Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 10** : 1–16.
ISSN-0037-8712
doi:10.5269/bspm.79204

Common Coupled Fixed Point Result for Integral Type Contraction Using w-compatibility in Partial Metric Spaces

Hemant Kumar Nashine*, Gurucharan Singh Saluja and Rabha W. Ibrahim

ABSTRACT: The purpose of this work is to demonstrate the common coupled fixed point result of integral type contraction utilizing w-compatibility in the context of partial metric spaces. Furthermore, we present various corollaries to the established result. The findings in this study extend and generalize various previously reported conclusions from the current literature. An application is demonstrated using the modified fractional integral operator and the specific Rabotnov function, followed by an illustration.

Keywords: Common coupled fixed point, integral type contraction, partial metric space, w - compatible mapping.

Contents

1. Introduction

In 1994, *Matthews* [24] introduced the notion of a partial metric space (PMS) which is a generalized metric space in which each object does not necessarily have to have a zero distance from itself. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation (see, e.g., [13], [28] and some others). Introducing partial metric space, *Matthews* proved the partial metric version of Banach fixed point theorem [5]. *Matthews* [24], *Oltra and Valero* [27] and *Altun et al.* [2] proved some fixed point theorems in partial metric spaces for a single map. For more work on fixed and common fixed point theorems in partial metric spaces, we refer to ([9], [10], [16], [22], [26]).

The notion of a coupled fixed point was introduced by *Bhashkar and Lakshmikantham* [7] and they studied some coupled fixed point theorems in partially ordered complete metric spaces. They also proved mixed monotone property for the first time and gave their classical coupled fixed point theorem for mapping which satisfy the mixed monotone property. Later some authors proved coupled fixed and common coupled fixed point theorems (see [11], [12], [18], [31], [32]).

Recently, *Kim et al.* [20] proved some common coupled fixed point theorems for weak compatible mappings in the setting of partial metric spaces. Very recently, *Jain et al.* [14] proved some coupled and common coupled fixed theorems for various contractive type conditions in the setting of partial metric spaces.

Quite recently, Nashine et al. [25] proved some coupled and common coupled fixed point theorems using control function in the framework of partial metric spaces and also give some illustrative examples in support of the established results.

---

* Corresponding author.
2020 *Mathematics Subject Classification:* 47H10, 54H25.
Submitted September 27, 2025. Published March 12, 2026

2. Preliminaries

In this section, first we recall some basic definitions and lemmas which play a crucial role in the theory of partial metric spaces ($PMSs$).

Definition 2.1 *([24]) Let $\Theta \neq \emptyset$ be a set and $\mathcal{P}_{\mathcal{M}} \colon \Theta \times \Theta \to \mathbb{R}^+$ be a self mapping of Θ such that for all $u, v, s \in \Theta$ the followings are satisfied:*
$(pms1)$ $u = v \Leftrightarrow \mathcal{P}_{\mathcal{M}}(u,u) = \mathcal{P}_{\mathcal{M}}(u,v) = \mathcal{P}_{\mathcal{M}}(v,v)$,
$(pms2)$ $\mathcal{P}_{\mathcal{M}}(u,u) \leq \mathcal{P}_{\mathcal{M}}(u,v)$,
$(pms3)$ $\mathcal{P}_{\mathcal{M}}(u,v) = \mathcal{P}_{\mathcal{M}}(v,u)$,
$(pms4)$ $\mathcal{P}_{\mathcal{M}}(u,v) \leq \mathcal{P}_{\mathcal{M}}(u,s) + \mathcal{P}_{\mathcal{M}}(s,v) - \mathcal{P}_{\mathcal{M}}(s,s)$.
Then $\mathcal{P}_{\mathcal{M}}$ is called partial metric on Θ and the pair $(\Theta, \mathcal{P}_{\mathcal{M}})$ is called partial metric space (in short PMS).

Remark 2.1 *It is clear that if $\mathcal{P}_{\mathcal{M}}(u,v) = 0$, then from $(pms1)$, $(pms2)$, and $(pms3)$, $u = v$. But if $u = v$, $\mathcal{P}_{\mathcal{M}}(u,v)$ may not be 0.*

If $\mathcal{P}_{\mathcal{M}}$ is a partial metric on Θ, then the function $\mathcal{P}_{\mathcal{M}(d)} \colon \Theta \times \Theta \to \mathbb{R}^+$ given by

$$\mathcal{P}_{\mathcal{M}(d)}(u,v) = 2\mathcal{P}_{\mathcal{M}}(u,v) - \mathcal{P}_{\mathcal{M}}(u,u) - \mathcal{P}_{\mathcal{M}}(v,v), \tag{2.1}$$

is a metric on Θ.

Example 2.1 *([4]) Let $\Theta = \mathbb{R}^+$, where $\mathbb{R}^+ = [0, +\infty)$ and $\mathcal{P}_{\mathcal{M}} \colon \Theta \times \Theta \to \mathbb{R}^+$ be given by $\mathcal{P}_{\mathcal{M}}(p,q) = C \max\{p,q\}, C > 0$ for all $p, q \in \mathbb{R}^+$. Then $(\mathbb{R}^+, \mathcal{P}_{\mathcal{M}})$ is a partial metric space.*

Example 2.2 *([4]) Let I denote the set of all intervals $[u,v]$ for any real numbers $u \leq v$. Let $\mathcal{P}_{\mathcal{M}} \colon I \times I \to [0, \infty)$ be a function such that*

$$\mathcal{P}_{\mathcal{M}}\big([p,q],[n,m]\big) = \max\{q,m\} - \min\{p,n\}.$$

Then $(I, \mathcal{P}_{\mathcal{M}})$ is a partial metric space.

Example 2.3 *([8]) Let $\Theta = \mathbb{R}$ and $\mathcal{P}_{\mathcal{M}} \colon \Theta \times \Theta \to \mathbb{R}^+$ be given by $\mathcal{P}_{\mathcal{M}}(p,q) = e^{\max\{p,q\}}$ for all $p, q \in \mathbb{R}$. Then $(\Theta, \mathcal{P}_{\mathcal{M}})$ is a partial metric space.*

Numerous applications of this space has been extensively investigated by many authors (see, [21], [33] for details).

Each partial metric $\mathcal{P}_{\mathcal{M}}$ on Θ generates a T_0 topology τ_ρ on Θ, which has a base of the family of open $\mathcal{P}_{\mathcal{M}}$-balls $\{\mathcal{B}_{\mathcal{P}_M}(a,\delta) : a \in \Theta, \delta > 0\}$ where

$$\mathcal{B}_{\mathcal{P}_M}(a,\delta) = \{b \in \Theta : \mathcal{P}_{\mathcal{M}}(a,b) < \mathcal{P}_{\mathcal{M}}(a,a) + \delta\},$$

for all $a \in \Theta$ and $\delta > 0$.

Similarly, closed $\mathcal{P}_{\mathcal{M}}$-ball is defined as

$$\mathcal{B}_{\mathcal{P}_M}[a,\delta] = \{b \in \Theta : \mathcal{P}_{\mathcal{M}}(a,b) \leq \mathcal{P}_{\mathcal{M}}(a,a) + \delta\},$$

for all $a \in \Theta$ and $\delta > 0$.

Definition 2.2 *([24]) Let $(\Theta, \mathcal{P}_{\mathcal{M}})$ be a partial metric space and also $\{r_n\}$ is a sequence in Θ.*
At the time,
(1) $\mathcal{P}_{\mathcal{M}}(r,r) = \lim_{n\to\infty} \mathcal{P}_{\mathcal{M}}(r_n, r) \Leftrightarrow \{r_n\}$ *converges to a point* $r \in \Theta$,
(2) *if there exists* $\lim_{m,n\to\infty} \mathcal{P}_{\mathcal{M}}(r_m, r_n)$, *then* $\{r_n\}$ *is a Cauchy sequence.*

Definition 2.3 *([24]) A partial metric space $(\Theta, \mathcal{P}_{\mathcal{M}})$ is named to be complete if every Cauchy sequence $\{r_n\}$ in Θ converges to a point $r \in \Theta$ with respect to τ_ρ. Furthermore,*

$$\lim_{m,n\to\infty} \mathcal{P}_{\mathcal{M}}(r_m, r_n) = \lim_{n\to\infty} \mathcal{P}_{\mathcal{M}}(r_n, r) = \mathcal{P}_{\mathcal{M}}(r,r).$$

Definition 2.4 *([24]) A mapping* $\mathcal{Q}\colon \Theta \to \Theta$ *is said to be continuous at* $r_0 \in \Theta$ *if for every* $\delta > 0$*, there exists* $c > 0$ *such that* $\mathcal{Q}\big(\mathcal{B}_{\mathcal{P}_\mathcal{M}}(r_0, c)\big) \subset \mathcal{B}_{\mathcal{P}_\mathcal{M}}\big(\mathcal{Q}(r_0), \delta\big)$.

Lemma 2.1 *([3,24]) Let* $(\Theta, \mathcal{P}_\mathcal{M})$ *be a partial metric space. Then*

(a_1) *the sequence* $\{r_n\}$ *in* $(\Theta, \mathcal{P}_\mathcal{M})$ *is a Cauchy sequence* $\Leftrightarrow$ *it is a Cauchy sequence in the metric space* $(\Theta, \mathcal{P}_{\mathcal{M}(d)})$,

(a_2) $(\Theta, \mathcal{P}_\mathcal{M})$ *is complete* $\Leftrightarrow$ *the metric space* $(\Theta, \mathcal{P}_{\mathcal{M}(d)})$ *is complete. Besides,* $\lim_{n\to\infty} \mathcal{P}_{\mathcal{M}(d)}(r_n, r) = 0 \Leftrightarrow \mathcal{P}_\mathcal{M}(r, r) = \lim_{n\to\infty} \mathcal{P}_\mathcal{M}(r_n, r) = \lim_{n,m\to\infty} \mathcal{P}_\mathcal{M}(r_n, r_m)$.

Lemma 2.2 *([15]) Let* $(\Theta, \mathcal{P}_\mathcal{M})$ *be a partial metric space. Then*

(A) *if* $p, q \in \Theta$, $\mathcal{P}_\mathcal{M}(p, q) = 0$, *then* $p = q$,

(B) *if* $p \neq q$, *then* $\mathcal{P}_\mathcal{M}(p, q) > 0$.

Definition 2.5 *([3]) An element* $(p, q) \in \Theta\times\Theta$ *is called a coupled fixed point of a mapping* $\mathcal{T}\colon \Theta\times\Theta \to \Theta$ *if* $\mathcal{T}(p, q) = p$ *and* $\mathcal{T}(q, p) = q$.

Example 2.4 *Let* $\Theta = [0, +\infty)$ *and* $\mathcal{T}\colon \Theta \times \Theta \to \Theta$ *be defined by* $\mathcal{T}(p, q) = \frac{p+q}{3}$ *for all* $p, q \in \Theta$. *Then one can easily see that* $\mathcal{T}$ *has a unique coupled fixed point* $(0, 0)$.

Example 2.5 *Let* $\Theta = [0, +\infty)$ *and* $\mathcal{T}\colon \Theta \times \Theta \to \Theta$ *be defined by* $\mathcal{T}(p, q) = \frac{p+q}{2}$ *for all* $p, q \in \Theta$. *Then we see that* $\mathcal{T}$ *has two coupled fixed point* $(0, 0)$ *and* $(1, 1)$, *that is, the coupled fixed point is not unique.*

Definition 2.6 *([1], [19]) An element* $(p, q) \in \Theta \times \Theta$ *is called*

(g_1) *a coupled coincidence point of mappings* $\mathcal{T}\colon \Theta \times \Theta \to \Theta$ *and* $f\colon \Theta \to \Theta$ *if* $f(p) = \mathcal{T}(p, q)$ *and* $f(q) = \mathcal{T}(q, p)$, *and* (fp, fq) *is called a coupled point of coincidence,*

(g_2) *a common coupled fixed point of mappings* $\mathcal{T}\colon \Theta \times \Theta \to \Theta$ *and* $f\colon \Theta \to \Theta$ *if* $p = f(p) = \mathcal{T}(p, q)$ *and* $q = f(q) = \mathcal{T}(q, p)$.

Definition 2.7 *([1]) The mappings* $\mathcal{T}\colon \Theta \times \Theta \to \Theta$ *and* $f\colon \Theta \to \Theta$ *are called weakly compatible if* $f(\mathcal{T}(p, q)) = \mathcal{T}(fp, fq)$ *and* $f(\mathcal{T}(q, p)) = \mathcal{T}(fq, fp)$ *for all* $p, q \in \Theta$, *whenever* $f(p) = \mathcal{T}(p, q)$ *and* $f(q) = \mathcal{T}(q, p)$.

Example 2.6 *Let* $\Theta = [0, 3]$ *endowed with* $\mathcal{P}_\mathcal{M}(p, q) = \max\{p, q\}$ *for all* $p, q \in \Theta$. *Define* $\mathcal{T}\colon \Theta\times\Theta \to \Theta$ *and* $f\colon \Theta \to \Theta$ *by*

$$\mathcal{T}(p, q) = \begin{cases} p + q, & \text{if } p, q \in [0, 1), \\ 3, & \text{otherwise}, \end{cases}$$

for all $p, q \in \Theta$ *and*

$$f(p) = \begin{cases} p, & \text{if } p \in [0, 1), \\ 3, & \text{if } p \in [1, 3], \end{cases}$$

for all $p \in \Theta$. *Then for any* $p, q \in [1, 3]$,

$$\mathcal{T}(fp, fq) = \mathcal{T}(3, 3) = 3 = f(\mathcal{T}(p, q)) = f(3) = 3.$$

Similarly, we have

$$\mathcal{T}(fq, fp) = \mathcal{T}(3, 3) = 3 = f(\mathcal{T}(q, p)) = f(3) = 3.$$

Thus,

$$\mathcal{T}(fp, fq) = f(\mathcal{T}(p, q)) \ \text{ and } \ \mathcal{T}(fq, fp) = f(\mathcal{T}(q, p)).$$

This shows that the mappings $\mathcal{T}$ *and* f *are weakly compatible on* $[0, 3]$.

Example 2.7 *Let* $\Theta = \mathbb{R}$ *endowed with the usual metric* $\mathcal{P}_\mathcal{M}(p, q) = \max\{p, q\}$ *for all* $p, q \in \Theta$. *Define* $\mathcal{T}\colon \Theta \times \Theta \to \Theta$ *and* $f\colon \Theta \to \Theta$ *by* $\mathcal{T}(p, q) = p + q$ *and* $f(p) = p^2$ *for all* $p, q \in \Theta$. *Then* $\mathcal{T}$ *and* f *are not weakly compatible maps on* $\mathbb{R}$, *since*

$$\mathcal{T}(fp, fq) = \mathcal{T}(p^2, q^2) = p^2 + q^2, \ \text{ but } \ f(\mathcal{T}(p, q)) = f(p + q) = (p + q)^2.$$

Therefore,

$$\mathcal{T}(fp, fq) \neq f(\mathcal{T}(p, q)).$$

Hence the mappings $\mathcal{T}$ *and* f *are not weakly compatible on* $\mathbb{R}$.

Definition 2.8 *([17]) Let $\Theta \subset (-\infty, \infty)$. The function $\phi \colon \Theta \to \Theta$ is called sub-additive integrable function if and only if for all $c, d \in \Theta$*

$$\int_0^{c+d} \phi(t)\, dt \leq \int_0^{c} \phi(t)\, dt + \int_0^{d} \phi(t)\, dt.$$

Example 2.8 *([17]) Let $\Theta = (0, \infty)$, $d(x, y) = |x - y|$, and $\phi(t) = \frac{1}{t+1}$ for all $t > 0$, then for all $c, d \in \Theta$,*

$$\int_0^{c+d} \frac{dt}{t+1} = ln(c+d+1),\ \int_0^{c} \frac{dt}{t+1} = ln(c+1),\ \int_0^{d} \frac{dt}{t+1} = ln(d+1).$$

Since $c\,d \geq 0$, then $c+d+1 \leq c+d+1+cd = (c+1)(d+1)$. Therefore

$$ln(c+d+1) \leq ln((c+1)(d+1)) = ln(c+1) + ln(d+1).$$

This shows that ϕ is an example of sub-additive integrable function.

Example 2.9 *Let $\Theta = (1, \infty)$, and $\phi(t) = e^t$ for all $t > 0$. Then the function ϕ is not sub-additive integrable function.*

Definition 2.9 *([23]) Let $\{r_n\}_{n\in\mathbb{N}}$ be a non-negative sequence such that $\lim_{n\to\infty} r_n = a$. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \phi(t)\, dt = \int_0^{a} \phi(t)\, dt,$$

where $\phi \colon [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)dt > 0$

Definition 2.10 *([23]) Let $\{r_n\}_{n\in\mathbb{N}}$ be a non-negative sequence. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \phi(t)\, dt = 0,$$

if and only if $\lim_{n\to\infty} r_n = 0$, where $\phi \colon [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)dt > 0$

In 2002, Branciari [6] introduced the concept of contractive mapping of integral type and obtained the following fixed point result for the mapping T defined on a complete metric space (Θ, ρ):

Theorem 2.1 *([6]) Let (Θ, ρ) be a complete metric space, $h \in [0, 1)$, and let $T \colon \Theta \to \Theta$ be a mapping such that for each $\alpha, \beta \in \Theta$,*

$$\int_0^{\rho(T\alpha, T\beta)} \phi(t)dt \leq h \int_0^{\rho(\alpha, \beta)} \phi(t)dt,$$

where $\phi \colon [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)dt > 0$, then T has a unique fixed point $p \in \Theta$, such that for each $q \in \Theta$, $\lim_{n\to\infty} T^n(q) = p$.

After the paper of Branciari [6], various research works have been carried out on generalizing contractive condition of integral type for different contractive conditions and spaces satisfying many known properties (see, for example [29], [30]).

The aim of this paper is to prove common coupled fixed point theorem of integral type contraction using w-compatibility condition in the setting of partial metric spaces. In addition, we provide some consequences of the established results. Our results extend, generalize and enrich several previous works from the existing literature.

3. Main Results

In this section, we shall prove some common coupled fixed point theorems of integral type contractions in the setting of partial metric spaces.

Theorem 3.1 *Let $(\Theta, \mathcal{P}_{\mathcal{M}})$ be a complete partial metric space. Let $T\colon \Theta\times\Theta \to \Theta$ and $R\colon \Theta \to \Theta$ be two mappings satisfying the following conditions:*

$$\int_0^{\mathcal{P}_{\mathcal{M}}(T(f,g),T(j,k))} \phi(t)dt \leq \mu \int_0^{\Delta_M^P(f,g,j,k)} \phi(t)dt, \tag{3.1}$$

for all $f, g, j, k \in \Theta$, where

$$\begin{aligned}\Delta_M^P(f,g,j,k) &= \max\Big\{\mathcal{P}_{\mathcal{M}}(Rf,Rj), \mathcal{P}_{\mathcal{M}}(Rf,T(f,g)), \mathcal{P}_{\mathcal{M}}(Rj,T(j,k)),\\ &\qquad \frac{\mathcal{P}_{\mathcal{M}}(Rj,T(f,g))}{1+\mathcal{P}_{\mathcal{M}}(Rf,Rj)}, \frac{\mathcal{P}_{\mathcal{M}}(Rf,T(f,g))}{1+\mathcal{P}_{\mathcal{M}}(Rf,Rj)}\Big\},\end{aligned}$$

$\mu \in [0,1)$ is a constant, $\phi\colon [0,+\infty) \to [0,+\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0,+\infty)$, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)dt > 0$. Also, suppose $T(\Theta\times\Theta) \subseteq R(\Theta)$ and $R(\Theta)$ is a complete subspace of Θ. Then there exist $r_1, r_2 \in \Theta$ such that $Rr_1 = T(r_1, r_2)$ and $Rr_2 = T(r_2, r_1)$, that is, T and R have a unique coupled coincidence point. Moreover, if the pair (T,R) is w-compatible, then T and R have a unique common coupled fixed point in Θ with $\mathcal{P}_{\mathcal{M}}(t,t) = 0$ for some $t \in \Theta$.

Proof: Let f_0, g_0 be two arbitrary elements of Θ. Since $T(\Theta\times\Theta) \subseteq R(\Theta)$, we can choose $f_0, g_0 \in \Theta$, such that $Rf_1 = T(f_0, g_0)$ and $Rg_1 = T(g_0, f_0)$. Continuing this process, we can construct two sequences $\{f_n\}$ and $\{g_n\}$ in Θ such that $Rf_{n+1} = T(f_n, g_n)$ and $Rg_{n+1} = T(g_n, f_n)$ for all $n \geq 0$. Then, from equation (3.1) and using $(pms3)$, $(pms4)$, we have

$$\begin{aligned}\int_0^{\mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n+1})} \phi(t)dt &= \int_0^{\mathcal{P}_{\mathcal{M}}(T(f_{n-1},g_{n-1}),T(f_n,g_n))} \phi(t)dt\\ &\leq \mu \int_0^{\Delta_M^P(f_{n-1},g_{n-1},f_n,g_n)} \phi(t)dt,\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}\Delta_M^P(f_{n-1},g_{n-1},f_n,g_n) &= \max\Big\{\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n), \mathcal{P}_{\mathcal{M}}(Rf_{n-1},T(f_{n-1},g_{n-1})), \mathcal{P}_{\mathcal{M}}(Rf_n,T(f_n,g_n)),\\ &\qquad \frac{\mathcal{P}_{\mathcal{M}}(Rf_n,T(f_{n-1},g_{n-1}))}{1+\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n)}, \frac{\mathcal{P}_{\mathcal{M}}(Rf_{n-1},T(f_{n-1},g_{n-1}))}{1+\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n)}\Big\}\\ &= \max\Big\{\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n), \mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n), \mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n+1}),\\ &\qquad \frac{\mathcal{P}_{\mathcal{M}}(Rf_n,Rf_n)}{1+\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n)}, \frac{\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n)}{1+\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n)}\Big\}\\ &= \max\Big\{\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n), \mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n), \mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n+1}), 0,\\ &\qquad \frac{\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n)}{1+\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n)}\Big\}\\ &= \max\Big\{\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n), \mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n+1})\Big\},\end{aligned}$$

since

$$\frac{\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n)}{1+\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n)} \leq \mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n).$$

Using the above value in equation (3.2), we obtain

$$\int_0^{\mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n+1}))} \phi(t)dt \le \mu \int_0^{\max\left\{\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n),\mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n+1})\right\}} \phi(t)dt. \tag{3.3}$$

Now, we have the following cases.

If $\max\left\{\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n),\mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n+1})\right\} = \mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n+1})$, then from equation (3.3), we obtain

$$\int_0^{\mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n+1}))} \phi(t)dt \le \mu \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n+1})} \phi(t)dt,$$

which is a contradiction, since $0 \le \mu < 1$. Thus, we conclude that $\mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n+1}) < \mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n)$.

Hence from equation (3.3), we obtain

$$\int_0^{\mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n+1}))} \phi(t)dt \le \mu \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n)} \phi(t)dt. \tag{3.4}$$

Again using equation (3.4), we obtain

$$\int_0^{\mathcal{P}_{\mathcal{M}}(Rf_{n-1},Rf_n))} \phi(t)dt \le \mu \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_{n-2},Rf_{n-1})} \phi(t)dt. \tag{3.5}$$

Using equation (3.5) in equation (3.4), we obtain

$$\int_0^{\mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n+1}))} \phi(t)dt \le \mu^2 \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_{n-2},Rf_{n-1})} \phi(t)dt. \tag{3.6}$$

Continuing the above process, we obtain

$$\begin{aligned} \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n+1}))} \phi(t)dt &\le \mu^2 \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_{n-2},Rf_{n-1})} \phi(t)dt \\ &\le \mu^3 \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_{n-3},Rf_{n-2})} \phi(t)dt \\ &\vdots \\ &\le \mu^n \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_0,Rf_1)} \phi(t)dt. \end{aligned} \tag{3.7}$$

Likewise, one can show that

$$\begin{aligned} \int_0^{\mathcal{P}_{\mathcal{M}}(Rg_n,Rg_{n+1}))} \phi(t)dt &= \int_0^{\mathcal{P}_{\mathcal{M}}(T(g_{n-1},f_{n-1}),T(g_n,f_n))} \phi(t)dt \\ &\le \mu^n \int_0^{\mathcal{P}_{\mathcal{M}}(Rg_0,Rg_1)} \phi(t)dt. \end{aligned} \tag{3.8}$$

From equations (3.7) and (3.8), we obtain

$$\begin{aligned} &\int_0^{\mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n+1}))} \phi(t)dt + \int_0^{\mathcal{P}_{\mathcal{M}}(Rg_n,Rg_{n+1}))} \phi(t)dt \\ &\le \mu^n \Big[\int_0^{\mathcal{P}_{\mathcal{M}}(Rf_0,Rf_1)} \phi(t)dt + \int_0^{\mathcal{P}_{\mathcal{M}}(Rg_0,Rg_1)} \phi(t)dt\Big]. \end{aligned} \tag{3.9}$$

Let

$$C_n = \mathcal{P}_{\mathcal{M}}(Rf_n, Rf_{n+1}) \text{ and } D_n = \mathcal{P}_{\mathcal{M}}(Rg_n, Rg_{n+1}). \tag{3.10}$$

Then from equations (3.9) and (3.10), we obtain

$$\int_0^{C_n} \phi(t)dt + \int_0^{D_n} \phi(t)dt \leq \mu^n \Big[\int_0^{C_0} \phi(t)dt + \int_0^{D_0} \phi(t)dt \Big]. \tag{3.11}$$

Let

$$V_n = \int_0^{C_n} \phi(t)dt + \int_0^{D_n} \phi(t)dt.$$

Then from equation (3.11), we obtain

$$V_n \leq \mu^n V_0, \tag{3.12}$$

for all $n \in \mathbb{N}$.

If $V_0 = 0$, then we have

$$\int_0^{\mathcal{P}_{\mathcal{M}}(Rf_0, Rf_1)} \phi(t)dt + \int_0^{\mathcal{P}_{\mathcal{M}}(Rg_0, Rg_1)} \phi(t)dt = 0. \tag{3.13}$$

This implies that

$$\int_0^{\mathcal{P}_{\mathcal{M}}(Rf_0, Rf_1)} \phi(t)dt = 0 \text{ and } \int_0^{\mathcal{P}_{\mathcal{M}}(Rg_0, Rg_1)} \phi(t)dt = 0. \tag{3.14}$$

Hence by the property of integral ϕ, we obtain $\mathcal{P}_{\mathcal{M}}(Rf_0, Rf_1) = 0$ and $\mathcal{P}_{\mathcal{M}}(Rg_0, Rg_1) = 0$. Hence, from Remark 2.1, we get $Rf_0 = Rf_1 = T(f_0, g_0)$ and $Rg_0 = Rg_1 = T(g_0, f_0)$. This shows that (Rf_0, Rg_0) is a coupled fixed point of T and R. Now, we assume that $V_0 > 0$. For each $n \geq m$, where $n, m \in \mathbb{N}$, by using condition $(pms4)$ and sub-additivity of integral ϕ, we have

$$\begin{aligned} \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_n, Rf_m)} \phi(t)dt &\leq \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_n, Rf_{n-1}) + \mathcal{P}_{\mathcal{M}}(Rf_{n-1}, Rf_{n-2}) + \cdots + \mathcal{P}_{\mathcal{M}}(Rf_{m+1}, Rf_m)} \phi(t)dt \\ &\leq \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_n, Rf_{n-1})} \phi(t)dt + \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_{n-1}, Rf_{n-2})} \phi(t)dt + \dots \\ &\quad + \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_{m+1}, Rf_m)} \phi(t)dt \\ &\leq (\mu^{n-1} + \mu^{n-2} + \cdots + \mu^m) \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_1, Rf_0)} \phi(t)dt \\ &\leq \Big(\frac{\mu^m}{1-\mu}\Big) \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_1, Rf_0)} \phi(t)dt \\ &\to 0 \text{ as } m \to \infty. \end{aligned} \tag{3.15}$$

By similar fashion, we can show that

$$\int_0^{\mathcal{P}_{\mathcal{M}}(Rg_n, Rg_m)} \phi(t)dt \to 0 \text{ as } n, m \to \infty. \tag{3.16}$$

From equations (3.15) and (3.16), regarding the property of integral ϕ, we obtain

$$\lim_{n,m\to\infty} \mathcal{P}_{\mathcal{M}}(Rf_n, Rf_m) = 0 \text{ and } \lim_{n,m\to\infty} \mathcal{P}_{\mathcal{M}}(Rg_n, Rg_m) = 0. \tag{3.17}$$

Due to (2.1), we have $\mathcal{P}_{\mathcal{M}(d)}(Rf,Rg) \leq 2\mathcal{P}_{\mathcal{M}}(Rf,Rg)$. Therefore, using equation (3.17), we obtain

$$\lim_{n,m\to\infty} \mathcal{P}_{\mathcal{M}(d)}(Rf_n, Rf_m) = 0. \tag{3.18}$$

Likewise, we obtain

$$\lim_{n,m\to\infty} \mathcal{P}_{\mathcal{M}(d)}(Rg_n, Rg_m) = 0. \tag{3.19}$$

Thus, by Lemma 2.1, $\{Rf_n\}$ and $\{Rg_n\}$ are Cauchy sequences in both $(\Theta, \mathcal{P}_{\mathcal{M}(d)})$ and $(\Theta, \mathcal{P}_{\mathcal{M}})$. Since $(\Theta, \mathcal{P}_{\mathcal{M}})$ is a complete partial metric space, so there exist $r_1, r_2 \in \Theta$ such that

$$\lim_{n\to\infty} \mathcal{P}_{\mathcal{M}}(Rf_n, Rr_1) = \mathcal{P}_{\mathcal{M}}(Rr_1, Rr_1), \tag{3.20}$$

and

$$\lim_{n\to\infty} \mathcal{P}_{\mathcal{M}}(Rg_n, Rr_2) = \mathcal{P}_{\mathcal{M}}(Rr_2, Rr_2). \tag{3.21}$$

Since $\lim_{n,m\to\infty} \mathcal{P}_{\mathcal{M}}(Rf_n, Rf_m) = 0$ and $\lim_{n,m\to\infty} \mathcal{P}_{\mathcal{M}}(Rg_n, Rg_m) = 0$, then again by Lemma 2.1, we have

$$\mathcal{P}_{\mathcal{M}}(Rr_1, Rr_1) = 0 \text{ and } \mathcal{P}_{\mathcal{M}}(Rr_2, Rr_2) = 0. \tag{3.22}$$

Now, using equation (3.1), condition $(pms3)$ and sub-additivity of integral ϕ, we have

$$\begin{aligned}
\int_0^{\mathcal{P}_{\mathcal{M}}(T(r_1,r_2),Rr_1)} \phi(t)dt &\leq \int_0^{\mathcal{P}_{\mathcal{M}}(T(r_1,r_2),Rf_{n+1})+\mathcal{P}_{\mathcal{M}}(Rf_{n+1},Rr_1)} \phi(t)dt \\
&\leq \int_0^{\mathcal{P}_{\mathcal{M}}(T(f_n,g_n),T(r_1,r_2))+\mathcal{P}_{\mathcal{M}}(Rf_{n+1},Rr_1)} \phi(t)dt \\
&\leq \int_0^{\mathcal{P}_{\mathcal{M}}(T(f_n,g_n),T(r_1,r_2))} \phi(t)dt + \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_{n+1},Rr_1)} \phi(t)dt \\
&\leq \mu \int_0^{\Delta_M^P(f_n,g_n,r_1,r_2)} \phi(t)dt + \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_{n+1},Rr_1)} \phi(t)dt,
\end{aligned} \tag{3.23}$$

where

$$\begin{aligned}
\Delta_M^P(f_n,g_n,r_1,r_2) &= \max\Big\{\mathcal{P}_{\mathcal{M}}(Rf_n,Rr_1), \mathcal{P}_{\mathcal{M}}(Rf_n,T(f_n,g_n)), \mathcal{P}_{\mathcal{M}}(Rr_1,T(r_1,r_2)), \\
&\qquad \frac{\mathcal{P}_{\mathcal{M}}(Rr_1,T(f_n,g_n))}{1+\mathcal{P}_{\mathcal{M}}(Rf_n,Rr_1)}, \frac{\mathcal{P}_{\mathcal{M}}(Rf_n,T(f_n,g_n))}{1+\mathcal{P}_{\mathcal{M}}(Rf_n,Rr_1)}\Big\} \\
&= \max\Big\{\mathcal{P}_{\mathcal{M}}(Rf_n,Rr_1), \mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n-1}), \mathcal{P}_{\mathcal{M}}(Rr_1,T(r_1,r_2)), \\
&\qquad \frac{\mathcal{P}_{\mathcal{M}}(Rr_1,Rf_{n-1})}{1+\mathcal{P}_{\mathcal{M}}(Rf_n,Rr_1)}, \frac{\mathcal{P}_{\mathcal{M}}(Rf_n,Rf_{n-1})}{1+\mathcal{P}_{\mathcal{M}}(Rf_n,Rr_1)}\Big\}.
\end{aligned}$$

Letting $n \to \infty$ in the above and using equations (3.20)-(3.22), we obtain

$$\begin{aligned}
\Delta_M^P(f_n,g_n,r_1,r_2) &= \max\Big\{0, 0, \mathcal{P}_{\mathcal{M}}(Rr_1,T(r_1,r_2)), 0, 0\Big\} \\
&= \mathcal{P}_{\mathcal{M}}(Rr_1,T(r_1,r_2)).
\end{aligned} \tag{3.24}$$

From equations (3.23) and (3.24), we obtain

$$\begin{aligned}
\int_0^{\mathcal{P}_{\mathcal{M}}(T(r_1,r_2),Rr_1)} \phi(t)dt &\leq \mu \int_0^{\mathcal{P}_{\mathcal{M}}(Rr_1,T(r_1,r_2))} \phi(t)dt \\
&\quad + \int_0^{\mathcal{P}_{\mathcal{M}}(Rf_{n+1},Rr_1)} \phi(t)dt.
\end{aligned} \tag{3.25}$$

Letting $n \to \infty$ in equation (3.25) and using Definition 2.9, equation (3.22) and $(pms3)$, we obtain

$$\int_0^{\mathcal{P}_{\mathcal{M}}(T(r_1,r_2),Rr_1)} \phi(t)dt \quad \leq \quad \mu \int_0^{\mathcal{P}_{\mathcal{M}}(T(r_1,r_2),Rr_1)} \phi(t)dt, \tag{3.26}$$

which is a contradiction, since $0 \leq \mu < 1$. Hence, we can conclude that

$$\int_0^{\mathcal{P}_{\mathcal{M}}(T(r_1,r_2),Rr_1)} \phi(t)dt = 0. \tag{3.27}$$

Now, by the property of integral ϕ, we obtain $\mathcal{P}_{\mathcal{M}}(T(r_1,r_2),Rr_1) = 0$ and so $Rr_1 = T(r_1,r_2)$. By similar way, one can prove that $Rr_2 = T(r_2,r_1)$.

Since by assumption, the pair (T,R) is weakly compatible (w-compatible), so by weak compatibility of T and R, we have

$$R(T(r_1,r_2)) = T(Rr_1,Rr_2) \text{ and } R(T(r_2,r_1)) = T(Rr_2,Rr_1).$$

Hence (Rr_1,Rr_2) is a common coupled fixed point of T and R.

Now, we show that the common coupled fixed point of T and R is unique. Assume that (Rz_1,Rz_2) is another common coupled fixed point of T and R with $Rr_1 \neq Rz_1$ and $Rr_2 \neq Rz_2$, that is, $(Rr_1,Rr_2) \neq (Rz_1,Rz_2)$. Then from equation (3.1) and using equation (3.22) and condition $(pms3)$, we have

$$\begin{aligned}
\int_0^{\mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1)} \phi(t)dt &= \int_0^{\mathcal{P}_{\mathcal{M}}(T(r_1,r_2),T(z_1,z_2))} \phi(t)dt \\
&\leq \mu \int_0^{\Delta_M^P(r_1,r_2,z_1,z_2)} \phi(t)dt,
\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}
\Delta_M^P(r_1,r_2,z_1,z_2) &= \max\Big\{\mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1), \mathcal{P}_{\mathcal{M}}(Rr_1,T(r_1,r_2)), \mathcal{P}_{\mathcal{M}}(Rz_1,T(z_1,z_2)), \\
&\qquad \frac{\mathcal{P}_{\mathcal{M}}(Rz_1,T(r_1,r_2))}{1+\mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1)}, \frac{\mathcal{P}_{\mathcal{M}}(Rr_1,T(r_1,r_2))}{1+\mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1)}\Big\} \\
&= \max\Big\{\mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1), \mathcal{P}_{\mathcal{M}}(Rr_1,Rr_1), \mathcal{P}_{\mathcal{M}}(Rz_1,Rz_1), \\
&\qquad \frac{\mathcal{P}_{\mathcal{M}}(Rz_1,Rr_1)}{1+\mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1)}, \frac{\mathcal{P}_{\mathcal{M}}(Rr_1,Rr_1)}{1+\mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1)}\Big\} \\
&= \max\Big\{\mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1), 0, 0, \mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1), 0\Big\} \\
&= \mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1),
\end{aligned}$$

since

$$\frac{\mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1)}{1+\mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1)} \leq \mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1).$$

Using this in equation (3.28), we obtain

$$\int_0^{\mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1)} \phi(t)dt \leq \mu \int_0^{\mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1)} \phi(t)dt,$$

which is a contradiction, since $0 \leq \mu < 1$. Hence, we can conclude that

$$\int_0^{\mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1)} \phi(t)dt = 0. \tag{3.29}$$

Now, regarding the property of integral ϕ, we obtain $\mathcal{P}_{\mathcal{M}}(Rr_1,Rz_1) = 0$ and so $Rr_1 = Rz_1$. Similarly, we can prove that $Rr_2 = Rz_2$. Thus, we have $(Rr_1,Rr_2) = (Rz_1,Rz_2)$ which shows that (Rr_1,Rr_2) is a unique common coupled fixed point of T and R. The proof is completed. □

If we take $\phi(t) = 1$ for all $t \geq 0$ in Theorem 3.1, then we have the following result.

Corollary 3.1 *Let* $(\Theta, \mathcal{P}_{\mathcal{M}})$ *be a complete partial metric space. Let* $T\colon \Theta\times\Theta \to \Theta$ *and* $R\colon \Theta \to \Theta$ *be two mappings satisfying the following contractive condition:*

$$\mathcal{P}_{\mathcal{M}}(T(f,g), T(j,k)) \leq \mu\, \Delta_M^P(f,g,j,k),$$

for all $f,g,j,k \in \Theta$, *where*

$$\begin{aligned}\Delta_M^P(f,g,j,k) &= \max\Big\{\mathcal{P}_{\mathcal{M}}(Rf,Rj), \mathcal{P}_{\mathcal{M}}(Rf,T(f,g)), \mathcal{P}_{\mathcal{M}}(Rj,T(j,k)),\\ &\qquad \frac{\mathcal{P}_{\mathcal{M}}(Rj,T(f,g))}{1+\mathcal{P}_{\mathcal{M}}(Rf,Rj)}, \frac{\mathcal{P}_{\mathcal{M}}(Rf,T(f,g))}{1+\mathcal{P}_{\mathcal{M}}(Rf,Rj)}\Big\},\end{aligned}$$

$\mu \in [0,1)$ *is a constant. Also, suppose* $T(\Theta\times\Theta) \subseteq R(\Theta)$ *and* $R(\Theta)$ *is a complete subspace of* Θ. *Then there exist* $r_1, r_2 \in \Theta$ *such that* $Rr_1 = T(r_1, r_2)$ *and* $Rr_2 = T(r_2, r_1)$, *that is,* T *and* R *have a unique coupled coincidence point. Moreover, if* T *and* R *are* w*-compatible, then* T *and* R *have a unique common coupled fixed point in* Θ *with* $\mathcal{P}_{\mathcal{M}}(t,t) = 0$ *for some* $t \in \Theta$.

Remark 3.1 *Corollary 3.1 generalizes the results of Kim et al. [20] for more general contractive condition.*

If we take $R = I$ (the identity map on Θ) in Corollary 3.1, then we have the following result.

Corollary 3.2 *Let* $(\Theta, \mathcal{P}_{\mathcal{M}})$ *be a complete partial metric space. Let* $T\colon \Theta\times\Theta \to \Theta$ *be a mapping satisfying the following contractive condition:*

$$\mathcal{P}_{\mathcal{M}}(T(f,g), T(j,k)) \leq \mu\, \Delta_{M'}^{P'}(f,g,j,k),$$

for all $f,g,j,k \in \Theta$, *where*

$$\begin{aligned}\Delta_{M'}^{P'}(f,g,j,k) &= \max\Big\{\mathcal{P}_{\mathcal{M}}(f,j), \mathcal{P}_{\mathcal{M}}(f,T(f,g)), \mathcal{P}_{\mathcal{M}}(j,T(j,k)),\\ &\qquad \frac{\mathcal{P}_{\mathcal{M}}(j,T(f,g))}{1+\mathcal{P}_{\mathcal{M}}(f,j)}, \frac{\mathcal{P}_{\mathcal{M}}(f,T(f,g))}{1+\mathcal{P}_{\mathcal{M}}(f,j)}\Big\}\end{aligned}$$

and $\mu \in [0,1)$ *is a constant. Then there exist* $r_1, r_2 \in \Theta$ *such that* $r_1 = T(r_1, r_2)$ *and* $r_2 = T(r_2, r_1)$, *that is,* T *has has a unique coupled fixed point in* Θ *with* $\mathcal{P}_{\mathcal{M}}(t,t) = 0$ *for some* $t \in \Theta$.

Remark 3.2 *Corollary 3.2 generalizes the results of H. Aydi [3] for more general contractive condition.*

If we take $R = I$ (the identity map on Θ) and

$$\max\Big\{\mathcal{P}_{\mathcal{M}}(f,j), \mathcal{P}_{\mathcal{M}}(f,T(f,g)), \mathcal{P}_{\mathcal{M}}(j,T(j,k)), \frac{\mathcal{P}_{\mathcal{M}}(j,T(f,g))}{1+\mathcal{P}_{\mathcal{M}}(f,j)},$$

$$\frac{\mathcal{P}_{\mathcal{M}}(f,T(f,g))}{1+\mathcal{P}_{\mathcal{M}}(f,j)}\Big\} = \mathcal{P}_{\mathcal{M}}(f,j),$$

in Theorem 3.1, then we have the following result.

Corollary 3.3 *Let* $(\Theta, \mathcal{P}_{\mathcal{M}})$ *be a complete partial metric space. Let* $T\colon \Theta\times\Theta \to \Theta$ *be a mapping satisfying the following contractive condition:*

$$\int_0^{\mathcal{P}_{\mathcal{M}}(T(f,g),T(j,k))} \phi(t)dt \leq \mu \int_0^{\mathcal{P}_{\mathcal{M}}(f,j)} \phi(t)dt, \tag{3.30}$$

for all $f,g,j,k \in \Theta$, *where* $\mu \in [0,1)$ *is a constant and* $\phi\colon [0,+\infty) \to [0,+\infty)$ *is a Lebesgue-integrable mapping which is summable on each compact subset of* $[0,+\infty)$, *and such that for each* $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)dt > 0$. *Then* T *has a unique coupled fixed point in* Θ *with* $\mathcal{P}_{\mathcal{M}}(t,t) = 0$ *for some* $t \in \Theta$.

If we define $T(f,f) = Qf$ in Corollary 3.3, then we have the following result.

Corollary 3.4 *Let $(\Theta, \mathcal{P}_{\mathcal{M}})$ be a complete partial metric space. Let $Q\colon \Theta \to \Theta$ be a mapping satisfying the following contractive condition:*

$$\int_0^{\mathcal{P}_{\mathcal{M}}(Qf,Qj))} \phi(t)dt \leq \mu \int_0^{\mathcal{P}_{\mathcal{M}}(f,j)} \phi(t)dt, \tag{3.31}$$

for all $f, j \in \Theta$, where $\mu \in [0,1)$ is a constant and $\phi\colon [0,+\infty) \to [0,+\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0,+\infty)$, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)dt > 0$. Then Q has a unique fixed point in Θ with $\mathcal{P}_{\mathcal{M}}(t,t) = 0$ for some $t \in \Theta$.

Proof: Taking $f = g$ and $j = k$ in Corollary 3.3, then inequality (3.30) coincides with the inequality (3.31). Thus, we have the conclusion of Corollary 3.4 from Corollary 3.3. □

Remark 3.3 *Corollary 3.4 extends and generalizes Theorem 2.1 of Branciari [6] from metric spaces to the setting of partial metric spaces.*

Remark 3.4 *If we take $\phi(t) = 1$ for all $t \in [0,+\infty)$ in Corollary 3.4, then we obtain Theorem 5.3 of Matthews [24] as follows.*

Corollary 3.5 *([24], Theorem 5.3) Let $(\Theta, \mathcal{P}_{\mathcal{M}})$ be a complete partial metric space. Let $Q\colon \Theta \to \Theta$ be a mapping satisfying the following contractive condition:*

$$\mathcal{P}_{\mathcal{M}}(Qf,Qj)) \leq \mu \mathcal{P}_{\mathcal{M}}(f,j), \tag{3.32}$$

for all $f, j \in \Theta$, where $\mu \in [0,1)$ is a constant. Then T has a unique fixed point $\alpha \in \Theta$ and $\mathcal{P}_{\mathcal{M}}(\alpha,\alpha) = 0$.

Remark 3.5 *Corollary 3.5 extends and generalizes well-known Banach contraction mapping principle [5] from metric spaces to the setting of partial metric spaces.*

If we take $\phi(t) = 1$ for all $t \geq 0$ in Corollary 3.3, then we have the following result.

Corollary 3.6 *Let $(\Theta, \mathcal{P}_{\mathcal{M}})$ be a complete partial metric space. Let $T\colon \Theta \times \Theta \to \Theta$ be a mapping satisfying the following contractive condition:*

$$\mathcal{P}_{\mathcal{M}}(T(f,g), T(j,k)) \leq \mu \mathcal{P}_{\mathcal{M}}(f,j), \tag{3.33}$$

for all $f, g, j, k \in \Theta$, where $\mu \in [0,1)$ is a constant. Then T has a unique coupled fixed point in Θ with $\mathcal{P}_{\mathcal{M}}(t,t) = 0$ for some $t \in \Theta$.

Another auxiliary result of Corollary 3.6 is the following.

Corollary 3.7 *Let $(\Theta, \mathcal{P}_{\mathcal{M}})$ be a complete partial metric space. Let $T\colon \Theta \times \Theta \to \Theta$ be a mapping satisfying the following contractive condition:*

$$\mathcal{P}_{\mathcal{M}}(T(f,g), T(j,k)) \leq \mu \mathcal{P}_{\mathcal{M}}(f,j) + \nu \mathcal{P}_{\mathcal{M}}(g,k), \tag{3.34}$$

for all $f, g, j, k \in \Theta$, where μ, ν are nonnegative constants such that $\mu + \nu < 1$. Then T has a unique coupled fixed point in Θ with $\mathcal{P}_{\mathcal{M}}(t,t) = 0$ for some $t \in \Theta$.

If we take $\mu = \nu = b$ where $b \in [0,1)$ in Corollary 3.7, then we have the following result.

Corollary 3.8 *Let $(\Theta, \mathcal{P}_{\mathcal{M}})$ be a complete partial metric space. Let $T\colon \Theta \times \Theta \to \Theta$ be a mapping satisfying the following contractive condition:*

$$\mathcal{P}_{\mathcal{M}}(T(f,g), T(j,k)) \leq \frac{b}{2} \left[\mathcal{P}_{\mathcal{M}}(f,j) + \mathcal{P}_{\mathcal{M}}(g,k)\right], \tag{3.35}$$

for all $f, g, j, k \in \Theta$, where $b \in [0,1)$ is a constant. Then T has a unique coupled fixed point in Θ with $\mathcal{P}_{\mathcal{M}}(t,t) = 0$ for some $t \in \Theta$.

Example 3.1 *Let* $\Theta = \mathbb{R}$. *Let* $\mathcal{P}_{\mathcal{M}} \colon \Theta \times \Theta \to \mathbb{R}$ *be defined by* $\mathcal{P}_{\mathcal{M}}(f, g) = \max\{f, g\}$ *for all* $f, g \in \Theta$. *Then the partial metric space* $(\Theta, \mathcal{P}_{\mathcal{M}})$ *is complete because* $(\Theta, \mathcal{P}_{\mathcal{M}(d)})$ *is complete. Indeed, for any* $f, g \in \Theta$,

$$\begin{aligned} \mathcal{P}_{\mathcal{M}(d)}(f, g) &= 2\mathcal{P}_{\mathcal{M}}(f, g) - \mathcal{P}_{\mathcal{M}}(f, f) - \mathcal{P}_{\mathcal{M}}(g, g) \\ &= 2\max\{f, g\} - (f + g) = |f - g|. \end{aligned}$$

Thus, $(\Theta, \mathcal{P}_{\mathcal{M}(d)})$ *is the Euclidean metric space which is complete. Consider the mapping* $T \colon \Theta \times \Theta \to \Theta$ *defined by* $T(f, g) = \frac{f+g}{6}$. *Now, for any* $f, g, j, k \in \Theta$, *we have*

$$\begin{aligned} \mathcal{P}_{\mathcal{M}}(T(f, g), T(j, k)) &= \frac{1}{6}\max\{f + g, j + k\} \\ &\leq \frac{1}{6}[\max\{f, j\} + \max\{g, k\}] \\ &= \frac{1}{6}[\mathcal{P}_{\mathcal{M}}(f, j) + p(g, k)], \end{aligned}$$

which is the contractive condition of Corollary 3.8 for $b = 1/3 < 1$. *Therefore, by Corollary 3.8,* T *has a unique coupled fixed point, which is* $(0, 0)$.

But if the mapping $T \colon \Theta \times \Theta \to \Theta$ *is given by* $T(f, g) = \frac{f+g}{2}$, *then* T *satisfies contractive condition of Corollary 3.8 for* $b = 1$,

$$\begin{aligned} \mathcal{P}_{\mathcal{M}}(T(f, g), T(j, k)) &= \frac{1}{2}\max\{f + g, j + k\} \\ &\leq \frac{1}{2}[\max\{f, j\} + \max\{g, k\}] \\ &= \frac{1}{2}[\mathcal{P}_{\mathcal{M}}(f, j) + p(g, k)], \end{aligned}$$

In this case $(0, 0)$ *and* $(1, 1)$ *are both coupled fixed points of* T, *and hence, the coupled fixed point of* T *is not unique. This shows that the condition* $b < 1$ *in Corollary 3.8 cannot be omitted in the statement of the aforesaid result.*

Example 3.2 *Let* $\Theta = \mathbb{R}$. *Let* $\mathcal{P}_{\mathcal{M}} \colon \Theta \times \Theta \to \mathbb{R}$ *be defined by* $\mathcal{P}_{\mathcal{M}}(f, g) = \max\{f, g\}$ *for all* $f, g \in \Theta$. *Then the partial metric space* $(\Theta, \mathcal{P}_{\mathcal{M}})$ *is complete because* $(\Theta, \mathcal{P}_{\mathcal{M}(d)})$ *is complete. Indeed, for any* $f, g \in \Theta$,

$$\begin{aligned} \mathcal{P}_{\mathcal{M}(d)}(f, g) &= 2\mathcal{P}_{\mathcal{M}}(f, g) - \mathcal{P}_{\mathcal{M}}(f, f) - \mathcal{P}_{\mathcal{M}}(g, g) \\ &= 2\max\{f, g\} - (f + g) = |f - g|. \end{aligned}$$

Thus, $(\Theta, \mathcal{P}_{\mathcal{M}(d)})$ *is the Euclidean metric space which is complete. Consider the mapping* $T \colon \Theta \times \Theta \to \Theta$ *defined by* $T(f, g) = \frac{3f-g}{5}$ *and let* $\mathcal{P}_{\mathcal{M}}(f, g) = |f - g|$ *for all* $f, g \in \Theta$. *Let us take* $f \neq j$ *and* $g = k$ *in the inequality of Corollary 3.8. Hence* $u = |f - j| > 0$. *Now, using inequality of Corollary 3.8, we have*

$$\begin{aligned} \frac{3}{5}u &= \frac{3|f - j|}{5} = \mathcal{P}_{\mathcal{M}}(T(f, g), T(j, k)) \\ &\leq \frac{b}{2}\big[|f - j| + |g - k|\big] = \frac{b}{2}(|f - j|) \\ &= \frac{b}{2}u, \end{aligned}$$

or,

$$\frac{3}{5}u \leq \frac{b}{2}u,$$

that is,

$$b \geq \frac{6}{5},$$

which is a contradiction, since by hypothesis $0 < b < 1$. *Hence the Corollary 3.8 is not applicable to the operator* T *in order to prove that* $(0, 0)$ *is a unique coupled fixed point of* T.

4. Application by using modified fractional integral

Recently, a modification of fractional calculus is given in [34] by introducing the following integral operator

$$Q^{\alpha}_{\lambda}f(x) := Qf(x) = \int_0^x f(s)[R]^{\alpha}_{\lambda}(s)ds, \tag{4.1}$$

where $[R]^{\alpha}_{\lambda}(s)$ represents Rabotnov function (see [35] for recent information)

$$[R]^{\alpha}_{\lambda}(s) = \exp(-\lambda(x-s)^{\alpha}), \quad \alpha \in (0,1),\, \lambda \in \mathbb{R}^+,\, x \geq s.$$

Correspondingly, the fractional derivative is given by

$$D^{\alpha}_{\lambda}f(x) = \int_0^x f'(s)[R]^{\alpha}_{\lambda}(s)ds. \tag{4.2}$$

Assume the following assumptions:

[**A1**] we let $\Theta = \mathbb{R}^+$, where $\mathbb{R}^+ = [0, +\infty)$ and $\mathcal{P}_{\mathcal{M}}\colon \Theta \times \Theta \to \mathbb{R}^+$ be given by $\mathcal{P}_{\mathcal{M}}(f,j) = C\max\{f,j\}, C > 0$ for all $f, j \in \mathbb{R}^+$.

[**A2**] We consider

$$C_{\alpha,\lambda} := \max_{x \in J=[0,T]} |[R]^{\alpha}_{\lambda}(x)|.$$

By the definition of $\mathcal{P}_{\mathcal{M}}(Qf, Qj))$, we have

$$\begin{aligned}
\mathcal{P}_{\mathcal{M}}(Qf,Qj) &= C \max_{x\in J=[0,T]}\{Qf,Qj\} \\
&= C\max\left\{\int_0^x f(s)[R]^{\alpha}_{\lambda}(s)ds, \int_0^x j(s)[R]^{\alpha}_{\lambda}(s)ds\right\} \\
&\leq C \max_{x\in J=[0,T]}\{f,j\} \max_{x\in J=[0,T]} |[R]^{\alpha}_{\lambda}(x)|T \\
&\leq CC_{\alpha,\lambda}T \max_{x\in J=[0,T]}\{f,j\} \\
&:= \bar{C} \max_{x\in J=[0,T]}\{f,j\}.
\end{aligned}$$

Our aim is to use Corollary 3.4, where

$$\int_0^{\mathcal{P}_{\mathcal{M}}(Qf,Qj)} \phi(t)dt \leq \mu \int_0^{\mathcal{P}_{\mathcal{M}}(f,j)} \phi(t)dt.$$

By the assumption of ϕ (see Corollary 3.4), we have

$$\begin{aligned}
\int_0^{\mathcal{P}_{\mathcal{M}}(Qf,Qj)} \phi(t)dt &= \int_0^{C\max_{x\in J=[0,T]}\{Qf,Qj\}} \phi(t)dt \\
&\leq \int_0^{\bar{C}\max_{x\in J=[0,T]}\{f,j\}} \phi(t)dt \\
&= \int_0^{\mathcal{P}_{\mathcal{M}}(f,j)} \phi(t)dt,
\end{aligned}$$

where $\mu = 1$. Hence, we obtain the following result.

Proposition 4.1 *If the assumptions [**A1**] and [**A2**] are applied then*

$$\int_0^{\mathcal{P}_{\mathcal{M}}(Qf,Qj)} \phi(t)dt \leq \int_0^{\mathcal{P}_{\mathcal{M}}(f,j)} \phi(t)dt,$$

where

$$Qf(x) = \int_0^x f(s)[R]^{\alpha}_{\lambda}(s)ds$$

and $\mathcal{P}_{\mathcal{M}}(f,j) = C\max\{f,j\}, C > 0$ *for all* $f, j \in \mathbb{R}^+$.

Example 4.1 *Consider the following data* $\phi(x) = 1, C = 1, f(x) = \cos(x), j(x) = \sin(x)$ *where* $x \in J = [0, T]$ *such that*

$$T \geq \frac{1}{C_{\alpha,\lambda}} := \frac{1}{\max_{x\in J} |[R]_{\lambda}^{\alpha}(x)|}.$$

A computation yields

$$\begin{aligned}
\mathcal{P}_{\mathcal{M}}(Qf, Qj) &= C \max_{x\in J=[0,T]} \{Qf, Qj\} \\
&= C \max \left\{ \int_0^x \cos(s)[R]_{\lambda}^{\alpha}(s)ds, \int_0^x \sin(s)[R]_{\lambda}^{\alpha}(s)ds \right\} \\
&\leq C \max_{x\in J=[0,T]} \{\cos(x), \sin(x)\} \max_{x\in J=[0,T]} |[R]_{\lambda}^{\alpha}(x)|T \\
&\leq C_{\alpha,\lambda}T \max_{x\in J=[0,T]} \{\cos(x), \sin(x)\}, \quad C_{\alpha,\lambda}T \geq 1 \\
&:= \bar{C} \max_{x\in J=[0,T]} \{\cos(x), \sin(x)\}.
\end{aligned}$$

As a consequence of Proposition 4.1, we have

$$\int_0^{\mathcal{P}_{\mathcal{M}}(Q\cos(x),Q\sin(x))} dt \leq \int_0^{\mathcal{P}_{\mathcal{M}}(\cos(x),\sin(x))} dt.$$

5. Conclusion

In this paper, we establish common coupled fixed point result of integral type contraction for a pair of w-compatible mappings in the setting of partial metric spaces. Furthermore, we provide some consequences of the established result as corollaries. Also, an illustrative example and application of integral equation are given. Our results extend and generalize several previous works from the existing literature (see, for example, [3,5,6,20,24] and many others). Application is presented by utilize the modified fractional integral operator involving the special Rabotnov function. Numerical example is introduced.

References

1. Abbas, M., Khan, M. A. and Radenović, S., *Common coupled fixed point theorems in cone metric spaces for w-compatible mappings*, Appl. Math. Comput., 217(1), 195-202, (2010).
2. Altun, I., Sola, F. and Simsek, H., *Generalized contractions on partial metric spaces*, Topology and its Appl., 157, 2778-2785, (2010).
3. Aydi, H., *Some coupled fixed point results on partial metric spaces*, International J. Math. Math. Sci., 2011, Article ID 647091, 11 pages.
4. Aydi, H., Abbas, M. and Vetro, C., *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*, Topology and Its Appl., 159, No. 14, 3234-3242, (2012).
5. Banach, S., *Sur les operation dans les ensembles abstraits et leur application aux equation integrals*, Fund. Math., 3, 133-181, (1922).
6. Branciari, A., *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci., 29(9), 531-536, (2002).
7. Bhaskar, T. G. and Lakshmikantham, V., *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Analysis: TMA, 65(7), 1379-1393, (2006).
8. Chandok, S., Kumar, D. and Khan, M. S., *Some results in partial metric space using auxiliary functions*, Applied Math. E-Notes, 15, 233-242, (2015).
9. Dhawan, P. and Kaur, J., *Some common fixed point theorems in ordered partial metric spaces via $\mathcal{F}$-generalized contractive type mappings*, Mathematics, 7, 193, (2019), doi:10.3390/math7020193.
10. Eke, K. S. and Oghonyon, J. G., *Some fixed point theorems in ordered partial metric spaces with applications*, Cogent Math. Stat. (2018), 5:1509426, https://doi.org/10.1080/25742558.2018.1509426.
11. Gregorio, R. O., *Coupled fixed point theorems in partially ordered metric spaces*, Adv. Fixed Point Theory, 9(1), 17-28, (2019).

12. Hazarika, B., Arab, R. and Kumam, P., *Coupled fixed point theorems in partially ordered metric spaces via mixed g-monotone property*, J. Fixed Point Theory Appl., 21, Art. No. 1, (2019).
13. Heckmann, R., *Approximation of metric spaces by partial metric spaces*, Appl. Categ. Structures, 7(1-2), 71-83, (1999).
14. Jain, R., Saluja, G. S., Nashine, H. K. and Agarwal, R. P., *Coupled and common coupled fixed point theorems for contractive type conditions in partial metric spaces*, Advances Math. Sci. Appl., 32(1), 73-96, (2023).
15. Karapinar, E., Shatanawi, W. and Tas, K., *Fixed point theorems on partial metric spaces involving rational expressions*, Miskolc Math. Notes, 14, 135-142, (2013).
16. Karapınar, E. and Agarwal, R. P., *Fixed Point Theory in Generalized Metric Spaces*, Springer, 2022.
17. Khojasteh, F., Goodarzi, Z. and Razani, A., *Some fixed point theorems of integral type contraction in cone metric spaces*, Fixed Point Theory Appl., (2010), Article ID 189684, 13 pages, 2010.
18. Khomdram, B., Rohen, Y. and Singh, T. C., *Coupled fixed point theorems in G_b-metric space satisfying some rational contractive conditions*, Springer Plus (2016), 5:1261. Doi:10.1186/s40064-016-2925-7.
19. Kim, J. K. and Chandok, S., *Coupled common fixed point theorems for generalized nonlinear contraction mappings with the mixed monotone property in partially ordered metric spaces*, Fixed Point Theory Appl. (2013), 2013:307.
20. Kim, J. K., Okeke, G. A. and Lim, W. H., *Common coupled fixed point theorems for w-compatible mappings in partial metric spaces*, Global J. Pure Appl. Math., 13(2), 519-536, (2017).
21. Künzi, H. P. A., *Nonsymmetric distances and their associated topologies about the origins of basic ideas in the area of asymptotic topology*, Handbook of the History Gen. Topology (eds. C.E. Aull and R. Lowen), Kluwer Acad. Publ., 3, 853-868, (2001).
22. Kumar, D., Sadat, S., Lee, J. R. and Park, C., *Some theorems in partial metric spaces using auxiliary functions*, AIMS Maths., 6(7), 6734-6748, (2021).
23. Liu, Z., Li, J. and Kang, S. M., *Fixed point theorems of contractive mappings of integral type*, Fixed Point Theory Appl., 2013, 2013:300, 17 pp.
24. Matthews, S. G., *Partial metric topology*, Proceedings of the 8th summer conference on topology and its applications, Annals of the New York Academy of Sciences, 728, 183-197, (1994).
25. Nashine, H. K., Saluja, G. S., Rao, G. V. V. J. and Lim, W. H., *Control function based coupled and common coupled fixed point theorems in partial metric spaces*, Nonlinear Functional Anal. Appl., 29(2), 559-580, (2024).
26. Nazam, M., Hamid, Z., Sulami, H. Al and Hussain, A., *Common fixed point theorems in the partial b-metric spaces and an application to the system of boundary value problems*, J. Function Spaces, 2021, Art. ID 7777754, 11 pages. https://doi.org/10.1155/2021//7777754.
27. Oltra, S. and Valero, O., *Banach's fixed point theorem for partial metric spaces*, Rend. Ist. Mat. Univ. Trieste, 36(1-2), 17-26, (2004).
28. Olaleru, J. O., Okeke, G. A. and Akewe, H., *Coupled fixed point theorems for generalized φ-mappings satisfying contractive condition of integral type on cone metric spaces*, Int. J. Math. Model. Comput., 2(2), 87-98, (2012).
29. Rahman, M. U., Sarwar, M. and Rahman, M. U., *Fixed point results of Altman integral type mappings in S-metric spaces*, Int. J. Anal. Appl., 10(1), 58-63, (2016).
30. Rashwan, R. A. and Hammad, H. A., *Common fixed point theorems for weakly compatible mappings satisfying a general contractive condition of integral type*, Palestine J. Math., 8(2), 114-126, (2019).
31. Sharma, Y. and Jain, S., *Coupled fixed point theorems in modular metric spaces endowed with a graph*, Kyunpook Math. J., 61, 441-453, (2021). https://doi.org/10.5666/KMJ.2021.61.2.441.
32. Shatanawi, W., Samet, B. and Abbas, M., *Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces*, Math. Comput. Modell., 55(3-4), 680-687, (2012).
33. Valero, O., *On Banach fixed point theorems for partial metric spaces*, Appl. Gen. Topl., 6(2), 229-240, (2005).
34. Yavuz, M. and Sene, N., *Fundamental calculus of the fractional derivative defined with Rabotnov exponential kernel and application to nonlinear dispersive wave model*, Journal of Ocean Engineering and Science, 6(2), 196-205, (2021).
35. Ibrahim, R. W., Salahshour, S. and Pall-Szabo, A. O., *Analysis of a Normalized Structure of a Complex Fractal-Fractional Integral Transform Using Special Functions*, Axioms 13(8), 522, (2024).

Hemant Kumar Nashine,

Mathematics Division, School of Advanced Sciences and Languages, VIT Bhopal University,

Kothrikalan, Sehore, Madhya Pradesh - 466114, India.

E-mail address: `hknashine@gmail.com`, `hemantkumar.nashine@vitbhopal.ac.in`

and

G. S. Saluja,
H.N. 3/1005, Geeta Nagar, Raipur, Raipur - 492001 (C.G.), India.
E-mail address: `saluja1963@gmail.com`

and

Rabha W. Ibrahim,
Information and Communication Technology Research Group, Scientific Research Center,
Al-Ayen, University, Thi-Qar, Iraq.
E-mail address: `rabha@alayen.edu.iq`