma_{n+2})) \\ &\quad \leqslant (\rho(\tilde\sigma_{n},\tilde\sigma_{n+1}))^{\eta+\theta}(\rho(\tilde\sigma_{n+1},\tilde\sigma_{n+2}))^{\theta}\left[\Phi(\rho(\tilde\sigma_{n},\tilde\sigma_{n+1}),\rho(\tilde\sigma_{n+1},\tilde\sigma_{n+2}))\right]^{1-\eta-2\theta}. \end{aligned} \tag{2.22}$$

Case 1: If $\exists\, n_0 \in \mathbb{N}$ s.t. $\tilde\sigma_{n_0} = \tilde\sigma_{n_0+1}$, then by Eq. (2.22),

$$\Phi(\rho(\tilde\sigma_{n_0+1},\tilde\sigma_{n_0+2}),\rho(\tilde\sigma_{n_0+1},\tilde\sigma_{n_0+2})) = 0, \tag{2.23}$$

and as Φ is monotonically increasing, we obtain $\rho(\tilde\sigma_{n_0+1},\tilde\sigma_{n_0+2}) = 0$. So, $\tilde\sigma_{n_0} = \tilde\sigma_{n_0+1} = \tilde\sigma_{n_0+2}$. We follow the same procedure, we obtain $\tilde\sigma_n = \tilde\sigma_{n+1}$, for any integer $n \geqslant n_0$. The sequence $(\tilde\sigma_n)_{n\geqslant 0}$ being stationary from a certain rank, it is therefore convergent.

Case 2: Let $n \in \mathbb{N}$, $\tilde\sigma_n \neq \tilde\sigma_{n+1}$. Suppose that $\rho(\tilde\sigma_n,\tilde\sigma_{n+1}) < \rho(\tilde\sigma_{n+1},\tilde\sigma_{n+2})$, for some $n \in \mathbb{N}$. Then,

$$\Phi(\rho(\tilde\sigma_n,\tilde\sigma_{n+1}),\rho(\tilde\sigma_{n+1},\tilde\sigma_{n+2})) < \Phi(\rho(\tilde\sigma_{n+1},\tilde\sigma_{n+2}),\rho(\tilde\sigma_{n+1},\tilde\sigma_{n+2})). \tag{2.24}$$

From inequality (2.10), one has

$$\begin{aligned} &\Phi(\rho(\tilde\sigma_{n+1},\tilde\sigma_{n+2}),\rho(\tilde\sigma_{n+1},\tilde\sigma_{n+2})) \\ &\quad < (\rho(\tilde\sigma_{n},\tilde\sigma_{n+1}))^{\eta+\theta}(\rho(\tilde\sigma_{n+1},\tilde\sigma_{n+2}))^{\theta}\left(\Phi(\rho(\tilde\sigma_{n+1},\tilde\sigma_{n+2}),\rho(\tilde\sigma_{n+1},\tilde\sigma_{n+2}))\right)^{1-\eta-2\theta}. \end{aligned} \tag{2.25}$$

We obtain

$$\left(\Phi(\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2}),\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2}))\right)^{\eta+2\theta}\left(\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2})\right)^{-\theta}<\left(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1})\right)^{\eta+\theta}, \tag{2.26}$$

As by hypothesis, for all $\mathfrak{s}\in\mathbb{R}_{>0}$, $\Phi(\mathfrak{s},\mathfrak{s})\geqslant\mathfrak{s}$, then thanks to Eq. (2.26),

$$\left(\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2})\right)^{\eta+\theta}<\left(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1})\right)^{\eta+\theta}. \tag{2.27}$$

Hence, $\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2})<\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1})$, which is a contradiction. Thus, $\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2})\leqslant\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1})$, for $n\in\mathbb{N}$. Therefore, $(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}))_{n\geqslant 0}$ is non-increasing sequence and bounded below. There exists $\tau\geqslant 0$ s.t. $\lim_{n\to+\infty}\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1})=\tau$. Assuming that $\varpi>0$, we can find $\aleph\in\mathbb{N}$ s.t. for all positive integers $n\geqslant\aleph$, $\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1})>0$. Additionally, $\forall\, n\in\mathbb{N}$, one has

$$\begin{aligned}&\Phi(\rho(a_{n+1},\tilde{\sigma}_{n+2}),\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2}))\\&\quad\leqslant\psi(\rho(\tilde{\sigma}_{n},\mathfrak{t}_{n+1}))(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}))^{\eta+2\theta}\left[\Phi(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}),\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2}))\right]^{1-\eta-2\theta},\end{aligned} \tag{2.28}$$

in other words,

$$\begin{aligned}&\left[\Phi(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}),\ \rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}))\right]^{\eta+2\theta}\Phi(\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2}),\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2}))\\&\qquad\leqslant\psi(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}))(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}))^{\eta+2\theta}\Phi(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}),\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1})).\end{aligned} \tag{2.29}$$

And consequently,

$$\begin{aligned}&(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}))^{\eta+2\theta}\Phi(\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2}),\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2}))\\&\qquad\leqslant\psi(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}))(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}))^{\eta+2\theta}\Phi\left(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}),\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1})\right).\end{aligned} \tag{2.30}$$

Since $\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1})>0$, for all $n\geqslant\aleph$,

$$\Phi(\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2}),\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2}))\leqslant\psi(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}))\Phi(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}),\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1})). \tag{2.31}$$

We have,

$$\limsup_{n\to+\infty}\Phi(\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2}),\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2}))\geqslant\limsup_{n\to+\infty}\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2})=r>0, \tag{2.32}$$

and so

$$\begin{aligned}&\limsup_{n\to+\infty}\Phi(\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2}),\rho(\tilde{\sigma}_{n+1},\tilde{\sigma}_{n+2}))\\&\qquad\leqslant\limsup_{n\to+\infty}\psi(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}))\limsup_{n\to+\infty}\Phi\left(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}),\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1})\right),\end{aligned} \tag{2.33}$$

which shows that

$$1\leqslant\limsup_{n\to+\infty}\psi(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}))\implies\limsup_{n\to+\infty}\psi(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}))=1. \tag{2.34}$$

Since, $\psi\in\Psi$, we consider a subsequence $(\rho(\tilde{\sigma}_{n_k},\tilde{\sigma}_{n_k+1}))_{k\geqslant 0}$ of $(\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1}))_{n\geqslant 0}$ s.t.

$$\lim_{k\to+\infty}\rho(\tilde{\sigma}_{n_k},\tilde{\sigma}_{n_k+1})=0, \tag{2.35}$$

so $\varpi=0$, which is a contradiction. Hence, $\lim_{n\to+\infty}\rho(\tilde{\sigma}_{n},\tilde{\sigma}_{n+1})=0$.

• Suppose that $(a_n)_{n\geqslant 0}$ is not a Cauchy sequence. Therefore, there $\exists\,\varepsilon>0$ s.t. we can find two subsequences $(m_k)_{k\geqslant 0}$ and $(n_k)_{k\geqslant 0}$ of positive integers satisfying $n_k>m_k>k$ s.t.

$$\zeta_k:=\rho(\tilde{\sigma}_{m_k},\tilde{\sigma}_{n_k})\geqslant\varepsilon,\quad\rho(\tilde{\sigma}_{m_k},\tilde{\sigma}_{n_k-1})<\varepsilon,\qquad k\in\mathbb{N}. \tag{2.36}$$

Set $\nu_n := \rho(x_{n+1}, x_n),\ \forall\, n \in \mathbb{N}$. We notice that, $\forall\, k \in \mathbb{N}$,

$$\begin{aligned}
\rho(\tilde{\sigma}_{m_k+2}, \tilde{\sigma}_{n_k}) &\leqslant \Phi(\nu_{m_k+1}, \rho(\tilde{\sigma}_{m_k+1}, \tilde{\sigma}_{n_k})) \leqslant \Phi(\nu_{m_k+1}, \Phi(\nu_{m_k}, \rho(\tilde{\sigma}_{m_k}, \tilde{\sigma}_{n_k}))) \\
&\leqslant \Phi(\rho(\tilde{\sigma}_0, \tilde{\sigma}_1), \Phi(\rho(\tilde{\sigma}_0, \tilde{\sigma}_1), \zeta_k)), \\
\rho(\tilde{\sigma}_{m_k+1}, \tilde{\sigma}_{n_k+2}) &\leqslant \Phi(\nu_{m_k}, \Phi(\rho(\tilde{\sigma}_{m_k}, \tilde{\sigma}_{n_k+2})) \leqslant \Phi(\nu_{m_k}, \Phi(\zeta_k, \Phi(\nu_{n_k}, \nu_{n_k+1}))) \\
&\leqslant \Phi\left(\rho(\tilde{\sigma}_0, \tilde{\sigma}_1), \Phi(\zeta_k, \Phi(\rho(\tilde{\sigma}_0, \tilde{\sigma}_1), \rho(\tilde{\sigma}_0, \tilde{\sigma}_1)))\right), \\
\rho(\tilde{\sigma}_{m_k+1}, \tilde{\sigma}_{n_k+1})) &\leqslant \Phi(\nu_{m_k}, \phi(\zeta_k, \nu_{n_k})) \leqslant \Phi(\rho(\tilde{\sigma}_0, \tilde{\sigma}_1), \phi(\zeta_k, \rho(\tilde{\sigma}_0, \tilde{\sigma}_1))), \\
\rho(\tilde{\sigma}_{m_k+1}, \tilde{\sigma}_{n_k}) &\leqslant \Phi(\nu_{m_k}, \zeta_k) \leqslant \Phi(\rho(\tilde{\sigma}_0, \tilde{\sigma}_1), \zeta_k).
\end{aligned} \tag{2.37}$$

Since $\zeta_k \leqslant \Phi(\rho(\tilde{\sigma}_{m_k}, \tilde{\sigma}_{n_k-1}), \rho(\tilde{\sigma}_{n_k-1}, \tilde{\sigma}_{n_k})) < \Phi(\varepsilon, \rho(\tilde{\sigma}_0, \tilde{\sigma}_1))$, then $\limsup_{k\to+\infty} \zeta_k < +\infty$,

$$\limsup_{k\to+\infty} \rho(\tilde{\sigma}_{m_k+1}, \tilde{\sigma}_{n_k+1}), \quad \limsup_{k\to+\infty} \rho(\tilde{\sigma}_{m_k+1}, \tilde{\sigma}_{n_k}), \quad \limsup_{k\to+\infty} \rho(\tilde{\sigma}_{m_k}, \tilde{\sigma}_{n_k+1}), \tag{2.38}$$

$\limsup_{k\to+\infty} \rho(\tilde{\sigma}_{m_k+2}, \tilde{\sigma}_{n_k})$ and $\limsup_{k\to+\infty} \rho(\tilde{\sigma}_{m_k+1}, \tilde{\sigma}_{n_k+2})$ are also finite. α being triangular, one has $\alpha(a_{n_k}, \tilde{\sigma}_{m_k+1}) \geqslant 1$, for all $k \in \mathbb{N}$, and as

$$\rho(\tilde{\sigma}_{n_k+1}, V\tilde{\sigma}_{n_k}) = \operatorname{dist}(E_1, E_2), \qquad \rho(\tilde{\sigma}_{m_k+2}, S\tilde{\sigma}_{m_k+1}) = \operatorname{dist}(E_1, E_2), \tag{2.39}$$

then

$$\begin{aligned}
&\Phi(\rho(\tilde{\sigma}_{n_k+1}, \tilde{\sigma}_{m_k+2}), \rho(\tilde{\sigma}_{n_k+1}, \tilde{\sigma}_{m_k+2})) \\
&\quad \leqslant \psi(\rho(\tilde{\sigma}_{n_k}, \tilde{\sigma}_{m_k+1}))(\rho(\tilde{\sigma}_{n_k}, \tilde{\sigma}_{m_k+1}))^{\eta}(\rho(\tilde{\sigma}_{n_k}, \tilde{\sigma}_{n_k+1}))^{\theta} \\
&\qquad \cdot (\rho(\tilde{\sigma}_{m_k+1}, \tilde{\sigma}_{m_k+2}))^{\theta} \left[\rho(\tilde{\sigma}_{n_k+1}, \tilde{\sigma}_{m_k+1}) + \rho(\tilde{\sigma}_{n_k}, \tilde{\sigma}_{m_k+2})\right]^{1-\eta-2\theta}.
\end{aligned} \tag{2.40}$$

As, $\lim_{k\to+\infty} \rho(\tilde{\sigma}_{n_k}, \tilde{\sigma}_{n_k+1}) = 0 = \lim_{k\to+\infty} \rho(\tilde{\sigma}_{m_k}, \tilde{\sigma}_{m_k+1})$, and $\eta, \theta > 0$ and $\eta + 2\theta < 1$, then $\limsup_{k\to+\infty} \rho(\tilde{\sigma}_{n_k+1}, \tilde{\sigma}_{m_k+2}) = 0$. We have

$$\varepsilon \leqslant \zeta_k < \Phi\left(\rho(\tilde{\sigma}_{n_k}, \tilde{\sigma}_{n_k+1}), \Phi(\rho(\tilde{\sigma}_{n_k+1}, \tilde{\sigma}_{m_k+1}), \rho(\tilde{\sigma}_{m_k+1}, \tilde{\sigma}_{m_k}))\right). \tag{2.41}$$

Hence, taking the limit as $k \to +\infty$, will have $\varepsilon \leqslant \lim_{k\to+\infty} \zeta_k \leqslant \Phi(0, \Phi(0,0))) = \Phi(0,0) = 0$. We reach a contradiction by observing that $\varepsilon = 0$. Consequently, $(\tilde{\sigma}_n)_{n\geqslant 0}$ is a Cauchy sequence in E_{1_0}. Since E_{1_0} is a closed set, the sequence $(\tilde{\sigma}_n)_{n\geqslant 0}$ converges to an element $\kappa^* \in E_{1_0}$.

- $VE_{1_0} \subset E_{2_0}$, $SE_{1_0} \subset E_{2_0}$ and $\kappa^* \in E_{1_0}$, there exists $(\mu_1, \mu_2) \in E_1^2$ s.t.

$$\rho(\mu_1, V\kappa^*) = \operatorname{dist}(E_1, E_2), \qquad \rho(\mu_2, S\kappa^*) = \operatorname{dist}(E_1, E_2). \tag{2.42}$$

As the sequence $(\tilde{\sigma}_n)_{n\geqslant 0}$ is α-regular and $\alpha(\tilde{\sigma}_n, \tilde{\sigma}_{n+1}) \geqslant 1$, there exists a subsequence $(\tilde{\sigma}_{n_k})_{k\geqslant 0}$ of $(\tilde{\sigma}_n)_{n\geqslant 0}$ s.t. $\min\{\alpha(\tilde{\sigma}_{n_k}, \kappa^*), \alpha(\kappa^*, \tilde{\sigma}_{n_k}) \geqslant 1, \ \ \forall\, k \in \mathbb{N}$. Furthermore,

$$\rho(\tilde{\sigma}_{n_k+1}, V\tilde{\sigma}_{n_k}) = \operatorname{dist}(E_1, E_2), \qquad \rho(\mu_2, S\kappa^*) = \operatorname{dist}(E_2, E_2), \tag{2.43}$$

for any $k \in \mathbb{N}$ and as (V, S) is an α-generalized G_{pi} contraction pair type HR, one has

$$\begin{aligned}
\Phi(\rho(\tilde{\sigma}_{n_k+1}, \mu_2), \rho(\tilde{\sigma}_{n_k+1}, \mu_2)) &\leqslant (\rho(\tilde{\sigma}_{n_k}, \kappa^*))^{\eta} \left(\rho(\tilde{\sigma}_{n_k}, \tilde{\sigma}_{n_k+1})\right)^{\theta} \\
&\qquad \cdot (\rho(\kappa^*, \mu_2))^{\theta} \left[\rho(\tilde{\sigma}_{n_k}, \mu_2) + \rho(\kappa^*, \tilde{\sigma}_{n_k+1})\right]^{1-\eta-2\theta}.
\end{aligned} \tag{2.44}$$

As $\rho(\tilde{\sigma}_{n_k}, \mu_2) \leqslant \Phi(\rho(\tilde{\sigma}_{n_k}, \kappa^*), \rho(\kappa^*, \mu_2))$, for all $k \in \mathbb{N}$, then $\limsup_{k\to+\infty} \rho(\tilde{\sigma}_{n_k}, \mu_2) < +\infty$. By taking the limit as $k \to +\infty$ in the previous inequality, we deduce that

$$\lim_{k\to+\infty} \Phi(\rho(\tilde{\sigma}_{n_k+1}, \mu_2), \rho(\tilde{\sigma}_{n_k+1}, \mu_2)) = 0. \tag{2.45}$$

Because

$$\Phi(\rho(\tilde{\sigma}_{n_k+1}, \mu_2), \rho(\tilde{\sigma}_{n_k+1}, \mu_2)) \geqslant \rho(\tilde{\sigma}_{n_k+1}, \mu_2), \qquad k \in \mathbb{N}, \tag{2.46}$$

then $\mu_2 = \lim_{k\to+\infty} \tilde{\sigma}_{n_k+1} = \kappa^*$. Thus, $\upsilon = \kappa^*$ and $\rho(\kappa^*, S\kappa^*) = \operatorname{dist}(E_1, E_2)$. Similarly, one can show that $\rho(\kappa^*, V\mathfrak{t}^*) = \operatorname{dist}(E_1, E_2)$. □

Corollary 2.1 *Consider a complete regular* SMS $(\mathfrak{Q}, \rho, \Phi)$*, and let E be a non-empty closed subset of $\mathfrak{Q}$. Let $V, S : E \to E$ be two mappings satisfying*

$$\begin{aligned}\alpha(\kappa, \mathfrak{s})\Phi(\rho(S\kappa, V\mathfrak{s}), \rho(S\kappa, V\mathfrak{s})) &\leqslant \psi(\rho(\kappa, \mathfrak{s}))(\rho(\kappa, \mathfrak{s}))^{\eta}(\rho(\kappa, S\kappa))^{\theta}(\rho(\mathfrak{s}, V\mathfrak{s}))^{\theta}\\ &\quad\cdot [\rho(\kappa, V\mathfrak{s}) + \rho(\mathfrak{s}, S\kappa)]^{1-\eta-2\theta},\end{aligned} \tag{2.47}$$

for all $\kappa, \mathfrak{s} \in E$, where $0 < \eta, \theta < 1$ with $\eta + 2\theta < 1$, and the same conditions (ii), (iii) *and* (iv) *of Theorem 2.2. Then, there exists a point $\kappa^* \in A$ s.t. $S\kappa^* = \kappa^* = V\kappa^*$.*

Proof: Thanks to Eq. (2.47), the pair (V, S) is a α-generalized G_{pi} contraction type HR with $E_1 = E_2 = E$. □

The following represents an illustration of the previous result.

Example 2.1 *Let $\mathfrak{Q} = \mathbb{R}^2$ with the following distance*

$$\rho((\tilde{\sigma}_1, \mathfrak{s}_1), (\tilde{\sigma}_2, \mathfrak{s}_2)) = |\tilde{\sigma}_1 - \tilde{\sigma}_2| + |\mathfrak{s}_1 - \mathfrak{s}_2|, \tag{2.48}$$

and assume that

$$\begin{aligned} E_1 &= \left\{(0, \tfrac{1}{2})\ (0, \tfrac{1}{3})\ (0, \tfrac{1}{4})\ (0, \tfrac{1}{5})\ (0, \tfrac{1}{6})\ (0, \tfrac{1}{7})\ (0, \tfrac{1}{9})\right\}\\ &\quad \cup \Big\{(0, n) : n \in \mathbb{N}\Big\},\\ E_2 &= \left\{(1, 0)\ (1, \tfrac{1}{2})\ (1, \tfrac{1}{3})\ (1, \tfrac{1}{4})\ (1, \tfrac{1}{5})\ (1, \tfrac{1}{6})\ (1, \tfrac{1}{7})\ (1, \tfrac{1}{9})\right\}\\ &\quad \cup \Big\{(1, n)\ :\ n \in \mathbb{N}, n \geqslant 5\Big\}. \end{aligned} \tag{2.49}$$

It is easily verified that $(\mathfrak{Q}, \rho)$ is a complete MS *and* $\operatorname{dist}(E_1, E_2) = 1$*. Consider the mappings $V, S : E_1 \to E_2$ as follow*

$$\begin{cases} S(0,0) = (1,0),\\ S(0, \frac{1}{k}) = (1,0),\\ S(0,1) = (1,0),\\ S(0,2) = (1, \frac{1}{2}),\\ S(0,3) = (1, \frac{1}{4}),\\ S(0,4) = (1, \frac{1}{6}),\\ S(0,n) = (1,0), \end{cases} \qquad \begin{cases} V(0,0) = (1,0),\\ V(0, \frac{1}{k}) = (1,0), & k \in \{2,3,4,5,6,7,9\},\\ V(0,1) = (1, \frac{1}{9}),\\ V(0,2) = (1, \frac{1}{3}),\\ V(0,3) = (1, \frac{1}{5}),\\ V(0,4) = (1, \frac{1}{7}),\\ V(0,n) = (1,n), & n \in \mathbb{N},\ n \geqslant 5. \end{cases} \tag{2.50}$$

Also, define a function $\alpha : \mathfrak{Q}^2 \to \mathbb{R}_{\geq 0}$ by

$$\begin{aligned}&\alpha((\tilde{\sigma}_1, \mathfrak{s}_1), (\tilde{\sigma}_2, \mathfrak{s}_2))\\ &\quad = \begin{cases} 1 + \frac{1}{9} = \frac{10}{9} & (\tilde{\sigma}_1, \mathfrak{s}_1), (\tilde{\sigma}_2, \mathfrak{s}_2) \in \{(0,k)\ :\ k \in \{1,2,3,4\}\},\\ & (\tilde{\sigma}_1, \mathfrak{s}_1) \neq (\tilde{\sigma}_2, \mathfrak{s}_2),\\ 1, & (\tilde{\sigma}_1, \mathfrak{s}_1), (\tilde{\sigma}_2, \mathfrak{s}_2) \in \left\{(0,0), (0, \frac{1}{k})\ :\ k \in \{2,3,4,5,6,7,9\}\right\},\\ 0 & \text{otherwise}. \end{cases}\end{aligned} \tag{2.51}$$

The function α *is triangular and any sequence of* E_1 *is* α*-regular. One can check that* $S(E_{1_0}) \subset E_{2_0}$ *and* $V(E_{1_0}) \subset E_{2_0}$*, where*

$$\begin{aligned} E_{1_0} &= \left\{(0,0)\,\left(0,\tfrac{1}{2}\right)\,\left(0,\tfrac{1}{3}\right)\,\left(0,\tfrac{1}{4}\right)\,\left(0,\tfrac{1}{5}\right)\,\left(0,\tfrac{1}{6}\right)\,\left(0,\tfrac{1}{7}\right)\,\left(0,\tfrac{1}{9}\right)\right\} \cup \Big\{(0,n)\,:\, n\in\mathbb{N},\, n\geqslant 5\Big\},\\ E_{2_0} &= \left\{(1,0)\,\left(1,\tfrac{1}{2}\right)\,\left(1,\tfrac{1}{3}\right)\,\left(1,\tfrac{1}{4}\right)\,\left(1,\tfrac{1}{5}\right)\,\left(1,\tfrac{1}{6}\right)\,\left(1,\tfrac{1}{7}\right)\,\left(1,\tfrac{1}{9}\right)\right\} \cup \Big\{(1,n)\,:\, n\in\mathbb{N},\, n\geqslant 5\Big\}. \end{aligned} \tag{2.52}$$

Choose $\eta = \frac{1}{3}$ *and* $\theta = \frac{1}{4}$*, then* $\eta + 2\theta = \frac{5}{6} < 1$.

• *First task is to demonstrate that* (V,S) *fulfills the properties of being an* α*-generalized* G_{pi} *contraction pair type* HR*, with* $\psi : \mathbb{R}^{\geq 0} \to [0,1)$ *defined by* $\psi(\kappa) = \frac{3}{3+\kappa}$ *whenever* $\kappa \in]0,+\infty[$ *and* $\psi(0) = 0$*. We notice that* $\psi \in \Psi$*, because if* $\limsup_{n\to+\infty} \psi(\tilde{\sigma}_n) = 1$*, then there exists a subsequence* $(\tilde{\sigma}_{n_k})_{k\geqslant 0}$ *of* $(\tilde{\sigma}_n)_{n\geqslant 0}$ *s.t.* $\lim_{k\to+\infty} \psi(\tilde{\sigma}_{n_k}) = 0$*. We mention that* $\psi(1), \psi(2), \psi(3), \psi(\frac{3}{2})$ *belong to* $[\frac{1}{2}, +\infty[$*, which we will use later. Take* $\kappa = (0,\tilde{\sigma}_2)$*,* $\mathfrak{s} = (0,\mathfrak{s}_2)$*,* $\mu = (0,\mu_2)$*,* $\upsilon = (0,\upsilon_2)$ *four elements of* E_1 *s.t.*

$$\begin{cases} \alpha\left((0,\tilde{\sigma}_2),(0,\mathfrak{s}_2)\right) \geqslant 1,\\ \rho((\mu, V\kappa) = \operatorname{dist}(E_1,E_2) = 1,\\ \rho(\upsilon, S\mathfrak{s}) = \operatorname{dist}(E_1,E_2) = 1, \end{cases} \tag{2.53}$$

verifying that

$$\begin{aligned} &\psi(\rho(\kappa,\mathfrak{s}))(\rho(\kappa,\mathfrak{s}))^{\eta}(\rho(\kappa,\mu_1))^{\theta}(\rho(\mathfrak{s},\mu_2))^{\theta}\left[\rho(\kappa,\mu_2)+\rho(\mathfrak{s},\mu_1)\right]^{1-\eta-2\theta}\\ &\qquad\qquad \geqslant \alpha(\kappa,\mathfrak{s})\ \Phi(\rho(\mu_1,\mu_2),\rho(\mu_1,\mu_2)), \end{aligned} \tag{2.54}$$

where $\Phi(u,v) = u+v$*, for all* $u,v \in \mathbb{R}_{>0}$*. Table 2.1 summarizes the calculations to verify that* (V,S) *is a* α*-generalized* G_{pi} *contraction pair type* HR*. Denote*

$$\begin{aligned} \mathcal{G} &= \psi(\rho(\kappa,\mathfrak{s}))(\rho(\kappa,\mathfrak{s}))^{\eta}(\rho(\kappa,\mu_1))^{\theta}(\rho(\mathfrak{s},\mu_2))^{\theta}\left[\rho(\kappa,\mu_2)+\rho(\mathfrak{s},\mu_1)\right]^{1-\eta-2\theta},\\ \mathcal{F} &= \alpha(\kappa,\mathfrak{s})\ \Phi(\rho(\mu_1,\mu_2),\rho(\mu_1,\mu_2)). \end{aligned} \tag{2.55}$$

κ	$\mathfrak{s}$	μ_1	μ_2	$\mathcal{G} \geqslant \mathcal{F}$	
$(0,1)$	$(0,2)$	$(0,0)$	$(0,\frac{1}{3})$	$\psi(1)\ \left(\frac{5}{3}\right)^{\frac{1}{4}}\left(\frac{8}{3}\right)^{\frac{1}{6}} \approx 1.0035$	$\frac{20}{27} = \left(\frac{10}{9}\right)\ \left(\frac{2}{3}\right) = 2\alpha(\kappa,\mathfrak{s})\ \rho(\mu_1,\mu_2)$
$(0,1)$	$(0,3)$	$(0,0)$	$(0,\frac{1}{5})$	$\psi(2)\ 2^{\eta}\ \left(\frac{14}{5}\right)^{\theta}\ \left(\frac{19}{5}\right)^{1-\eta-2\theta} \approx 1.22155$	$\frac{20}{45} = \left(\frac{10}{9}\right)\ \left(\frac{2}{5}\right) = 2\alpha(\kappa,\mathfrak{s})\ \rho(\mu_1,\mu_2)$
$(0,1)$	$(0,4)$	$(0,0)$	$(0,\frac{1}{7})$	$\psi(3)\ 3^{\eta}\ \left(\frac{27}{7}\right)^{\theta}\left(\frac{34}{7}\right)^{1-\eta-2\theta} \approx 1.31514$	$\frac{20}{63} = \left(\frac{10}{9}\right)\ \left(\frac{2}{7}\right) = 2\alpha(a,b)\ \rho(\mu_1,\mu_2)$
$(0,2)$	$(0,1)$	$(0,\frac{1}{2})$	$(0,\frac{1}{9})$	$\psi\left(\frac{3}{2}\right)\ \left(\frac{3}{2}\right)^{\eta}\ \left(\frac{8}{9}\right)^{\theta}\left(\frac{25}{9}\right)^{1-\eta-2\theta} \approx 0.87855$	$\frac{70}{81} = \left(\frac{10}{9}\right)\ \left(\frac{7}{9}\right) = 2\alpha(\kappa,\mathfrak{s})\ \rho(\mu_1,\mu_2)$
$(0,2)$	$(0,3)$	$(0,\frac{1}{2})$	$(0,\frac{1}{5})$	$\psi\left(\frac{3}{2}\right)\ \left(\frac{3}{2}\right)^{\eta}\ \left(\frac{14}{5}\right)^{\theta}\left(\frac{43}{10}\right)^{1-\eta-2\theta} \approx 1.25884$	$\frac{2}{3} = \left(\frac{10}{9}\right)\ \left(\frac{6}{10}\right) = 2\alpha(\kappa,\mathfrak{s})\ \rho(\mu_1,\mu_2)$
$(0,2)$	$(0,4)$	$(0,\frac{1}{2})$	$(0,\frac{1}{7})$	$\psi(2)2^{\eta}\ \left(\frac{3}{2}\right)^{\theta}\ \left(\frac{27}{7}\right)^{\theta}\left(\frac{75}{14}\right)^{1-\eta-2\theta} \approx 1.55086$	$\frac{50}{63} = \left(\frac{10}{9}\right)\ \left(\frac{5}{7}\right) = 2\alpha(\kappa,\mathfrak{s})\ \rho(\mu_1,\mu_2)$
$(0,3)$	$(0,1)$	$(0,\frac{1}{4})$	$(0,\frac{1}{9})$	$\psi(2)2^{\eta}\ \left(\frac{11}{4}\right)^{\theta}\ \left(\frac{8}{9}\right)^{\theta}\left(\frac{131}{36}\right)^{1-\eta-2\theta} \approx 1.17228$	$\frac{25}{162} = \left(\frac{10}{9}\right)\ \left(\frac{5}{18}\right) = 2\alpha(\kappa,\mathfrak{s})\ \rho(\mu_1,\mu_2)$
$(0,3)$	$(0,2)$	$(0,\frac{1}{4})$	$(0,\frac{1}{3})$	$\psi(1)\ \left(\frac{11}{4}\right)^{\theta}\ \left(\frac{5}{3}\right)^{\theta}\left(\frac{53}{12}\right)^{1-\eta-2\theta} \approx 1.40563$	$\frac{5}{27} = \left(\frac{10}{9}\right)\ \left(\frac{1}{6}\right) = 2\alpha(\kappa,\mathfrak{s})\ \rho(\mu_1,\mu_2)$
$(0,3)$	$(0,4)$	$(0,\frac{1}{4})$	$(0,\frac{1}{7})$	$\psi(1)\ \left(\frac{11}{4}\right)^{\theta}\ \left(\frac{27}{7}\right)^{\theta}\left(\frac{185}{28}\right)^{1-\eta-2\theta} \approx 1.85408$	$\frac{5}{21} = \left(\frac{10}{9}\right)\ \left(\frac{3}{14}\right) = 2\alpha(\kappa,\mathfrak{s})\ \rho(\mu_1,\mu_2)$
$(0,4)$	$(0,1)$	$(0,\frac{1}{6})$	$(0,\frac{1}{9})$	$\psi(3)3^{\eta}\ \left(\frac{23}{6}\right)^{\theta}\ \left(\frac{8}{9}\right)^{\theta}\left(\frac{255}{54}\right)^{1-\eta-2\theta} \approx 1.26903$	$\frac{10}{81} = \left(\frac{10}{9}\right)\ \left(\frac{3}{27}\right) = 2\alpha(\kappa,\mathfrak{s})\ \rho(\mu_1,\mu_2)$
$(0,4)$	$(0,2)$	$(0,\frac{1}{6})$	$(0,\frac{1}{3})$	$\psi(2)2^{\eta}\ \left(\frac{23}{6}\right)^{\theta}\ \left(\frac{5}{3}\right)^{\theta}\left(\frac{11}{2}\right)^{1-\eta-2\theta} \approx 1.59677$	$\frac{10}{27} = \left(\frac{10}{9}\right)\ \left(\frac{1}{3}\right) = 2\alpha(\kappa,\mathfrak{s})\ \rho(\mu_1,\mu_2)$
$(0,4)$	$(0,3)$	$(0,\frac{1}{6})$	$(0,\frac{1}{5})$	$\psi(1)\ \left(\frac{23}{6}\right)^{\theta}\ \left(\frac{14}{5}\right)^{\theta}\left(\frac{199}{30}\right)^{1-\eta-2\theta} \approx 1.86080$	$\frac{4}{27} = \left(\frac{10}{9}\right)\ \left(\frac{2}{15}\right) = 2\alpha(\kappa,\mathfrak{s})\ \rho(\mu_1,\mu_2)$

Table 1: Calculations of (V,S).

Moreover, for $\kappa = (0,\tilde{\sigma}_1)$ *and* $\mathfrak{s} = (0,\mathfrak{s}_2)$*, where* $\tilde{\sigma}_1, \mathfrak{s}_2 \in \{0, \frac{1}{k}\}$*, with* $k \in \{2,3,4,5,6,7,9\}$*, one has* $\mu = \upsilon = (0,0)$*, so*

$$\begin{aligned} 2\alpha(\kappa,\mathfrak{s})\rho(\mu_1,\mu_2) = 0 \leqslant\ & \psi(\rho(\kappa,\mathfrak{s}))(\rho(\kappa,\mathfrak{s}))^{\eta}\\ & \cdot \rho(\kappa,\mu_1)^{\theta}(\rho(\mathfrak{s},\mu_2))^{\theta}\left[\rho(\kappa,\mu_2)+\rho(\mathfrak{s},\mu_1)\right]^{1-\eta-2\theta}. \end{aligned} \tag{2.56}$$

Thus, (V,S) is a α-generalized $\mathsf{G}_{\rm pi}$ contraction pair type HR*. We confirm that (V,S) is α-proximal admissible. For example, if one takes $(0,2)$ and $(0,3)$. Moreover, if $\mu_1=(0,u_2)\in E_1$ and $\mu_2=(0,v_2)\in E_1$ s.t.*

$$\begin{cases}\alpha((0,2),(0,3))\geqslant 1,\\ \rho((\mu_1,V(0,2))=\operatorname{dist}(E_1,E_2)=1,\\ \rho(\mu_2,S(0,3))=\operatorname{dist}(E_1,E_2)=1,\end{cases}\implies \mu_1=\left(0,\tfrac{1}{2}\right),\ \upsilon=\left(0,\tfrac{1}{5}\right). \tag{2.57}$$

Then, $\min\{\alpha(\mu_1,\mu_2),\,\alpha(\mu_2,\mu_1)\}=\frac{10}{9}>1$. If one takes $(0,0)$ and $(0,\frac{1}{2})$. Moreover, if $\mu_1=(0,u_2)\in E_1$ and $\mu_2=(0,v_2)\in E_1$ s.t.

$$\begin{cases}\alpha\left((0,0),\left(0,\tfrac{1}{2}\right)\right)\geqslant 1,\\ \rho(\mu_1,V(0,0))=\operatorname{dist}(E_1,E_2)=1,\\ \rho\left(\mu_2,S\left(0,\tfrac{1}{2}\right)\right)=\operatorname{dist}(E_1,E_2)=1,\end{cases}\implies \mu_1=\mu_2=(0,0). \tag{2.58}$$

Then, $\min\{\alpha(\mu_1,\mu_2),\alpha(\mu_2,\mu_1)\}=1$. For condition (iv)*, take $(\tilde\sigma_0,\tilde\sigma_1)=((0,\frac{1}{k}),(0,0))\in E_1^2$, where $k\in\{2,3,4,5,6,7,9\}$. Then,*

$$\min\left\{\alpha(\tilde\sigma_0,\tilde\sigma_1),\,\alpha(\tilde\sigma_1,\tilde\sigma_0)\right\}\geqslant 1,\qquad \rho(\tilde\sigma_1,V\tilde\sigma_0)=\operatorname{dist}(E_1,E_2). \tag{2.59}$$

Thus, the hypotheses of Theorem 2.2 are fulfilled and

$$\rho((0,0),V(0,0))=\rho((0,0),S(0,0))=\operatorname{dist}(E_1,E_2). \tag{2.60}$$

For the uniqueness of the CBPP of V and S in Theorem 2.2 and Corollary 2.1, we introduce the following necessary condition $CB(V,S)$: for all $\kappa,\mathfrak{s}\in E_1$,

$$\begin{cases}\rho(\kappa,V\kappa))=\rho(\kappa,S\kappa)=\operatorname{dist}(E_1,E_2),\\ \rho(\mathfrak{s},V\mathfrak{s}))=\rho(\mathfrak{s},S\mathfrak{s})=\operatorname{dist}(E_1,E_2),\end{cases}\implies \alpha(\kappa,\mathfrak{s})\geqslant 1. \tag{2.61}$$

Theorem 2.3 *Adding the condition $CB(V,S)$ to the hypothesis of Theorems 2.2 (resp. Corollary 2.1), there exists an unique* CBPP *of V and S.*

Proof: Suppose (V,S) is a α-generalized $\mathsf{G}_{\rm pi}$ contraction pair type HR. Let $\tilde\sigma_1,\tilde\sigma_2\in E_1$ s.t.

$$\rho(\tilde\sigma_1,V\tilde\sigma_1)=\rho(\tilde\sigma_1,S\tilde\sigma_1)=\operatorname{dist}(E_1,E_2),\qquad \rho(\tilde\sigma_2,V\tilde\sigma_2)=\rho(\tilde\sigma 2,S\tilde\sigma_2)=\operatorname{dist}(E_1,E_2). \tag{2.62}$$

By the condition $CB(V,S)$, one has $\alpha(\tilde\sigma_1,\tilde\sigma_2)\geqslant 1$. Then

$$\begin{aligned}\Phi(\rho(\tilde\sigma_1,\tilde\sigma_2),\rho(\tilde\sigma_1,\tilde\sigma_2))&=\alpha(\tilde\sigma_1,\tilde\sigma_2)\ \Phi(\rho(\tilde\sigma_1,\tilde\sigma_2),\rho(\tilde\sigma_1,\tilde\sigma_2))\\ &\leqslant \psi(\rho(\tilde\sigma_1,\tilde\sigma_2))(\rho(\tilde\sigma_1,\tilde\sigma_2))^{\eta}(\rho(\tilde\sigma_1,\tilde\sigma_1))^{\theta}\\ &\quad\cdot[\rho(\tilde\sigma_1,\tilde\sigma_2)+\rho(\tilde\sigma_2,\tilde\sigma_1)]^{1-\eta-2\theta},\end{aligned} \tag{2.63}$$

which give $\tilde\sigma_1=\tilde\sigma_2$. □

3. Application to $\mathbb{FDE}$s

Let $b,v\in\mathbb{R}$ with $b<v$. Consider the complete regular SMS $(\mathfrak{Q},\rho,\Phi)$, where $\mathfrak{Q}=C^1(\mathcal{I})$, $\mathcal{I}:=[b,v]$. The SM, ρ is defined by

$$\rho(w_1,w_1)=\|w_1-w_2\|_\infty^2=\left(\sup_{\kappa\in\mathcal{I}}|w_1(\kappa)-w_2(\kappa)|\right)^2, \tag{3.1}$$

and the regular map Φ associated with ρ is given $\forall\,(\mathfrak{r}_1,\mathfrak{r}_2)\in\mathbb{R}^2_{>0}$, by $\Phi(\mathfrak{r}_1,\mathfrak{r}_2)=2(\mathfrak{r}_1+\mathfrak{r}_2)$. We consider an increasing and positive monotone function ψ on $\mathcal{I}$ s.t. ψ' is continuous on $\mathcal{I}\setminus\{b,v\}$.

Definition 3.1 ([39]) *Let $h \in \mathfrak{Q}$.*

(i) *The integral*

$$\mathcal{I}_{b^+}^{p;\psi} h(\kappa) = \int_b^{\kappa} \tfrac{(\psi(\kappa)-\psi(s))^{p-1}}{\Gamma(p)} \psi'(s) h(s)\,\mathrm{d}s, \qquad \kappa > b, \tag{3.2}$$

is referred to as the left-sided fractional integral of h of order $p > 0$ on $\mathcal{I}$, with respect to function ψ;

(ii) *If for each $\kappa \in \mathcal{I}$, $\psi'(\kappa) \neq 0$, then left-sided-Hilfer fractional derivative of order p and type $0 \leqslant q \leqslant 1$ of h is expressed as follows*

$$^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi} h(\kappa) = \mathcal{I}_{b^+}^{q(1-p);\psi} \left(\tfrac{1}{\psi'(\kappa)} \tfrac{\mathrm{d}}{\mathrm{d}\kappa} \right) \mathcal{I}_{b^+}^{(1-q)(1-p);\psi} h(\kappa), \qquad \kappa > b,\ 0 < p < 1. \tag{3.3}$$

It can be written as

$$^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi} h(\kappa) = \mathcal{I}_{b^+}^{\gamma-p;\psi} \mathcal{D}_{b^+}^{\gamma;\psi} h(\kappa), \qquad \gamma = p + q(1-p), \tag{3.4}$$

where $\mathcal{D}_{b^+}^{\gamma;\psi}$ is the ψ-Riemann-Liouville (RL) fractional derivative

$$\mathcal{D}_{b^+}^{\gamma;\psi} h(\kappa) = \left(\tfrac{1}{\psi'(\kappa)} \tfrac{\mathrm{d}}{\mathrm{d}\kappa} \right) \mathcal{I}_{b^+}^{1-\gamma;\psi} h(\kappa). \tag{3.5}$$

Theorem 3.1 ([33]) *If $h \in \mathfrak{Q}$, $0 < p < 1$ and $0 \leqslant q \leqslant 1$, then*

$$\mathcal{I}_{b^+}^{p;\psi}\ {}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi} h(\kappa) = h(\kappa) - \tfrac{(\psi(\kappa)-\psi(b))^{\gamma-1}}{\Gamma(\gamma)} \mathcal{I}_{b^+}^{(1-p)(1-q);\psi} h(b), \quad \gamma = p + q(1-p). \tag{3.6}$$

The following results pertain to fractional derivatives.

Lemma 3.1 ([39]) . *For all $p, q \geqslant 0$ and $h \in L^1(b, v)$,*

$$\mathcal{I}_{b^+}^{p;\psi} \mathcal{I}_{b^+}^{q;\psi} h(\kappa) = \mathcal{I}_{b^+}^{p+q;\psi} h(\kappa), \qquad \kappa \in [b, v]. \tag{3.7}$$

Lemma 3.2 ([39]) *For $\kappa > 0$, $p > 0$ and $\delta > 0$, we have*

$$\mathcal{I}_{b^+}^{p;\psi} (\psi(\kappa) - \psi(b))^{\delta-1} = \tfrac{\Gamma(\delta)}{\Gamma(p+\delta)} (\psi(\kappa) - \psi(b))^{p+\delta-1}. \tag{3.8}$$

Theorem 3.2 ([33]) *Let $h \in \mathfrak{Q}$, $p > 0$ and $0 \leqslant q \leqslant 1$. Also, ${}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi}\ \mathcal{I}_{b^+}^{p;\psi} h(\kappa) = h(\kappa)$.*

We consider a coupled system of right sided-Hilfer $\mathbb{FDE}$ with arbitrary order under initial conditions as form

$$\begin{cases} {}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi} \mu_1(\kappa) = \lambda_1(\kappa, \mu_2(\kappa)), \\ {}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi} \mu_2(\kappa) = \lambda_2(\kappa, \mu_1(\kappa)), \end{cases} \tag{3.9}$$

for $b < \kappa \leq v$, under conditions

$$\mathcal{I}_{b^+}^{(1-p)(1-q);\psi} \mu_1(b) = \mathcal{I}_{b^+}^{(1-p)(1-q);\psi} \mu_2(b) = a_0, \tag{3.10}$$

where $\mu_1 \in \mathfrak{Q}$, $a_0 \in \mathbb{R}_{\leq 0}$ and $\lambda_1, \lambda_2 \in C(\mathcal{I} \times \mathbb{R})$. Set $\gamma = p + q - pq$.

Lemma 3.3 *The system (3.9) is equivalent to the integral equations, $\kappa \in \mathcal{I}$,*

$$\begin{cases} \mu_1(\kappa) = \tfrac{(\psi(\kappa)-\psi(b))^{\gamma-1}}{\Gamma(\gamma)} a_0 + \displaystyle\int_b^{\kappa} \psi'(r) \tfrac{(\psi(\kappa)-\psi(r))^{p-1}}{\Gamma(p)} \lambda_1(r, \mu_2(r))\,\mathrm{d}r, \\ \mu_2(\kappa) = \tfrac{(\psi(\kappa)-\psi(b))^{\gamma-1}}{\Gamma(\gamma)} a_0 + \displaystyle\int_b^{\kappa} \psi'(r) \tfrac{(\psi(\kappa)-\psi(r))^{p-1}}{\Gamma(p)} \lambda_2(r, \mu_1(r))\,\mathrm{d}r. \end{cases} \tag{3.11}$$

Proof: The maps $\kappa \mapsto \lambda_1(\kappa, \mu_2(\kappa))$ and $\kappa \mapsto \lambda_2(\kappa, \mu_1(\kappa))$ are continuous on $\mathcal{I}$. Taking the integral operator $\mathcal{I}_{b^+}^{p;\psi}(\cdot)$ on (3.9), for $\kappa \in \mathcal{I} \setminus \{b\}$, one has

$$\begin{cases} \mathcal{I}_{b^+}^{p;\psi}\ {}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi}\mu_1(\kappa) = \mathcal{I}_{b^+}^{p;\psi}\lambda_1(\kappa, \mu_2(\kappa)), \\ \mathcal{I}_{b^+}^{p;\psi}\ {}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi}\mu_2(\kappa) = \mathcal{I}_{b^+}^{p;\psi}\lambda_2(\kappa, \mu_1(\kappa)). \end{cases} \tag{3.12}$$

According to [33, Theorem 7],

$$\begin{cases} \mathcal{I}_{b^+}^{p;\psi}\ {}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi}\mu_1(\kappa) = \mu_1(\kappa) - \frac{(\psi(\kappa)-\psi(b))^{\gamma-1}}{\Gamma(\gamma)}\mathcal{I}_{b^+}^{1-\gamma;\psi}\mu_1(b), \\ \mathcal{I}_{b^+}^{p;\psi}\ {}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi}\mu_2(\kappa) = \mu_2(\kappa) - \frac{(\psi(\kappa)-\psi(b))^{\gamma-1}}{\Gamma(\gamma)}\mathcal{I}_{b^+}^{1-\gamma;\psi}\mu_2(b). \end{cases} \tag{3.13}$$

Then,

$$\begin{cases} \mu_1(\kappa) = \frac{(\psi(\kappa)-\psi(b))^{\gamma-1}}{\Gamma(\gamma)}\mathcal{I}_{b^+}^{1-\gamma;\psi}\mu_1(b) + \mathcal{I}_{b^+}^{p;\psi}\lambda_1(\kappa, \mu_2(\kappa)), \\ \mu_2(\kappa) = \frac{(\psi(\kappa)-\psi(b))^{\gamma-1}}{\Gamma(\gamma)}\mathcal{I}_{b^+}^{1-\gamma;\psi}\mu_2(b) + \mathcal{I}_{b^+}^{p;\psi}\lambda_2(\kappa, \mu_1(\kappa)). \end{cases} \tag{3.14}$$

The two initial conditions (3.10), for $\kappa \in \mathcal{I} \setminus \{b\}$, lead to

$$\begin{cases} \mu(\kappa) = \frac{(\psi(\kappa)-\psi(b))^{\gamma-1}}{\Gamma(\gamma)}a_0 + \displaystyle\int_b^\kappa \psi'(r)\frac{(\psi(\kappa)-\psi(r))^{p-1}}{\Gamma(p)}\lambda_1(r, \mu_2(r))\,\mathrm{d}r, \\ \mu_2(\kappa) = \frac{(\psi(\kappa)-\psi(b))^{\gamma-1}}{\Gamma(\gamma)}a_0 + \displaystyle\int_b^\kappa \psi'(r)\frac{(\psi(\kappa)-\psi(r))^{p-1}}{\Gamma(p)}\lambda_2(r, \mu_1(r))\,\mathrm{d}r. \end{cases} \tag{3.15}$$

Furthermore, thanks to the ψ-Hilfer fractional derivative ${}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi}(\cdot)$ on Eqs. (3.9) and since

$${}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi}\left[\frac{(\psi(\kappa)-\psi(b))^{\gamma-1}}{\Gamma(\gamma)}\mathcal{I}_{v^+}^{1-\gamma;\psi}a_0\right] = 0, \tag{3.16}$$

one has

$$\begin{cases} {}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi}\mu_1(\kappa) = {}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi}\mathcal{I}_{b^+}^{p;\psi}\lambda_1(\mathfrak{t}, \mu_2(\kappa)), \\ {}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi}\mu_2(\kappa) = {}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi}\mathcal{I}_{b^+}^{p;\psi}\lambda_2(\kappa, \mu_1(\kappa)). \end{cases} \tag{3.17}$$

Using [33, Theorem 2], we obtain

$$\begin{cases} {}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi}\mu_1(\kappa) = \lambda_1(\kappa, \mu_2(\kappa)), \\ {}^{\mathsf{H}}\mathbb{D}_{b^+}^{p,q;\psi}\mu_2(\kappa) = \lambda_1(\kappa, \mu_1(\kappa)). \end{cases} \tag{3.18}$$

□

The set $\mathcal{M} = \left\{\mu \in \mathfrak{Q} : \mu \text{ is increasing on } \mathcal{I} \text{ and } \mathcal{I}_{b^+}^{(1-p)(1-q);\psi}\mu(b) = a_0\right\}$ is non-empty. Consider the maps S and V defined on $\mathcal{M}$, for $\kappa \in \mathcal{I}$, by

$$\begin{cases} S\mu_1(\kappa) = \frac{(\psi(\kappa)-\psi(b))^{\gamma-1}}{\Gamma(\gamma)}a_0 + \displaystyle\int_b^\kappa \psi'(r)\frac{(\psi(\kappa)-\psi(r))^{p-1}}{\Gamma(p)}\lambda_1(r, \mu_1(r))\,\mathrm{d}r, \\ V\mu_2(\kappa) = \frac{(\psi(\kappa)-\psi(b))^{\gamma-1}}{\Gamma(\gamma)}a_0 + \displaystyle\int_b^\kappa \psi'(r)\frac{(\psi(\kappa)-\psi(r))^{p-1}}{\Gamma(p)}\lambda_2(r, \mu_2(r))\,\mathrm{d}r. \end{cases} \tag{3.19}$$

Lemma 3.4 *If λ_1 and λ_1 are increasing with respect to each variable, then for all $\mu_1, \mu_2 \in \mathcal{M}$, we have $S\mu_1 \in \mathcal{M}$ and $V\mu_2 \in \mathcal{M}$.*

Proof: Let $\mu_1, \mu_2 \in \mathcal{M}$ and set $\gamma = p + q - pq$. The mappings $S\mu_1$ and $V\mu_2$ belong to the class $\mathcal{C}^1$ on $\mathcal{I}$, because μ_1, μ_2 are in $\mathfrak{Q}$. Taking $\mathcal{I}_{b^+}^{1-\gamma;\psi}$ on both sides and applying Lemmas 3.1, 3.2, we get

$$\begin{aligned} \mathcal{I}_{b^+}^{1-\gamma;\psi}S\mu_1(\kappa) &= \tfrac{a_0}{\Gamma(\gamma)}\mathcal{I}_{b^+}^{1-\gamma;\psi}(\psi(\kappa)-\psi(b))^{\gamma-1} + \mathcal{I}_{b^+}^{1-\gamma;\psi}\mathcal{I}_{b^+}^{p;\psi}\lambda_1(\kappa, \mu_1(\kappa)) \\ &= a_0 + \mathcal{I}_{b^+}^{1-q(1-p);\psi}\lambda_1(\kappa, \mu_1(\kappa)), \\ \mathcal{I}_{b^+}^{1-\gamma;\psi}V\mu_2(\kappa) &= \tfrac{a_0}{\Gamma(\gamma)}\mathcal{I}_{b^+}^{1-\gamma;\psi}(\psi(\kappa)-\psi(b))^{\gamma-1} + \mathcal{I}_{b^+}^{1-\gamma;\psi}\mathcal{I}_{b^+}^{p;\psi}\lambda_2(\kappa, \mu_2(\kappa)) \\ &= a_0 + \mathcal{I}_{b^+}^{1-q(1-p);\psi}\lambda_2(\kappa, \mu_2(\kappa)), \end{aligned} \tag{3.20}$$

for $\kappa \in \mathcal{I}$. Since

$$\lim_{\kappa \to b^+} \mathcal{I}_{b^+}^{1-q(1-p);\psi} \lambda_1(\kappa, \mu_1(\kappa)) = \lim_{\kappa \to b^+} \mathcal{I}_{b^+}^{1-q(1-p);\psi} \lambda_2(\kappa, \mu_2(\kappa)) = 0, \tag{3.21}$$

then

$$\mathcal{I}_{b^+}^{(1-p)(1-q);\psi} S\mu_1(b) = a_0, \qquad \mathcal{I}_{b^+}^{(1-p)(1-q);\psi} V\mu_2(b) = a_0. \tag{3.22}$$

Let $b < \kappa_1 < \kappa_2 < v$. Given that μ_i is increasing and λ_i is increasing with respect to each variable, for $i = 1, 2$, it follows that $\mu_1(\kappa_1) \leqslant \mu_1(\kappa_2)$ and $\lambda_1(\kappa_1, \mu_1(\kappa_1)) \leqslant \lambda_1(\kappa_2, \mu_1(\mathfrak{t}_2))$. Similarly, $\lambda_2(\kappa_1, \mu_2(\kappa_1)) \leqslant \lambda_2(\kappa_2, \mu_2(\kappa_2))$. Therefore, $S\mu_1(\kappa_1) \leqslant S\mu_1(\kappa_2)$ and $S\mu_2(\kappa_1) \leqslant S\mu_2(\kappa_2)$. □

We introduce the following assumptions

(H1) $\lambda_i(\cdot, \mu_i(\cdot))$, $i = 1, 2$ is continuous and increasing with respect to each variable;

(H2) $\forall\, \kappa \in \mathcal{I}$ and $u_i \in \mathbb{R}$, $i = 1, 2$,

$$\begin{aligned} |\lambda_1(\kappa, u_1) - \lambda_2(\kappa, u_2)| &\leqslant \beta\, |u_1 - u_2|^\eta |u_1 - K\lambda_1(\kappa, u_1)|^\theta |u_2 - K\lambda_2(\kappa, u_2)|^\theta \\ &\quad \cdot \Big[|u_1 - K\lambda_2(\kappa, u_2)| + |u_2 - K\lambda_1(\kappa, u_1)| \Big]^{1-\eta-2\theta}, \end{aligned} \tag{3.23}$$

where $K = \frac{(\psi(v)-\psi(b))^q}{q\Gamma(q)}$, $0 < \beta < \frac{q\Gamma(q)}{2\sqrt{2}(\psi(v)-\psi(b))^q}$ and $\eta, \theta \in]0, 1[$ with $\eta + 2\theta < 1$;

(H3) $\forall\, \kappa \in \mathcal{I}$ and $u_i \in \mathbb{R}$, $i = 1, 2$,

$$|\lambda_1(\kappa, u_1) - \lambda_1(\kappa, u_2)| \leqslant |u_1 - u_2|, \qquad |\lambda_2(\kappa, u_1) - \lambda_2(\kappa, u_2)| \leqslant |u_1 - u_2|. \tag{3.24}$$

(H4) $K\lambda_1(\kappa, u_1) \leqslant u_1$ and $K\lambda_2(\kappa, u_1) \leqslant u_1$, for all $\kappa \in \mathcal{I}$ and $u_1 \in \mathbb{R}$.

Theorem 3.3 *If* (H1)-(H4) *hold, then the system* (3.9) *admits a unique solution* $\mu^* \in \mathcal{A} = \overline{\mathcal{M}}^d$ *verifying* $T\mu^* = S\mu^* = \mu^*$.

Proof: Step 1: Let $(\mu_1, \mu_2) \in \mathcal{M}^2$ and $\kappa \in \mathcal{I} \setminus \{b\}$. Then, assumption (H2) yields

$$\begin{aligned} |S\mu_1(\kappa) - S\mu_2(\kappa)| &= \left| \int_b^\kappa \psi'(s) \tfrac{(\psi(\kappa)-\psi(\tilde r))^{p-1}}{\Gamma(p)} (\lambda_1(\tilde r, \mu_1(\tilde r)) - \lambda_2(\tilde r, \mu_2(\tilde r)))\, \mathrm{d}\tilde r \right| \\ &\leqslant \int_b^\kappa \psi'(\tilde r) \tfrac{(\psi(\kappa)-\psi(\tilde r))^{p-1}}{\Gamma(p)} |\lambda_1(\tilde r, \mu_1(\tilde r)) - \lambda_2(\tilde r, \mu_2(\tilde r))|\, \mathrm{d}\tilde r \\ &\leqslant \tfrac{\beta}{\Gamma(p)} \int_b^\kappa \psi'(\tilde r)(\psi(\kappa) - \psi(\tilde r))^{p-1} |\mu_1(\tilde r) - \mu_2(\tilde r)|^\eta \\ &\quad \cdot |\mu_1(\tilde r) - K\lambda_1(\tilde r, \mu_1(\tilde r))|^\theta |\mu_2(\tilde r) - K\lambda_2(\tilde r, \mu_2(\tilde r))|^\theta \\ &\quad \cdot \Big[|\mu_1(\tilde r) - K\lambda_2(\tilde r, \mu_2(\tilde r))| \\ &\quad + |\mu_2(\tilde r) - K\lambda_1(\tilde r, \mu_1(\tilde r))| \Big]^{1-\eta-2\theta} \mathrm{d}\tilde r. \end{aligned} \tag{3.25}$$

Let $s \in]b, \kappa]$. By assumptions (H1) and (H4), μ_1 is increasing, λ_1 is increasing with respect to each variable and $a_0 \leqslant 0$, then

$$\begin{aligned} |\mu_1(s) - K\lambda_1(s, \mu_1(s))| &= \mu_1(s) - \tfrac{(\psi(v)-\psi(b))^p}{p\Gamma(q)} \lambda_1(s, \mu_1(s)) \\ &\leqslant \mu_1(s) - \frac{(\psi(s) - \psi(b))^p}{p\Gamma(q)} \lambda_1(s, \mu_1(s)) \\ &= \mu_1(s) - \int_b^s \psi^{'}(\tilde r) \tfrac{(\psi(s)-\psi(\tilde r))^{p-1}}{\Gamma(p)} \lambda_1(s, \mu_1(s))\, \mathrm{d}\tilde r \\ &\leqslant \mu_1(s) - \tfrac{a_0}{\Gamma(\gamma)} \mathcal{I}_{b^+}^{1-\gamma;\psi} (\psi(\kappa) - \psi(b))^{\gamma-1} \\ &\quad - \int_b^s \psi^{'}(\tilde r) \tfrac{(\psi(s)-\psi(\tilde r))^{p-1}}{\Gamma(p)} \lambda_1(\tilde r, \mu_1(\tilde r))\, \mathrm{d}\tilde r \\ &= \mu_1(s) - S\mu_1(s) \leqslant \|\mu_1 - S\mu_1\|_\infty. \end{aligned} \tag{3.26}$$

Likewise, we justify that $|\mu_2(s) - K\lambda_2(s,\mu_2(s))| \leqslant \|\mu_2 - V\mu_2\|_\infty$ and

$$|\mu_1(s) - K\lambda_2(s,\mu_2(s))| + |\mu_2(s) - K\lambda_1(s,\mu_1(s))| \leqslant \|\mu_1 - V\mu_2\|_\infty + \|\mu_2 - S\mu_1\|_\infty. \tag{3.27}$$

Thus, $\forall\, \kappa \in \mathcal{I} \setminus \{b\}$,

$$\begin{aligned} |S\mu_1(\kappa) - V\mu_2(\kappa)| &\leqslant \left(\tfrac{\beta}{\Gamma(p)} \int_b^\kappa \psi'(\tilde r)(\psi(\kappa) - \psi(\tilde r))^{p-1}\,\mathrm{d}\tilde r\right) \|\mu_1 - \mu_2\|_\infty^\eta \|\mu_1 - S\mu_1\|_\infty^\theta \\ &\quad \cdot \|\mu_2 - V\mu_2\|_\infty^\theta \,[\, \|\mu_1 - V\mu_2\|_\infty + \|\mu_2 - S\mu_1\|_\infty \,]^{1-\eta-2\theta} \\ &= \tfrac{\beta}{p\Gamma(q)}(\psi(\mathfrak{t}) - \psi(b))^p \|\mu_1 - \mu_2\|_\infty^\eta \|\mu_1 - S\mu_1\|_\infty^\theta \\ &\quad \cdot \|\mu_2 - V\mu_2\|_\infty^\theta \,[\, \|\mu_1 - V\mu_2\|_\infty + \|\mu_2 - S\mu_1\|_\infty \,]^{1-\eta-2\theta}. \end{aligned} \tag{3.28}$$

Hence,

$$\begin{aligned} \|S\mu_1 - V\mu_2\|_\infty^2 &\leqslant \left(\tfrac{\beta}{p\Gamma(p)}(\psi(\kappa) - \psi(b))^p\right)^2 \|\mu_1 - \mu_2\|_\infty^{2\eta} \|\mu_1 - S\mu_1\|_\infty^{2\theta} \\ &\quad \cdot \|\mu_2 - V\mu_2\, |_\infty^{2\theta} \,[\, \|\mu_1 - V\mu_2\|_\infty + \|\mu_2 - S\mu_1\|_\infty \,]^{2(1-\eta-2\theta)} \\ &\leqslant \left(\tfrac{\beta}{p\Gamma(p)}(\psi(\kappa) - \psi(b))^p\right)^2 \|\mu_1 - \mu_2\|_\infty^{2\eta} \|\mu_1 - S\mu_1\|_\infty^{2\theta} \\ &\quad \cdot \|\mu_2 - V\mu_2\|_\infty^{2\theta} \left[2(\|\mu_1 - V\mu_2\|_\infty^2 + \|\mu_2 - S\mu_1\|_\infty^2)\right]^{1-\eta-2\theta}. \end{aligned} \tag{3.29}$$

Therefore,

$$\begin{aligned} \rho(S\mu_1, V\mu_2) &\leqslant 2\left(\tfrac{\beta}{p\Gamma(p)}(\psi(v) - \psi(b))^q\right)^2 \rho(\mu_1,\mu_2)^\eta \rho(\mu_1, S\mu_1)^\theta \\ &\quad \cdot \rho(\mu_2, V\mu_2)^\theta \,[\, \rho(\mu_1, V\mu_2) + \rho(\mu_2, S\mu_1) \,]^{1-\eta-2\theta}. \end{aligned} \tag{3.30}$$

Step 2: Consider the functions ϕ defined on $\mathbb{R}_{>0}$ by

$$\phi(\mathfrak{t}) = 8\left[\tfrac{\beta}{q\Gamma(q)}(\psi(v) - \psi(b))^q\right]^2, \tag{3.31}$$

and $\alpha : \mathfrak{Q}^2 \to \mathbb{R}_{\geq 0}$ defined as $\alpha(\mu_1,\mu_2) = 1$ whenever $(\mu_1,\mu_2), (\mu_2,\mu_1) \in \mathcal{A}^2$ and $\alpha(\mu_1,\mu_2) = 0$ otherwise. By the assumption (H2), $\frac{2\sqrt{2}\beta}{p\Gamma(p)}(\psi(v) - \psi(b))^p < 1$, then $\phi \in \Psi$. The function α is triangular, any sequence $(\mu_n)_{n\geqslant 0}$ of $\mathcal{A}$ is α-regular, (V,S) is α-proximal admissible and the condition (iv) of Theorem 2.2 is verified. Let $\mu_1,\mu_2 \in \mathcal{A}$. There exist two sequences $(\mu_{1_n})_{n\geqslant 0}$ and $(\mu_{2_n})_{n\geqslant 0}$ of $\mathcal{M}$ s.t. $\lim\limits_{n\to+\infty} \|\mu_{1_n} - \mu_1\|_\infty = 0 = \lim\limits_{n\to+\infty} \|\mu_{2_n} - \mu_2\|_\infty$. Let $\kappa \in \mathcal{I}$. Thanks to the assumption (H3), we get

$$\begin{aligned} |S\mu_{1_n}(\kappa) - S\mu_1(\kappa)| &\leqslant \int_b^\kappa \psi'(\tilde r)\tfrac{(\psi(\kappa)-\psi(\tilde r))^{p-1}}{\Gamma(p)} |\lambda_1(\tilde r, \mu_1(\tilde r)) - \lambda_1(\tilde r, \mu_2(\tilde r))|\,\mathrm{d}\tilde r \\ &\leqslant \int_b^\kappa \psi'(\tilde r)\tfrac{(\psi(\kappa)-\psi(\tilde r))^{p-1}}{\Gamma(p)} |\mu_{1_n}(s) - \mu_1(\tilde r)|\,\mathrm{d}\tilde r \\ &\leqslant \tfrac{(\psi(v)-\psi(b))^p}{p\Gamma(p)} \|\mu_{1_n} - \mu_1\|_\infty. \end{aligned} \tag{3.32}$$

Hence,

$$\|S\mu_{1_n} - S\mu_1\|_\infty \leqslant \tfrac{1}{p\Gamma(p)}(\psi(v) - \psi(b))^p \|\mu_{1_n} - \mu_1\|_\infty, \tag{3.33}$$

and so $\lim\limits_{n\to+\infty} \|S\mu_{1_n} - S\mu_1\|_\infty = 0$. Similarly, $\lim\limits_{n\to+\infty} \|S\mu_{2_n} - S\mu_2\|_\infty = 0$. And since $(S\mu_{1_n}, S\mu_{2_n}) \in \mathcal{M}^2$, $\forall\, n \in \mathbb{N}$, then, $(S\mu_1, S\mu_2) \in \mathcal{A}^2$. By Eq. (3.30), for all $n \in \mathbb{N}$,

$$\begin{aligned} \|S\mu_{1_n} - V\mu_{2_n}\|_\infty &\leqslant 2\left(\tfrac{\beta}{p\Gamma(p)}(\psi(v) - \psi(b))^q\right)^2 \rho(\mu_{1_n},\mu_{2_n})^\eta \rho(\mu_{1_n}, S\mu_{1_n})^\theta \\ &\quad \cdot \rho(\mu_{2_n}, V\mu_{2_n})^\theta \,[\, \rho(\mu_{2_n}, V\mu_{2_n}) + \rho(\mu_{2_n}, S\mu_{1_n}) \,]^{1-\eta-2\theta}. \end{aligned} \tag{3.34}$$

Thus, we get

$$\|S\mu_1 - V\mu_2\|_\infty \leqslant 2\left(\frac{\beta}{p\Gamma(p)}(\psi(v)-\psi(b))^q\right)^2 \rho(\mu_1,\mu_2)^\eta \rho(\mu_1,S\mu_1)^\theta \cdot \rho(\mu_2,V\mu_2)^\theta\,[\,\rho(\mu_1,V\mu_2)+\rho(\mu_2,S\mu_1)\,]^{1-\eta-2\theta}. \tag{3.35}$$

Thus, the couple (V,S) is α-generalized G_{pi} contraction pair type HR on $\mathcal{A}$,

$$\alpha(S\mu_1,V\mu_2)\Phi\left(\rho(S\mu_1,V\mu_2),\rho(S\mu_1,V\mu_2)\right) \leqslant \phi(\rho(\mu_1,\mu_2))\rho(\mu_1,\mu_2)^\eta\rho(\mu_1,S\mu_1)^\theta \cdot \rho(\mu_2,V\mu_2)^\theta\,[\,\rho(\mu_1,V\mu_2)+\rho(\mu_2,S\mu_1)\,]^{1-\eta-2\theta}, \tag{3.36}$$

$\forall\,(\mu_1,\mu_2)\in\mathcal{A}^2$. Moreover, the condition $CB(V,S)$ is verified, because $\alpha(S\mu_1,V\mu_2)\geqslant 1$, for each $(\mu_1,\mu_2)\in\mathcal{A}^2$. Hence, all conditions of Corollary 2.1 are fulfilled. This concludes the proof. □

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