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Some Coupled Fixed Point Theorems in Partially Ordered Partial Metric Spaces and Application to Integral Equations

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ABSTRACT: In this paper, we prove coupled fixed point theorems using ψ-contractive conditions in partially ordered partial metric spaces. We also provide corollaries to the established conclusions. Furthermore, we provide several examples to support the established results. An application of the nonlinear integral equation is also provided. Several conclusions in the current literature are extended, generalized and enriched by our findings. Our results, in particular extend and generalize the findings of *Aydi* [8] and *Bhaskar and Lakshmikantham* [12].

Key Words: Coupled fixed point, ψ-contractive condition, partial metric space, partially ordered set.

Contents

1. Introduction

Matthews [23,24] introduced the concept of partial metric spaces to study the denotational semantics dataflow networks. In fact, a partial metric space is a generalization of usual metric spaces in which the self-distance need not be zero. It is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation (see, e.g., [17], [22], [30], [35], [38], [39]). Later, *Matthews* proved the partial metric version of Banach fixed point theorem [11]. In fact, partial metrics are more adaptable having broader topological properties than that of metrics create partial order. *Heckmann* [17] introduced the concept of weak partial metric function and established some fixed point results. *Oltra and Valero* [29] generalized the *Matthews* results in the sense of *O'Neil* [31] in complete partial metric space. *Abdeljawad et al.* [4] considered a general form of the weak ϕ-contraction and established some common fixed point results. Afterwards, many authors have conducted further research on fixed point theorems in the same class of spaces (see e.g. [2], [3], [6], [7], [9], [10], [15], [19], [20], [21], [41]).

Bhashkar and Lakshmikantham [12] (2006) established some coupled fixed point theorems on ordered metric spaces and give application in the existence and uniqueness of a solution for periodic boundary value problem (see, also [16]). Later on, *Ćirić and Lakshmikantham* [13] (2009) investigated some more coupled fixed point theorems in partially ordered sets. Further, many authors have obtained coupled fixed point results for mappings under various contractive conditions in the setting of metric spaces and generalized metric spaces (see [1], [5], [8], [25], [26], [27], [32], [33], [34], [36], [40]).

Recently, *Jain et al.* [18] *Nashine et al.* [28] (2024) proved some coupled and common coupled fixed point results by means of control function and provide some corollaries of the established results. Also give some examples to validate the results.

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In this article, we prove some coupled fixed point theorems via ψ-contractive condition in the setting of partially ordered partial metric spaces and give some consequences of the established result. Moreover, we provide some illustrative examples to validate the established results. An application to the nonlinear integral equation is also given. Our results extend, generalize and enrich several results from the existing literature.

2. Preliminaries

In this section, we recall the notion of partial metric space and some of its properties which will be useful in the main section to establish few results.

Definition 2.1 *([24]) Let $\mathcal{M}$ be a nonempty set. A partial metric on $\mathcal{M}$ is a function $P\colon \mathcal{M}\times\mathcal{M}\to[0,+\infty)$ such that for all $r,s,t\in\mathcal{M}$ the followings are satisfied:*

(1) $r=s \Leftrightarrow P(r,r)=P(r,s)=P(s,s)$,

(2) $P(r,r)\le P(r,s)$,

(3) $P(r,s)=P(s,r)$,

(4) $P(r,s)\le P(r,t)+P(t,s)-P(t,t)$.

Then P is called a partial metric on $\mathcal{M}$ and the pair $(\mathcal{M},P)$ is called a partial metric space (in short PMS).

It is clear that if $P(r,s)=0$, then from (1), (2), and (3), $r=s$. But if $r=s$, $P(r,s)$ may not be 0.

If P is a partial metric on $\mathcal{M}$, then the function $d_P\colon\mathcal{M}\times\mathcal{M}\to[0,+\infty)$ given by

$$d_P(r,s)=2P(r,s)-P(r,r)-P(s,s), \tag{2.1}$$

is a usual metric on $\mathcal{M}$.

Each partial metric P on $\mathcal{M}$ generates a T_0 topology τ_ρ on $\mathcal{M}$ with the family of open P-balls $\{B_P(r,\varepsilon): r\in\mathcal{M},\varepsilon>0\}$ where $B_P(r,\varepsilon)=\{s\in\mathcal{M}: P(r,s)<P(r,r)+\varepsilon\}$ for all $r\in\mathcal{M}$ and $\varepsilon>0$. Similarly, closed P-ball is defined as $B_P[r,\varepsilon]=\{s\in\mathcal{M}: P(r,s)\le P(r,r)+\varepsilon\}$ for all $r\in\mathcal{M}$ and $\varepsilon>0$.

Example 2.1 *([10]) Let $\mathcal{M}=[0,+\infty)$ and $P\colon\mathcal{M}\times\mathcal{M}\to[0,+\infty)$ be given by $P(r,s)=\max\{r,s\}$ for all $r,s\in\mathcal{M}$. Then $(\mathcal{M},P)$ is a partial metric space.*

Example 2.2 *([10]) Let $I=\mathcal{M}$, where I denote the set of all intervals $[r_1,s_1]$ for any real numbers $r_1\le s_1$. Let $P\colon\mathcal{M}\times\mathcal{M}\to[0,\infty)$ be a function such that $P\big([r_1,s_1],[r_2,s_2]\big)=\max\{s_1,s_2\}-\min\{r_1,r_2\}$. Then $(\mathcal{M},P)$ is a partial metric space.*

Example 2.3 *([14]) Let $\mathcal{M}=\mathbb{R}$ and $P\colon\mathcal{M}\times\mathcal{M}\to\mathbb{R}^+$ be given by $P(r,s)=e^{\max\{r,s\}}$ for all $r,s\in\mathcal{M}$. Then $(\mathcal{M},P)$ is a partial metric space.*

Definition 2.2 *([23]) Let $(\mathcal{M},P)$ be a partial metric space.*

(i) A sequence $\{s_n\}$ converges to a point $s\in\mathcal{M}$ if and only if $\lim_{n\to\infty}P(s,s_n)=P(s,s)$.

(ii) A sequence $\{s_n\}$ in $\mathcal{M}$ is called a Cauchy sequence if and only if $\lim_{m,n\to\infty} P(s_m,s_n)$ exists (and finite).

(iii) A partial metric space $(\mathcal{M},P)$ is said to be complete if every Cauchy sequence $\{s_n\}$ in $\mathcal{M}$ converges, with respect to τ_ρ, to a point $s\in\mathcal{M}$, such that, $\lim_{m,n\to\infty}P(s_m,s_n)=P(s,s)$.

(iv) A mapping $F\colon\mathcal{M}\to\mathcal{M}$ is said to be continuous at $s_0\in\mathcal{M}$ if for every $\varepsilon>0$, there exists $\delta>0$ such that $F\big(B_P(s_0,\delta)\big)\subset B_P\big(F(s_0),\varepsilon\big)$.

Definition 2.3 *([23]) A partial metric space $(\mathcal{M},P)$ is said to be complete if every Cauchy sequence $\{s_n\}$ in $\mathcal{M}$ converges to a point $s\in\mathcal{M}$ with respect to τ_ρ. Furthermore,*

$$\lim_{m,n\to\infty}P(s_m,s_n)=\lim_{n\to\infty}P(s_n,s)=P(s,s).$$

Definition 2.4 *(Control function) Let Ψ be the set of all functions $\psi\colon [0,+\infty) \to [0,+\infty)$ with the properties*

(Ψ_1) ψ is continuous and non-decreasing,

(Ψ_2) $\psi(t) < t$ for each $t > 0$.

Obviously, if $\psi \in \Psi$, then $\psi(0) = 0$ and $\psi(t) \le t$ for all $t \ge 0$.

Definition 2.5 *([12]) Let $(\mathcal{M}, \le)$ be a partially ordered set. The mapping $V\colon \mathcal{M}\times\mathcal{M} \to \mathcal{M}$ is said to have the mixed monotone property if $V(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in \mathcal{M}$,*

$$x_1, x_2 \in \mathcal{M},\;\; x_1 \le x_2 \;\Rightarrow\; V(x_1, y) \le V(x_2, y),$$

and

$$y_1, y_2 \in \mathcal{M},\;\; y_1 \le y_2 \;\Rightarrow\; V(x, y_1) \ge V(x, y_2).$$

Definition 2.6 *([12,13]) An element $(x,y) \in \mathcal{M}\times\mathcal{M}$ is said to be a coupled fixed point of the mapping $V\colon \mathcal{M}\times\mathcal{M} \to \mathcal{M}$ if $V(x,y) = x$ and $V(y,x) = y$.*

Example 2.4 *Let $\mathcal{M} = [0,+\infty)$ and $V\colon \mathcal{M}\times\mathcal{M} \to \mathcal{M}$ be defined by $V(x,y) = \frac{x+y}{3}$ for all $x, y \in \mathcal{M}$. Then one can easily see that V has a unique coupled fixed point $(0,0)$.*

Example 2.5 *Let $\mathcal{M} = [0,+\infty)$ and $V\colon \mathcal{M}\times\mathcal{M} \to \mathcal{M}$ be defined by $V(x,y) = \frac{x+y}{2}$ for all $x, y \in \mathcal{M}$. Then we see that V has two coupled fixed point $(0,0)$ and $(1,1)$, that is, the coupled fixed point is not unique.*

Lemma 2.1 *([8,23,24])*

(a_1) A sequence $\{s_n\}$ is Cauchy in a partial metric space $(\mathcal{M}, P)$ if and only if $\{s_n\}$ is Cauchy in a metric space $(\mathcal{M}, d_P)$ where

$$d_P(r,s) = 2P(r,s) - P(r,r) - P(s,s).$$

(a_2) A partial metric space $(\mathcal{M}, P)$ is complete if a metric space $(\mathcal{M}, d_P)$ is complete, i.e.,

$$\lim_{n\to\infty} d_P(s_n, s) = 0 \;\Leftrightarrow\; P(s,s) = \lim_{n\to\infty} P(s_n, s) = \lim_{n,m\to\infty} P(s_n, s_m).$$

Lemma 2.2 *([21]) Let $(\mathcal{M}, P)$ be a partial metric space.*

(b_1) If $r, s \in \mathcal{M}$, $P(r,s) = 0$, then $r = s$.

(b_2) If $r \ne s$, then $P(r,s) > 0$.

One of the characterization of continuity of mappings in partial metric spaces was given by *Samet et al.* [37] as follows.

Lemma 2.3 *(see [37]) Let $(\mathcal{M}, P)$ be a partial metric space. The function $H\colon \mathcal{M} \to \mathcal{M}$ is continuous if given a sequence $\{s_n\}_{n\in\mathbb{N}}$ and $s \in \mathcal{M}$ such that $P(s,s) = \lim_{n\to\infty} P(s, s_n)$, then $P(Hs, Hs) = \lim_{n\to\infty} P(Hs, Hs_n)$.*

Example 2.6 *(see [37]) Let $\mathcal{M} = [0,+\infty)$ endowed with the partial metric $P\colon \mathcal{M}\times\mathcal{M} \to [0,+\infty)$ defined $P(r,s) = \max\{r,s\}$ for all $r, s \in \mathcal{M}$. Let $H\colon \mathcal{M} \to \mathcal{M}$ be a non-decreasing function. If H is continuous with respect to the standard metric $d(r,s) = |r-s|$ for all $r, s \in \mathcal{M}$, then H is continuous with respect to the partial metric P.*

Lemma 2.4 *(see [14]) Let $s_n \to s$ as $n \to \infty$ in a partial metric space $(\mathcal{M}, P)$ where $P(s,s) = 0$. Then $\lim_{n\to\infty} P(s_n, z) = P(s,z)$ for all $z \in \mathcal{M}$.*

3. Main Results

In this section, we shall prove some coupled fixed point theorems for ψ-contractive condition in the setting of partially ordered partial metric spaces.

Theorem 3.1 *Let $(\mathcal{M}, P, \leq)$ be a partially ordered complete partial metric space. Suppose that the mapping $V\colon \mathcal{M}\times\mathcal{M}\to\mathcal{M}$ satisfies the following conditions:*

(1)

$$P(V(r,s),V(t,z)) \leq \psi\big(M_V(r,s,t,z)\big), \tag{3.1}$$

for all $r,s,t,z\in\mathcal{M}$, where

$$\begin{aligned} M_V(r,s,t,z) &= a_1\,P(r,t)+a_2\,P(s,z)+a_3\,P(r,V(r,s)) \\ &\quad +a_4\,P(t,V(t,z))+a_5\,P(t,V(r,s)), \end{aligned}$$

a_1,a_2,a_3,a_4,a_5 are nonnegative reals such that $a_1+a_2+a_3+a_4+a_5<1$ and $\psi\in\Psi$,

(2) *either V is continuous or*

(3) *$\mathcal{M}$ has the following properties*

(a') *if a non-decreasing sequence $\{r_n\}$ in $\mathcal{M}$ converges to some point $r\in\mathcal{M}$, then $r_n\leq r$ for all n,*

(a'') *if a non-increasing sequence $\{s_n\}$ in $\mathcal{M}$ converges to some point $s\in\mathcal{M}$, then $s\leq s_n$ for all n.*

If there exist two elements $r_0,s_0\in\mathcal{M}$ with $r_0\leq V(r_0,s_0)$ and $s_0\geq V(s_0,r_0)$, then V has a coupled fixed point in $\mathcal{M}$.

Proof: Let $r_0,s_0\in\mathcal{M}$ be such that $r_0\leq V(r_0,s_0)$ and $s_0\geq V(s_0,r_0)$. Let $r_1=V(r_0,s_0)$ and $s_1=V(s_0,r_0)$. Then $r_0\leq r_1$ and $s_0\geq s_1$. Again, let $r_2=V(r_1,s_1)$ and $s_2=V(s_1,r_1)$. Since V has the mixed monotone property on $\mathcal{M}$, then we have $r_1\leq r_2$ and $s_1\geq s_2$. Continuing the above process, we get two sequences $\{r_n\}$ and $\{s_n\}$ in $\mathcal{M}$ such that $r_{n+1}=V(r_n,s_n)$ and $s_{n+1}=V(s_n,r_n)$ for all $n\geq 0$ and

$$r_0\leq r_1\leq\cdots\leq r_n\leq r_{n+1}\leq\ldots,\quad s_0\geq s_1\geq\cdots\geq s_n\geq s_{n+1}\geq\ldots. \tag{3.2}$$

Now, using equation (3.1) with $r=r_{n-1}$, $s=s_{n-1}$, $t=r_n$ and $z=s_n$, we have

$$\begin{aligned} P(r_n,r_{n+1}) &= P\big(V(r_{n-1},s_{n-1}),V(r_n,s_n)\big) \\ &\leq \psi\big(M_V(r_{n-1},s_{n-1},r_n,s_n)\big), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} M_V(r_{n-1},s_{n-1},r_n,s_n) &= a_1\,P(r_{n-1},r_n)+a_2\,P(s_{n-1},s_n)+a_3\,P(r_{n-1},V(r_{n-1},s_{n-1})) \\ &\quad +a_4\,P(r_n,V(r_n,s_n))+a_5\,P(r_n,V(r_{n-1},s_{n-1})) \\ &= a_1\,P(r_{n-1},r_n)+a_2\,P(s_{n-1},s_n)+a_3\,P(r_{n-1},r_n) \\ &\quad +a_4\,P(r_n,r_{n+1})+a_5\,P(r_n,r_n) \\ &\leq a_1\,P(r_{n-1},r_n)+a_2\,P(s_{n-1},s_n)+a_3\,P(r_{n-1},r_n) \\ &\quad +a_4\,P(r_n,r_{n+1})+a_5\,P(r_n,r_{n+1}) \\ &= (a_1+a_3)P(r_{n-1},r_n)+a_2\,P(s_{n-1},s_n) \\ &\quad +(a_4+a_5)P(r_n,r_{n+1}). \end{aligned}$$

Using this and the property of ψ in equation (3.3), we obtain

$$\begin{aligned} P(r_n,r_{n+1}) &\leq \psi\big((a_1+a_3)P(r_{n-1},r_n)+a_2\,P(s_{n-1},s_n) \\ &\quad +(a_4+a_5)P(r_n,r_{n+1})\big) \\ &< (a_1+a_3)P(r_{n-1},r_n)+a_2\,P(s_{n-1},s_n) \\ &\quad +(a_4+a_5)P(r_n,r_{n+1}). \end{aligned} \tag{3.4}$$

By similar fashion one can obtain

$$\begin{aligned} P(s_n, s_{n+1}) &< (a_1 + a_3)P(s_{n-1}, s_n) + a_2\, P(r_{n-1}, r_n) \\ &\quad + (a_4 + a_5)P(s_n, s_{n+1}). \end{aligned} \tag{3.5}$$

From equations (3.4) and (3.5), we obtain

$$\begin{aligned} P(r_n, r_{n+1}) + P(s_n, s_{n+1}) &< (a_1 + a_3)[P(r_{n-1}, r_n) + P(s_{n-1}, s_n)] \\ &\quad + a_2\, [P(r_{n-1}, r_n) + P(s_{n-1}, s_n)] \\ &\quad + (a_4 + a_5)[P(r_n, r_{n+1}) + P(s_n, s_{n+1})]. \end{aligned} \tag{3.6}$$

Let $Y_n = P(r_n, r_{n+1}) + P(s_n, s_{n+1})$ for all $n \geq 0$. Then from equation (3.6), we obtain

$$\begin{aligned} Y_n &< (a_1 + a_3)Y_{n-1} + a_2 Y_{n-1} + (a_4 + a_5)Y_n \\ &= (a_1 + a_2 + a_3)Y_{n-1} + (a_4 + a_5)Y_n. \end{aligned}$$

This implies that

$$\begin{aligned} Y_n &< \Big(\frac{a_1 + a_2 + a_3}{1 - a_4 - a_5}\Big)Y_{n-1} \\ &= \gamma\, Y_{n-1}, \end{aligned}$$

where

$$\gamma = \Big(\frac{a_1 + a_2 + a_3}{1 - a_4 - a_5}\Big) < 1,$$

since by assumption $a_1 + a_2 + a_3 + a_4 + a_5 < 1$.

Continuing in the same way, we obtain

$$Y_n < \gamma\, Y_{n-1} < \gamma^2\, Y_{n-2} < \gamma^3\, Y_{n-3} < \cdots < \gamma^n\, Y_0. \tag{3.7}$$

If $Y_0 = 0$, then $P(r_0, r_1) + P(s_0, s_1) = 0$. Hence $P(r_0, r_1) = 0$ and $P(s_0, s_1) = 0$. Therefore by Lemma 2.2 (b_1), we get $r_0 = r_1 = V(r_0, s_0)$ and $s_0 = s_1 = V(s_0, r_0)$. This means that (r_0, s_0) is a coupled fixed point V. Now, assume that $Y_0 > 0$. For each $n \geq m$, where $n, m \in \mathbb{N}$, by using condition (4), we have

$$\begin{aligned} P(r_n, r_m) &\leq P(r_n, r_{n-1}) + P(r_{n-1}, r_{n-2}) + \cdots + P(r_{m+1}, r_m) \\ &\quad - P(r_{n-1}, r_{n-1}) - P(r_{n-2}, r_{n-2}) - \cdots - P(r_{m+1}, r_{m+1}) \\ &\leq P(r_n, r_{n-1}) + P(r_{n-1}, r_{n-2}) + \cdots + P(r_{m+1}, r_m). \end{aligned} \tag{3.8}$$

Similarly, one can obtain

$$P(s_n, s_m) \leq P(s_n, s_{n-1}) + P(s_{n-1}, s_{n-2}) + \cdots + P(s_{m+1}, s_m). \tag{3.9}$$

Thus,

$$\begin{aligned} Y_{nm} = P(r_n, r_m) + P(s_n, s_m) &\leq Y_{n-1} + Y_{n-2} + \cdots + Y_m \\ &\leq (\gamma^{n-1} + \gamma^{n-2} + \cdots + \gamma^m)Y_0 \\ &\leq \Big(\frac{\gamma^m}{1 - \gamma}\Big)Y_0. \end{aligned} \tag{3.10}$$

By definition of metric d_P, we have $d_P(r, s) \leq 2P(r, s)$, therefore for any $n \geq m$

$$\begin{aligned} d_P(r_n, r_m) + d_P(s_n, s_m) &\leq 2[P(r_n, r_m) + P(s_n, s_m)] = 2Y_{nm} \\ &\leq \Big(\frac{2\gamma^m}{1 - \gamma}\Big)Y_0, \end{aligned} \tag{3.11}$$

which implies that $\{r_n\}$ and $\{s_n\}$ are Cauchy sequences in $(\mathcal{M}, d_P)$ since $\gamma < 1$. Since the partial metric space $(\mathcal{M}, P)$ is complete, by Lemma 2.1 (a_2), the metric space $(\mathcal{M}, d_P)$ is also complete, so there exist $u, v \in \mathcal{M}$ such that

$$\lim_{n\to\infty} d_P(r_n, u) = \lim_{n\to\infty} d_P(s_n, v) = 0. \tag{3.12}$$

From Lemma 2.1 (a_2), we obtain

$$P(u, u) = \lim_{n\to\infty} P(r_n, u) = \lim_{n\to\infty} P(r_n, r_n), \tag{3.13}$$

and

$$P(v, v) = \lim_{n\to\infty} P(s_n, v) = \lim_{n\to\infty} P(s_n, s_n). \tag{3.14}$$

But, from condition (2) of PMS and equation (3.7), we have

$$P(r_n, r_n) \le P(r_n, r_{n+1}) \le Y_n \le \gamma^n Y_0, \tag{3.15}$$

and since $\gamma < 1$, hence letting $n \to \infty$, we get $\lim_{n\to\infty} P(r_n, r_n) = 0$. It follows that

$$P(u, u) = \lim_{n\to\infty} P(r_n, u) = \lim_{n\to\infty} P(r_n, r_n) = 0. \tag{3.16}$$

Similarly, we obtain

$$P(v, v) = \lim_{n\to\infty} P(s_n, v) = \lim_{n\to\infty} P(s_n, s_n) = 0. \tag{3.17}$$

Now, we show that $u = V(u, v)$ and $v = V(v, u)$. We shall distinguish the following cases.

Case (1): We now show that if the assumption (2) holds, then (u, v) is a coupled fixed point of V. As, we have

$$u = \lim_{n\to\infty} r_{n+1} = \lim_{n\to\infty} V(r_n, s_n) = V(\lim_{n\to\infty} r_n, \lim_{n\to\infty} s_n) = V(u, v),$$

and

$$v = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} V(s_n, r_n) = V(\lim_{n\to\infty} s_n, \lim_{n\to\infty} r_n) = V(v, u).$$

Thus, (u, v) is a coupled fixed point of V.

Case (2): Suppose now that the conditions $(3)(a')$ and $(3)(a'')$ of the theorem hold.
Since $r_n \to u$ and $s_n \to v$ as $n \to \infty$, then we have

$$\begin{aligned} P(V(u, v), u) &\le P(V(u, v), r_{n+1}) + P(r_{n+1}, u) - P(r_{n+1}, r_{n+1}) \\ &\le P(V(u, v), r_{n+1}) + P(r_{n+1}, u) \\ &= P(V(u, v), V(r_n, s_n)) + P(r_{n+1}, u) \\ &\le \psi\big(M_V(u, v, r_n, s_n)\big) + P(r_{n+1}, u), \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} M_V(u, v, r_n, s_n) &= a_1\, P(u, r_n) + a_2\, P(v, s_n) + a_3\, P(u, V(u, v)) \\ &\quad + a_4\, P(r_n, V(r_n, s_n)) + a_5\, P(r_n, V(u, v)) \\ &= a_1\, P(u, r_n) + a_2\, P(v, s_n) + a_3\, P(V(u, v), u) \\ &\quad + a_4\, P(r_n, r_{n+1}) + a_5\, P(V(u, v), r_n). \end{aligned} \tag{3.19}$$

Letting $n \to \infty$ in equation (3.19) and using equations (3.16)-(3.17), we obtain

$$\lim_{n\to\infty} M_V(u, v, r_n, s_n) = (a_3 + a_5) P(V(u, v), u). \tag{3.20}$$

Letting $n \to \infty$ in equation (3.18) and using equations (3.16), (3.20) and the property of ψ, we obtain

$$\begin{aligned} P(V(u,v),u) &\leq \psi\big((a_3+a_5)P(V(u,v),u)\big) \\ &< (a_3+a_5)P(V(u,v),u), \end{aligned}$$

which is a contradiction, since $a_3 + a_5 < 1$. Thus, $P(V(u,v),u) = 0$ and so by Lemma 2.2 (b_1), we get $V(u,v) = u$. Similarly, we can show that $V(v,u) = v$. This completes the proof. □

If we take $\psi(t) = kt$ for all $t > 0$ where $k \in (0,1)$, $ka_1 \to q_1$, $ka_2 \to q_2$, $ka_3 \to q_3$, $ka_4 \to q_4$ and $ka_5 \to q_5$, $q_i \in (0,1)$ for $i = 1,2,\ldots,5$ in Theorem 3.1, then we have the following result.

Corollary 3.1 *Let $(\mathcal{M}, P, \leq)$ be a partially ordered complete partial metric space. Suppose that the mapping $V : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ satisfies the following conditions:*

(1)

$$\begin{aligned} P(V(r,s),V(t,z)) \quad \leq \quad & q_1\,P(r,t) + q_2\,P(s,z) + q_3\,P(r,V(r,s)) \\ & + q_4\,P(t,V(t,z)) + q_5\,P(t,V(r,s)), \end{aligned} \tag{3.21}$$

for all $r,s,t,z \in \mathcal{M}$, where $q_1, q_2, \ldots, q_5$ are nonnegative reals such that $q_1 + q_2 + \cdots + q_5 < 1$,

(2) *either V is continuous or*

(3) *$\mathcal{M}$ has the following properties*

(a') *if a non-decreasing sequence $\{r_n\}$ in $\mathcal{M}$ converges to some point $r \in \mathcal{M}$, then $r_n \leq r$ for all n,*

(a'') *if a non-increasing sequence $\{s_n\}$ in $\mathcal{M}$ converges to some point $s \in \mathcal{M}$, then $s \leq s_n$ for all n.*

If there exist two elements $r_0, s_0 \in \mathcal{M}$ with $r_0 \leq V(r_0,s_0)$ and $s_0 \geq V(s_0,r_0)$, then V has a coupled fixed point in $\mathcal{M}$.

If we take $q_1 = k, q_2 = l$ and $q_3 = q_4 = q_5 = 0$ where $k, l \in (0,1)$ in Corollary 3.1, then we have the following result.

Corollary 3.2 *Let $(\mathcal{M}, P, \leq)$ be a partially ordered complete partial metric space. Suppose that the mapping $V : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ satisfies the following conditions:*

(1)

$$P(V(r,s),V(t,z)) \leq k\,P(r,t) + l\,P(s,z), \tag{3.22}$$

for all $r,s,t,z \in \mathcal{M}$, where k, l are nonnegative reals such that $k + l < 1$,

(2) *either V is continuous or*

(3) *$\mathcal{M}$ has the following properties*

(a') *if a non-decreasing sequence $\{r_n\}$ in $\mathcal{M}$ converges to some point $r \in \mathcal{M}$, then $r_n \leq r$ for all n,*

(a'') *if a non-increasing sequence $\{s_n\}$ in $\mathcal{M}$ converges to some point $s \in \mathcal{M}$, then $s \leq s_n$ for all n.*

If there exist two elements $r_0, s_0 \in \mathcal{M}$ with $r_0 \leq V(r_0,s_0)$ and $s_0 \geq V(s_0,r_0)$, then V has a coupled fixed point in $\mathcal{M}$.

If we take $k = l = n$ where $n \in (0,1)$ in Corollary 3.2, then we have the following result.

Corollary 3.3 *Let $(\mathcal{M}, P, \leq)$ be a partially ordered complete partial metric space. Suppose that the mapping $V : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ satisfies the following conditions:*

(1)

$$P(V(r,s),V(t,z)) \leq \frac{n}{2}\big[P(r,t) + P(s,z)\big], \tag{3.23}$$

for all $r,s,t,z \in \mathcal{M}$, where $n \in (0,1)$ is a constant,

(2) *either V is continuous or*

(3) *$\mathcal{M}$ has the following properties*

(a') *if a non-decreasing sequence $\{r_n\}$ in $\mathcal{M}$ converges to some point $r \in \mathcal{M}$, then $r_n \leq r$ for all n,*

(a'') *if a non-increasing sequence $\{s_n\}$ in $\mathcal{M}$ converges to some point $s \in \mathcal{M}$, then $s \leq s_n$ for all n.*

If there exist two elements $r_0, s_0 \in \mathcal{M}$ with $r_0 \leq V(r_0,s_0)$ and $s_0 \geq V(s_0,r_0)$, then V has a coupled fixed point in $\mathcal{M}$.

Remark 3.1 *Corollary 3.3 extends and generalizes Theorem 2.1 and Theorem 2.2 of [12] from partially ordered complete metric spaces to partially ordered complete partial metric spaces.*

Notice that if $(\mathcal{M}, \leq)$ is a partially ordered set, we endow the product space $\mathcal{M} \times \mathcal{M}$ with the partial order relation given by

$$(p, q) \leq (r, s) \quad \Leftrightarrow \quad r \geq p \text{ and } s \leq q.$$

We say that two pairs (x, y) and (u, v) are comparable, that is, every pair of elements has either a lower bound or an upper bound.

Now, we prove the uniqueness of a coupled fixed point in the setting of partially ordered complete partial metric spaces. Moreover, we study appropriate conditions to ensure that for a coupled fixed point (x, y) we have $x = y$.

Theorem 3.2 *In addition to the hypotheses of Theorem 3.1, suppose that, for every $(f, g), (j, k) \in \mathcal{M} \times \mathcal{M}$, there exists a pair $(p, q) \in \mathcal{M} \times \mathcal{M}$ such that (p, q) is comparable to (f, g) and (j, k). Then V has a unique coupled fixed point. Moreover $p(z, z) = 0$.*

Proof: Suppose that (u, v) and (r, s) are coupled fixed point of V, that is, $u = V(u, v)$, $v = V(v, u)$, $r = V(r, s)$ and $s = V(s, r)$.

Let (n, m) be an element of $\mathcal{M} \times \mathcal{M}$ comparable to both (u, v) and (r, s). Suppose that $(r, s) \geq (n, m)$ (the proof is similar in other cases). We consider the following two cases.

Case (A). If (u, v) and (r, s) are comparable, then we have

$$P(u, r) = P\big(V(u, v), V(r, s)\big) \leq \psi\Big(M_V(u, v, r, s)\Big), \tag{3.24}$$

where

$$\begin{aligned} M_V(u, v, r, s) &= a_1 P(u, r) + a_2 P(v, s) + a_3 P(u, V(u, v)) \\ &\quad + a_4 P(r, V(r, s)) + a_5 P(r, V(u, v)) \\ &= a_1 P(u, r) + a_2 P(v, s) + a_3 P(u, u) \\ &\quad + a_4 P(r, r) + a_5 P(r, u) \\ &= (a_1 + a_5) P(u, r) + a_2 P(v, s). \end{aligned}$$

Putting in equation (3.24) and using the property of ψ, we obtain

$$P(u, r) < (a_1 + a_5) P(u, r) + a_2 P(v, s). \tag{3.25}$$

Similarly, we have

$$P(v, s) < (a_1 + a_5) P(v, s) + a_2 P(u, r). \tag{3.26}$$

From equations (3.25) and (3.26), we obtain

$$P(u, r) + P(v, s) < (a_1 + a_2 + a_5)[P(u, r) + P(v, s)],$$

which is a contradiction, since $a_1 + a_2 + a_5 < 1$. Hence, $P(u, r) + P(v, s) = 0$, that is, $P(u, r) = 0$ and $P(v, s) = 0$ and so $u = r$ and $v = s$. This shows that the coupled fixed point of V is unique.

Case (B). Suppose now that (u, v) and (r, s) are not comparable, then there exists an element $(p, q) \in \mathcal{M} \times \mathcal{M}$ is comparable to both (u, v) and (r, s). Now, since by iteration $V^n(r, s) = r$, $V^n(s, r) = s$,

$V^n(u,v)=u$, $V^n(v,u)=v$, $V^n(p,q)=p$ and $V^n(q,p)=q$, we have

$$\begin{aligned} P\Big(\begin{pmatrix}u\\v\end{pmatrix},\begin{pmatrix}r\\s\end{pmatrix}\Big) &= P\Big(\begin{pmatrix}V^n(u,v)\\V^n(v,u)\end{pmatrix},\begin{pmatrix}V^n(r,s)\\V^n(s,r)\end{pmatrix}\Big)\\ &\le P\Big(\begin{pmatrix}V^n(u,v)\\V^n(v,u)\end{pmatrix},\begin{pmatrix}V^n(p,q)\\V^n(q,p)\end{pmatrix}\Big)\\ &\quad +P\Big(\begin{pmatrix}V^n(p,q)\\V^n(q,p)\end{pmatrix},\begin{pmatrix}V^n(r,s)\\V^n(s,r)\end{pmatrix}\Big)\\ &\le \psi\Big(M_V(u,v,p,q)\Big)+\psi\Big(M_V(v,u,q,p)\Big)\\ &\quad +\psi\Big(M_V(p,q,r,s)\Big)+\psi\Big(M_V(q,p,s,r)\Big). \end{aligned}$$

where

$$\begin{aligned} M_V(u,v,p,q) &= a_1\,P(u,p)+a_2\,P(v,q)+a_3\,P(u,V(u,v))\\ &\quad +a_4\,P(p,V(p,q))+a_5\,P(p,V(u,v))\\ &= a_1\,P(u,p)+a_2\,P(v,q)+a_3\,P(u,u)\\ &\quad +a_4\,P(p,p)+a_5\,P(p,u)=0. \end{aligned}$$

Similarly,

$$M_V(v,u,q,p)=0,\ M_V(p,q,r,s)=0 \text{ and } M_V(q,p,s,r)=0.$$

Using this in the above inequality and the property of ψ, we obtain

$$p\Big(\begin{pmatrix}u\\v\end{pmatrix},\begin{pmatrix}r\\s\end{pmatrix}\Big)=0.$$

Thus, $u=r$ and $v=s$. Hence, the coupled fixed point of V is unique. This completes the proof. □

Theorem 3.3 *In addition to the hypotheses of Theorem 3.1, suppose that r_0, s_0 in $\mathcal{M}$ are comparable, then the coupled fixed point $(r,s)\in\mathcal{M}\times\mathcal{M}$ satisfies $r=s$. Moreover $p(z,z)=0$.*

Proof: Recall that $r_0\in\mathcal{M}$ is such that $r_0\le V(r_0,s_0)$. Now, if $r_0\le s_0$, we claim that for all $n\in\mathbb{N}$, $r_n\le s_n$. Indeed, by the mixed monotone property of V,

$$r_1=V(r_0,s_0)\le V(s_0,r_0)=s_1.$$

Assume that $r_n\le s_n$ for some n. Now, consider

$$\begin{aligned} r_{n+1} &= V^{n+1}(r_0,s_0)=V\big(V^n(r_0,s_0),V^n(s_0,r_0)\big)\\ &= V(r_n,s_n)\le V(s_n,r_n)=s_{n+1}. \end{aligned}$$

Hence, $r_n\le s_n$ for all n. Taking the limit as $n\to\infty$, we get

$$r=\lim_{n\to\infty} r_n\le \lim_{n\to\infty} s_n=s.$$

From the contractive condition (3.1), we get

$$\begin{aligned} P(r,s) &= P(V(r,s),V(s,r))\\ &\le \psi\big(M_V(r,s,s,r)\big), \end{aligned}$$

where

$$\begin{aligned} M_V(r,s,s,r) &= a_1\,P(r,s)+a_2\,P(s,r)+a_3\,P(r,V(r,s))\\ &\quad +a_4\,P(s,V(s,r))+a_5\,P(s,V(r,s))\\ &= a_1\,P(r,s)+a_2\,P(s,r)+a_3\,P(r,r)\\ &\quad +a_4\,P(s,s)+a_5\,P(s,r)\\ &= (a_1+a_2+a_5)P(r,s). \end{aligned}$$

Using this in the above inequality and the property of ψ, we get

$$P(r,s) < (a_1 + a_2 + a_5)P(r,s).$$

Similarly, we obtain

$$P(s,r) < (a_1 + a_2 + a_5)P(s,r).$$

From equations (3.27) and (3.27), we obtain

$$P(r,s) + P(s,r) < (a_1 + a_2 + a_5)[P(r,s) + P(s,r)],$$

which is a contradiction, since $a_1 + a_2 + a_5 < 1$. Hence, $P(r,s) + P(s,r) = 0$ and so $P(r,s) = 0 = P(s,r)$. Therefore, by Lemma 2.2 (b_1), we get $r = s$.

Similarly, if $r_0 \geq s_0$, then it is possible to show $r_n \geq s_n$ for all n and that $P(r,s) = 0$. This completes the proof. □

Remark 3.2 *Theorem 3.2 and Theorem 3.3 extend and generalize Theorem 2.4 and Theorem 2.6 of [12] from partially ordered complete metric spaces to partially ordered complete partial metric spaces.*

Example 3.1 *Let $\mathcal{M} = [0,1]$. Then $(\mathcal{M}, \leq)$ is a partially ordered set with a natural ordering of real numbers. Let $P\colon \mathcal{M} \times \mathcal{M} \to [0,1]$ be defined by $P(r,s) = |r - s|$ for all $r, s \in \mathcal{M}$. Consider the mapping $V\colon \mathcal{M} \times \mathcal{M} \to [0,1]$ defined by*

$$V(r,s) = \begin{cases} \frac{r^2 - s^2 + 1}{3}, & \text{if } r \leq s, \\ \frac{1}{3}, & \text{if } r > s, \end{cases}$$

for all $r, s \in \mathcal{M}$. Then

(1) *$(\mathcal{M}, P)$ is a complete partial metric space since $(\mathcal{M}, d_P)$ is complete;*
(2) *V has the mixed monotone property;*
(3) *V is continuous;*
(4) *$0 \leq V(0,1)$ and $1 \geq V(1,0)$;*
(5) *there exists a constant $0 < n < 1$ such that*

$$P\big(V(r,s), V(t,z)\big) \leq \frac{n}{2}\,[P(r,t) + P(s,z)],$$

for all $r, s, t, z \in \mathcal{M}$ with $r \leq t$ and $s \geq z$. Thus, by Corollary 3.3, V has a coupled fixed point. Moreover, $(\frac{1}{3}, \frac{1}{3})$ is the unique coupled fixed point of V.

Proof: The proofs of $(1) - (4)$ are obvious.

For any $r \leq t$ and $s \geq z$, we have

$$P(r,t) = t - r, \quad P(s,z) = s - z.$$

The proof of (5) is divided into the following cases.

Case $(1')$. If $t \leq z$. In this case, $r \leq t \leq z \leq s$, and so

$$V(r,s) = \frac{r^2 - s^2 + 1}{3}, \quad V(t,z) = \frac{t^2 - z^2 + 1}{3}.$$

Hence, we get

$$\begin{aligned} P\big(V(r,s), V(t,z)\big) &= P\Big(\frac{r^2 - s^2 + 1}{3}, \frac{t^2 - z^2 + 1}{3}\Big) \\ &= \frac{1}{3}(t^2 - z^2 - r^2 + s^2) = \frac{1}{3}\,[(t^2 - r^2) + (s^2 - z^2)] \\ &\leq \frac{1}{3}\,[(t - r) + (s - z)] = \frac{1}{3}\,[P(r,t) + P(s,z)] \\ &= \frac{n}{2}\,[P(r,t) + P(s,z)], \end{aligned}$$

with $n = \frac{2}{3} < 1$.

Case (2′). If $t > z$. In this case, $r \leq t \leq s$, and so

$$V(r,s) = \frac{r^2 - s^2 + 1}{3}, \quad V(t,z) = \frac{1}{3}.$$

Hence, we get

$$\begin{aligned} P\big(V(r,s), V(t,z)\big) &= P\Big(\frac{r^2 - s^2 + 1}{3}, \frac{1}{3}\Big) = \frac{1}{3}(s^2 - r^2) \\ &\leq \frac{1}{3}(s^2 - r^2 + t^2 - z^2) = \frac{1}{3}\left[(t^2 - r^2) + (s^2 - z^2)\right] \\ &\leq \frac{1}{3}\left[(t - r) + (s - z)\right] = \frac{1}{3}\left[P(r,t) + P(s,z)\right] \\ &= \frac{n}{2}\left[P(r,t) + P(s,z)\right], \end{aligned}$$

with $n = \frac{2}{3} < 1$.

Case (3′). If $r > s$. In this case, $t \leq z \leq s$, and so

$$V(r,s) = \frac{1}{3}, \quad V(t,z) = \frac{t^2 - z^2 + 1}{3}.$$

Hence, we get

$$\begin{aligned} P\big(V(r,s), V(t,z)\big) &= P\Big(\frac{1}{3}, \frac{t^2 - z^2 + 1}{3}\Big) = \frac{1}{3}(t^2 - z^2) \\ &\leq \frac{1}{3}(t^2 - z^2 + s^2 - r^2) = \frac{1}{3}\left[(t^2 - r^2) + (s^2 - z^2)\right] \\ &\leq \frac{1}{3}\left[(t - r) + (s - z)\right] = \frac{1}{3}\left[P(r,t) + P(s,z)\right] \\ &= \frac{n}{2}\left[P(r,t) + P(s,z)\right], \end{aligned}$$

with $n = \frac{2}{3} < 1$.

Thus, in all the above cases, the condition (5) is satisfied. Since $\mathcal{M} = [0,1]$ is a totally ordered set, by Theorem 3.3, $(\frac{1}{3}, \frac{1}{3})$ is the unique coupled fixed point of V.

□

4. An Application to the Nonlinear Integral Equation

In this section, we study the existence of solution of the nonlinear integral equations, as an application of the coupled fixed point theorem proved in the main results.

Consider the following nonlinear integral equations:

$$\begin{aligned} r(t) &= g(t) + \int_0^T N(t,p)h(p, r(p), s(p))dp, \\ s(t) &= g(t) + \int_0^T N(t,p)h(p, s(p), r(p))dp, \end{aligned} \tag{4.1}$$

where $t \in I = [0,T]$, with $T > 0$.

We consider the space $\mathcal{M} = C(I,\mathbb{R})$ of continuous functions defined in I. Define $P\colon \mathcal{M}\times\mathcal{M} \to [0,+\infty)$ by

$$P(r,s) = \max_{t\in I} |r(t) - s(t)|,$$

for all $r, s \in \mathcal{M}$. Then $(\mathcal{M}, P)$ is a complete partial metric space.

Let $\mathcal{M} = C(I, \mathbb{R})$ with the natural partial order relation, that is, $r, s \in C(I, \mathbb{R})$,

$$r \leq s \;\Leftrightarrow\; r(t) \leq s(t),\; t \in I.$$

We consider the following conditions:

(1) the mapping $h\colon I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ are continuous;

(2) there exists a continuous $0 \leq n < 1$ such that

$$|h(p, r, s) - h(p, w, z) \leq \frac{n}{2}\,(|r - w| + |s - z|), \tag{4.2}$$

for all $r, s, w, z \in \mathcal{M}$ and for all $p \in I$;

(3) for all $t, p \in I$, there exists a continuous $N\colon I \times \mathbb{R} \to \mathbb{R}$ such that

$$\max_{t \in I} \int_0^T N(t, p)dp < 1, \tag{4.3}$$

(4) there exist $r_0, s_0 \in \mathcal{M}$ such that

$$\begin{aligned} r_0(t) &\leq g(t) + \int_0^T N(t, p)h(p, r_0(p), s_0(p))dp, \\ s_0(t) &\leq g(t) + \int_0^T N(t, p)h(p, s_0(p), r_0(p))dp, \end{aligned} \tag{4.4}$$

where $t \in I$.

Theorem 4.1 *Consider the Corollary 3.3 and assume that conditions* (1) - (4) *are satisfied. Then equation (4.1) has a unique solution in* $\mathcal{M} = C(I, \mathbb{R})$.

Proof: Define the mapping $V\colon \mathcal{M}^2 \to \mathcal{M}$, $(r, s) \to V(r, s)$, where

$$V(r, s)(t) = g(t) + \int_0^T N(t, p)h(p, r(p), s(p))dp,\; t \in I, \tag{4.5}$$

for all $r, s \in \mathcal{M}$.

Equation (4.1) can be stated as

$$r = V(r, s) \;\text{ and }\; s = V(s, r). \tag{4.6}$$

For $r, s, w, z \in \mathcal{M}$ be such that $r \leq w$ and $s \leq z$ and

$$\begin{aligned} V(r, s)(t) &= g(t) + \int_0^T N(t, p)h(p, r(p), s(p))dp \\ &\leq g(t) + \int_0^T N(t, p)h(p, w(p), z(p))dp \\ &= V(w, z)(t) \text{ for all } t \in I. \end{aligned} \tag{4.7}$$

From equations (4.2) and (4.3) for all $t \in I$, we have

$$\begin{aligned} P(V(r, s), V(w, z)) &= \max_{t \in I} |V(r, s)(t) - V(w, z)(t)| \\ &\leq \max_{t \in I} \int_0^T N(t, p)\Big|h(p, r(p), s(p)) - h(p, w(p), z(p))\Big|dp \\ &\leq \Big|h(p, r(p), s(p)) - h(p, w(p), z(p))\Big| \\ &\leq \frac{n}{2}(\max_{p \in I} |r(p) - w(p)| + \max_{p \in I} |s(p) - z(p)|) \\ &= \frac{n}{2}[P(r, w) + P(s, z)], \end{aligned}$$

where $0 \leq n < 1$.

So that

$$P(V(r,s),V(w,z)) \leq \frac{n}{2}\,[P(r,w)+P(s,z)].$$

Which is the contractive condition in Corollary 3.3. Thus V has a coupled fixed point in $\mathcal{M}$, that is, the system of nonlinear integral equation has a solution. Finally, let (r,s) be a coupled lower and upper solution of the integral equation (4.1), then by assumption (4) of the Theorem 4.1, we have $r \leq V(r,s) \leq V(s,r) \leq s$. Corollary 3.3 gives us that V has a coupled fixed point, say $(u,v) \in \mathcal{M}\times\mathcal{M}$. Since $r \leq s$, Theorem 3.3 says us that $u = v$ and this implies $u = V(u,u)$ and u is the unique solution of the integral equation (4.1). □

The aforesaid application is illustrated by the following example.

Example 4.1 *Let $\mathcal{M} = C([0,1],\mathbb{R})$, $h\colon I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g\colon I\to\mathbb{R}$. Now consider the following functional integral equation:*

$$r(t) = \frac{t^2}{1+t^4} + \int_0^1 \frac{sin\,p\,3^{-p}e^{-p}}{9(t+3)}\Big(\frac{|r(p)|}{1+|r(p)|}+\frac{|s(p)|}{1+|s(p)|}\Big)dp$$
$$s(t) = \frac{t^2}{1+t^4} + \int_0^1 \frac{sin\,p\,3^{-p}e^{-p}}{9(t+3)}\Big(\frac{|s(p)|}{1+|s(p)|}+\frac{|r(p)|}{1+|r(p)|}\Big)dp,$$

for all $r,s\in\mathcal{M}$ and $t\in I$. Observe that the above equation is a special case of equation (4.1) with

$$\begin{aligned} g(t) &= \frac{t^2}{1+t^4}.\\ N(t,p) &= \frac{3^{-p}e^{-p}}{t+3}.\\ h(p,r,s) &= \frac{sin\,p}{9}\Big(\frac{|r(p)|}{1+|r(p)|}+\frac{|s(p)|}{1+|s(p)|}\Big).\\ h(p,s,r) &= \frac{sin\,p}{9}\Big(\frac{|s(p)|}{1+|s(p)|}+\frac{|r(p)|}{1+|r(p)|}\Big). \end{aligned}$$

It is also easily seen that these functions are continuous.

For arbitrary $r,s,w,z\in\mathcal{M}$ and for all $p\in I$, we have

$$\begin{aligned} |h(p,r,s)-h(p,w,z)| &= \Big|\frac{sin\,p}{9}\Big(\frac{|r(p)|}{1+|r(p)|}+\frac{|s(p)|}{1+|s(p)|}\Big)\\ &\quad -\frac{sin\,p}{9}\Big(\frac{|w(p)|}{1+|w(p)|}+\frac{|z(p)|}{1+|z(p)|}\Big)\Big|\\ &\leq \frac{1}{9}\big(|r-w|+|s-z|\big) = \frac{n}{2}\big(|r-w|+|s-z|\big). \end{aligned}$$

Therefore, the function h satisfies equation (4.2) with $n=\frac{2}{9}<1$.

For all $t,p\in I$, there exists $N\colon I\times\mathbb{R}\to\mathbb{R}$ such that

$$\begin{aligned} \int_0^1 N(t,p)dp &= \int_0^1 \frac{3^{-p}e^{-p}}{t+3}dp\\ &= -\frac{1}{3}\Big(\frac{e^{-1}-3}{(ln\,3+1)(t+3)}\Big)\\ &= \Big(1-\frac{1}{3e}\Big)\frac{1}{(ln\,3+1)(t+3)}\\ &\leq 1-\frac{1}{3e}\leq\frac{9}{10}<1. \end{aligned}$$

We put $r_0(t) = \frac{5t^2}{7(1+t^4)}$, *we obtain*

$$\begin{aligned} r_0(t) &= \frac{5t^2}{7(1+t^4)} \leq \frac{t^2}{1+t^4} \\ &\leq \frac{t^2}{1+t^4} + \int_0^1 \frac{sin\, p}{9}\Big(\frac{|r(p)|}{1+|r(p)|} + \frac{|s(p)|}{1+|s(p)|}\Big)dp \\ &= g(t) + \int_0^T N(t,p)h(p, r_0(p), s_0(p))dp. \end{aligned}$$

Similarly, we have

$$s_0(t) \leq g(t) + \int_0^T N(t,p)h(p, s_0(p), r_0(p))dp.$$

This shows that equation (4.4) holds.

Hence the integral equation (4.1) has a unique solution in $\mathcal{M}$ *with* $\mathcal{M} = C([0,1], \mathbb{R})$.

5. Conclusion

In this article, we establish some coupled fixed point theorems for ψ-contractive condition in the setting of partially ordered partial metric spaces. Moreover, we give some corollaries of the established results and provide an illustrative example in support of the established result. An application to the nonlinear integral equation is also given. The results obtained in this paper extend and generalize several previous works from the existing literature.

References

1. Abbas, M., Khan, M. A. and Radenović, S., *Common coupled fixed point theorems in cone metric spaces for w-compatible mappings*, Appl. Math. Comput., 217(1), 195-202, (2010).
2. Abbas, M., Aydi, H. and Radenović, S., *Fixed point of T-Hardy-Rogers contractive mappings in ordered partial metric spaces*, Int. J. Math. Math. Sci., 2012, Art. ID 313675 (2012).
3. Abbas, M., Nazir, T. and Romaguera, S., *Fixed point results for generalized cyclic contraction mappings in partial metric spaces*, Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat. RACSAM, 106(1), 287-297, (2012).
4. Abdeljawad, T., Karapınar, E. and Tas, K., *Existence and uniqueness of a common fixed point on partial metric spaces*' Appl. Math. Lett., 24(11), 1894-1899, (2011).
5. Abdeljawad, T., *Coupled fixed point theorems for partially contractive type mappings*, Fixed Point Theory Appl., 2012, 148(2012).
6. Altun, I., Sola, F. and Simsek, H., *Generalized contractions on partial metric spaces*, Topology and its Appl., 157, 2778-2785, (2010).
7. Altun, I. and Erduran, A., *Fixed point theorems for monotone mappings on partial metric spaces*, Fixed Point Theory Appl., 2011, Article ID 508730 (2011). doi:10.1155/2011/508730
8. Aydi, H., *Some coupled fixed point results on partial metric spaces*, International J. Math. Math. Sci., 2011, Article ID 647091, 11 pages.
9. Aydi, H., *Fixed point results for weakly contractive mappings in ordered partial metric spaces*, J. Adv, Math. Stud., 4(2), 1-12, (2011).
10. Aydi, H., Abbas, M. and Vetro, C., *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*, Topology and Its Appl., 159, No. 14, 3234-3242, (2012).
11. Banach, S., *Sur les operation dans les ensembles abstraits et leur application aux equation integrals*, Fund. Math., 3, 133-181, (1922).
12. Bhaskar, T. G. and Lakshmikantham, V., *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Analysis: TMA, 65(7), 1379-1393, (2006).
13. Ciric, L. and Lakshmikantham, V., *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Analysis: TMA, 70(12), 4341-4349, (2009).
14. Chandok, S., Kumar, D. and Khan, M. S., *Some results in partial metric space using auxiliary functions*, Applied Math. E-Notes, 15, 233-242, (2015).
15. Erhan, I., Karapinar, E. and Turkoglu, D., *Different types Meir-Keeler contractions on partial metric spaces*, J. Comput. Anal. Appl., 14(6), 1000-1005, (2012).

16. Guo, D. and Lakshmikantham, V., *Coupled fixed point of nonlinear operator with application*, Nonlinear Anal. TMA., 11, 623-632, (1987).

17. Heckmann, R., *Approximation of metric spaces by partial metric spaces*, Appl. Categ. Structures, 7, No. 1-2, 71-83, (1999).

18. Jain, R., Saluja, G. S., Nashine, H. K. and Agarwal, R. P., *Coupled and common coupled fixed point theorems for contractive type conditions in partial metric spaces*, Advances Math. Sci. Appl., 32(1), 73-96, (2023).

19. Karapınar, E., *A note on common fixed point theorems in partial metric spaces*, Miskolc Math. Notes, 12(2), 185-191, (2011).

20. Karapınar, E. and Erhan, I. M, *Fixed point theorems for operators on partial metric spaces*, Appl. Math. Lett., 24(11), 1900-1904, (2011).

21. Karapinar, E., Shatanawi, W. and Tas, K., *Fixed point theorems on partial metric spaces involving rational expressions*, Miskolc Math. Notes, 14, 135-142, (2013).

22. Künzi, H. P. A., Pajoohesh, H. and Schellekens, M. P., *Partial quasi-metrics*, Theoritical Comput. Sci., 365(3), 237-246, (2006).

23. Matthews, S. G., *Partial metric topology*, Research report 2012, Dept. Computer Science, University of Warwick, 1992.

24. Matthews, S. G., *Partial metric topology*, Proceedings of the 8th summer conference on topology and its applications, Annals of the New York Academy of Sciences, 728, 183-197, (1994).

25. Nashine, H. K. and Shatanawi, W., *Coupled common fixed point theorems for pair of commuting in partially ordered complete metric spaces*, Comput. Math. Appl., 62, 1984-1993, (2011).

26. Nashine, H. K. and Kadelburg, Z., *Partially ordered metric spaces, rational contractive expressions and couple fixed points*, Nonlinear Functional Anal. Appl., 17(4), 471-489, (2012).

27. Nashine, H. K., *Coupled common fixed point results in ordered G-metric spaces*, J. Nonlinear Sci. Appl., 1, 1-13, (2012).

28. Nashine, H. K., Saluja, G. S., Rao, G. V. V. J. and Lim, W. H., *Control function based coupled and common coupled fixed point theorems in partial metric spaces*, Nonlinear Functional Anal. Appl., 29(2), 559-580, (2024).

29. Oltra, S. and Valero, O., *Banach's fixed point theorem for partial metric spaces*, Rend. Ist. Mat. Univ. Trieste, 36(1-2), 17-26, (2004).

30. Olaleru, J. O., Okeke, G. A. and Akewe, H., *Coupled fixed point theorems for generalized φ-mappings satisfying contractive condition of integral type on cone metric spaces*, Int. J. Math. Model. Comput., 2(2), 87-98, (2012).

31. O'Neil, S. J., *Partial metrics, valuations and domain theory*, in: Proc. 11th summer conference on general topology and application, in Annals of the New York Academy of Sciences, vol. 806, 304-315, (1996).

32. Radenoví́ć, S., *Coupled fixed point theorems for monotone mappings in partially ordered metric spaces*, Kragujevac J. Maths., 38(2), 249-257, (2014).

33. Rao, K. P. R., Kishore, G. N. V. and Luong, N. V., *A unique common coupled fixed point theorem for four maps under $\psi - \varphi$-contractive condition in partial metric spaces*, CUBO, 14(3), 115-127, (2012).

34. Rao, K. P. R., Kishore, G. N. V. and Raju, V. C. C., *A coupled fixed point theorem for two pairs of w-compatible maps using altering distance function in partial metric space*, J. Adv. Res. Pure Math., 4(4), 96-114, (2012).

35. Romaguera, S. and Schellekens, M. P., *Duality and quasi-normability for complexity spaces*, Appl. General Topology, 3, 91-112, (2002).

36. Samet, B. and Yazidi, H., *Coupled fixed point theorems for contraction involving rational expressions in partially ordered metric spaces*, priprint, arXiv:1005.3142v1 [math CN].

37. Samet, B., Rajović, M., Lazović, R. and Stoiljković, R., *Common fixed point results for nonlinear contractions in ordered partial metric spaces*, Fixed Point Theory Appl., (2011), 2011:71.

38. Schellekens, M. P., *The correspondence between partial metrics and semivaluations*, Theoritical Comput. Sci., 315(1), 135-149, (2004).

39. Schellekens, M. P., *Characterization of partial metrizability: domains are quantifiable*, Theoritical Comput. Sci., 305(1-3), 409-432, (2003).

40. Shatanawi, W., Samet, B. and Abbas, M., *Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces*, Math. Comput. Modell., 55, Nos. 3-4, 680-687, (2012).

41. Valero, O., *On Banach fixed point theorems for partial metric spaces*, Appl. Gen. Topl., 6(2), 229-240, (2005).

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