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Non-Convex Optimal Control for Wastewater Treatment Problem: Hamilton-Jacobi-Bellman Approach

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ABSTRACT: This paper deals with a non-convex optimal control problem, that models a wastewater treatment process, which aims at the degradation of pollutants by bacteria using dissolved oxygen. The object is to build an optimal control strategy, involving dilution rate, recycle rate, and aeration rate as control variables, to minimize the final value of the substrate biomass, running dissolved oxygen concentration and bacteria biomass, together with the cost of recycling and aeration. As results, we investigate the invariance and dissipation properties and determine the existence and uniqueness of a solution for the controlled dynamical system. Furthermore, because the dynamics are non-convex with respect to the controls, we use the Hamilton-Jacobi-Bellman equation and its viscosity solutions to show that the value function is the unique viscosity solution to this equation, which provides insight into the existence of an optimal control strategy achieving our goal. We finally, present some numerical simulations to support the theoretical outcomes.

Key Words: Wastewater treatment, invariance, optimal control, Hamilton-Jacobi-Bellmen equation, viscosity solution.

Contents

1. Introduction

The treatment method, known as activated sludge, uses microorganisms to break down pollutants in wastewater treatment which consist of two interconnected tanks: an aeration tank and a settling tank. Wastewater enters the aeration tank with a flow rate Q_{in} and an influent concentration S_{in}. In this tank, microorganisms metabolize the organic matter in the wastewater, transforming it into carbon dioxide, water, and new microbial biomass. The biological oxidation process of the pollutant substrate S, made by a single bacterial population x_b under aerobic conditions, occurs in the aeration tank with the consumption of the dissolved oxygen D_o. A portion of the settled sludge, containing bacteria x_r, is recirculated to the aeration tank to sustain the microbial population. a detailed descriptions of this process and its operational principles can be found in the literature, for instance, in [1,4,17,18,22,23].

The dominant objective here is to find optimal control strategy, using control variables composed of the dilution rare D, recycle rate r and the aeration rate W, that minimize the consumption of dissolved oxygen $D_o(t)$, the density of bacteria and the output of substrate together with the cost of aeration and recycle process, by minimizing the recycled rate r and the aeration rate W. We achieve this goal by first, analyzing invariance and dissipation in order to provide a physically relevant interpretation. Secondly, we

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use the viscosity solution for the Hamilton–Jacobi–Bellman (HJB) equation which allows to determine an optimal control.

Several works was interested by optimal control of wastewater treatment [3,7,9,11]. But in these works, either the Pontryagin maximum principle is evoked as necessaries conditions [14], or for sufficient conditions, the problem is formulated to exhibit convexity properties, in order to provide the existence of optimal solution directly [6,21]. However, the convexity in controlled dynamical system fails, generally, to occur, particularly, when multiple variables of control are involved. So, in such cases, the use of the the Hamilton–Jacobi–Bellman (HJB) equation becomes essential.

The HJB equations arise naturally in optimal control theory and dynamic programming [2,6,15,19,20, 21]. These equations are typically nonlinear and may lack smooth solutions, particularly when the value function exhibits discontinuities or when control constraints are required [5]. Classical solutions require the value function to be continuously differentiable, a condition that is often not satisfied in real-world problems due to inherent nonsmoothness [6]. The main motivation of this work stems from the non-convexity of the system, which is precisely due to the product $r \cdot D$ of two control variables: the recycle rate r and the dilution rate D. This structural nonlinearity prevents the direct application of standard existence results in optimal control theory [15,21], thereby motivating the adoption, in this work, the framework of viscosity solutions for HJB equations [2,5,15,19,20]. This approach is well-suited to handle the nonsmooth and nonlinear nature of such equations, enabling the analysis and characterization of optimal controls even when classical differentiability fails.

The paper is organized as follows: the first part is devoted to present the mathematical model and prove the invariance and dissipation to provide a physically meaningful interpretation of this system. The second part introduce viscosity solutions of the HJB equation, we present the value function and show that this value function is the unique continuous viscosity solution of the HJB equation, which allows us to conclude that the system admits an optimal control using the bellman principle. Finally, in the third part, we present numerical simulations to illustrate the system's behavior under the derived optimal control strategy. These results illustrate the theoretical findings and demonstrate the effectiveness of the control policy in meeting operational and environmental objectives.

2. Mathematical Modeling

2.1. Description of the model

The mathematical model discussed in this study, was originally proposed by Nejjari F [12] and Serhani [16], see also [8,22,23]. It models the biological degradation of pollutant by a single bacteria in an aerobic setting as described in the introduction, the mathematical formalism is presented as:

$$\begin{cases} \dot{S}(t) = -\dfrac{\mu(t)}{Y_f} x_b(t) - D(t)(1+r(t))S(t) + D(t)S_{in} \\ \dot{x}_b(t) = \mu(t)x_b(t) - D(t)(1+r(t))x_b(t) + r(t)D(t)x_r(t) \\ \dot{x}_r(t) = D(t)(1+r(t))x_b(t) - D(t)(\beta + r(t))x_r(t) \\ \dot{D}_o(t) = -K_0 \dfrac{\mu(t)}{Y_f} x_b(t) - D(t)(1+r(t))D_o(t) + D(t)D_{o_{in}} + \alpha W(t)[D_{o_{max}} - D_o(t)] \end{cases} \tag{2.1}$$

where x_b, S, x_r and D_o are the state variables representing densities of bacteria population, substrate, recycled bacteria, and dissolved oxygen, respectively. While $D(t)$ denotes the dilution rate, $r(t)$ stands for The recycle flow describing the rate of bacteria biomass recycled from the Settler to the Aerator and β denotes the rate of both waste and recycled bacteria that is discharged out of the station. S_{in} and $D_{o_{in}}$ correspond to the substrate and dissolved oxygen concentrations in the feed stream, respectively. The production kinetics of the cell mass are characterized by the specific growth rate $\mu(t)$ and Y_f the Yield factor. K_0 is a model constant, $D_{o_{max}}$ is a maximum amount of dissolved oxygen, α and W denotes the oxygen transfer and aeration rates, respectively.

The specific growth rate μ is a crucial parameter for characterizing biomass growth. It is influenced by a complex interplay of various physicochemical and biological factors, including biomass concentration, substrate concentration, dissolved oxygen concentration, pH, temperature, and the presence of inhibitors.

Various analytical expressions have been proposed to model this parameter, with the Monod law being the most widely recognized. In this work, we consider the specific growth rate to be a function of both substrate concentration and dissolved oxygen levels, along with various kinetic parameters. The kinetic formulation adopted is based on the framework outlined in the Olsson model [10].

$$\mu(S(t), D_o(t)) = \mu_{max} \frac{S(t)}{K_s + S(t)} \frac{D_o(t)}{K_{D_o} + D_o(t)}. \tag{2.2}$$

We denote the specific growth rate as $\mu(t) := \mu(S(t), D_o(t))$, where μ_{max} represents the maximum specific growth rate, K_s is the affinity constant describing how the degradation rate varies with the pollutant concentration S, and K_c denotes the saturation constant.

The model (2.1) can be rewritten as a general controlled dynamical system

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) & a.e.\ t \in [0, T] \\ x(0) = x_0; \end{cases} \tag{2.3}$$

where $x(\cdot) \in W^1([0,T], \mathbb{R}^4)$ the space of absolutely continuous functions defined on $[0,T]$ with values in $\mathbb{R}^4$. In addition by considering the admissible control vectors set defined by $U = [\underline{D}, \overline{D}] \times [\underline{r}, \overline{r}] \times [\underline{W}, \overline{W}]$, we have $u(\cdot) \in \mathcal{U}$ such that $\mathcal{U} = \{u : [0,\ T] \to U \text{ is measurable}\}$, and $f : \mathbb{R}^4 \times U \longrightarrow \mathbb{R}^4$

The state and control vectors are given by:

$$x = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T = \begin{pmatrix} S & x_b & x_r & D_o \end{pmatrix}^T \ ; \ u = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}^T = \begin{pmatrix} D & r & W \end{pmatrix}^T.$$

The system dynamics $f(x, u)$ are defined componentwise as follows:

$$f(x,u) = \begin{pmatrix} f_1(x,u) & f_2(x,u) & f_3(x,u) & f_4(x,u) \end{pmatrix}^T,$$

where

$$\begin{aligned} f_1(x,u) &= -\frac{\mu(t)}{Y_f} x_2 - D(t)(1+r)x_1 + D(t)S_{in}, \\ f_2(x,u) &= \mu(t)x_2 - D(t)(1+r)x_2 + rD(t)x_3, \\ f_3(x,u) &= D(t)(1+r)x_2 - D(t)(\beta+r)x_3, \\ f_4(x,u) &= -K_0 \frac{\mu(t)}{Y_f} x_2 - D(t)(1+r)x_4 + D(t)D_{o_{in}} + \alpha W(t)[D_{o_{max}} - x_4]. \end{aligned}$$

2.2. Invariance and dissipation

To provide a physically meaningful interpretation of this system, it is natural to consider whether the trajectories are positively viable; in other words, the question is whether a solution starting in $\mathbb{R}^4_+$ remains in that region for all future times. See [4].

Proposition 2.1 *Under the dynamics of* (2.3)*, the region* $\mathbb{R}^4_+$ *remains invariant.*

Proof: For the first component of dynamical system (2.1), we have

$$\begin{aligned} \dot{S}(t) &= -\frac{\mu(t)}{Y_f} x_b(t) - D(t)(1+r(t))S(t) + D(t)S_{in}, \\ &\geq \left(-\frac{1}{Y_f} \mu_{max} \frac{1}{K_s + S(t)} \frac{D_o(t)}{K_{D_o} + D_o(t)} x_b(t) - D(t)(1+r(t)) \right) S(t). \end{aligned}$$

Using the arguments of differential inequalities, we get

$$S(t) \geq S(0) exp \left(\int_0^t -\frac{1}{Y_f} \mu_{max} \frac{1}{K_s + S(s)} \frac{D_o(s)}{K_{D_o} + D_o(s)} x_b(s) - D(s)(1+r(s)) ds \right) t.$$

Hence, if $S(0) \geq 0$, then $S(t) \geq 0$, for all $t \geq 0$.
Similarly, from the second component of dynamical system (2.1), we get

$$\dot{D}_o(t) \geq \left(-\frac{K_0}{Y_f}\mu_{max}\frac{S(t)}{K_s+S(t)}\frac{1}{K_{D_o}+D_o(t)}x_b(t) - D(t)(1+r(t)) - \alpha W(t)\right)D_o(t).$$

Then

$$D_o(t) \geq D_o(0)exp\left(\int_0^t -\frac{K_0}{Y_f}\mu_{max}\frac{S(s)}{K_s+S(s)}\frac{1}{K_{D_o}+D_o(s)}x_b(s) - D(s)(1+r(s)) - \alpha W(s)ds\right)t$$

which implies that $D_o(t) \geq 0$, for $t \geq 0$, if $D_o(0) \geq 0$.

On the other hand, for x_b and x_r, it turns out to prove that the trajectory can't violate the boundaries of $\mathbb{R}_+^4$. Indeed, for $x_r = 0$, if $x_b > 0$, the $\dot{x_r} = D(t)(1+r(t))x_b(t) > 0$ and hence the vector field is pointed towards the interior of $\mathbb{R}_+^4$. Now, if $x_b = 0$ and $x_r > 0$, then $\dot{x_b} = r(t)D(t)x_r(t) > 0$, similarly the vector field is pointed towards the interior of $\mathbb{R}_+^4$.
We conclude that trajectories starting at $\mathbb{R}_+^4$ remains in it thereafter. □

The second property that provides a physically meaningful interpretation of the system relates to the boundedness of its trajectories.

Proposition 2.2 *All trajectories of* (2.3) *starting in*

$$\Omega = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}_+^4 / Y_f x_1 + \left(\frac{\beta}{2\overline{r}} + 1\right)\ x_2 + x_3 + \frac{Y_f\beta}{2K_0\overline{r}}x_4 \leq \frac{\gamma}{M}\},$$

remain in Ω *thereafter.*
With

$$\gamma = Y_f\overline{D}S_{in} + \frac{Y_f\beta}{2K_0\overline{r}}\left[\overline{D}D_{o_{in}} + \alpha\overline{W}D_{o_{max}}\right],$$

and

$$0 < M < \min\left(Y_f\underline{D}(1+\underline{r}), \underline{D}(1+\underline{r})\frac{\beta}{2\overline{r}}, \underline{D}\beta, \frac{Y_f\beta}{2K_0\overline{r}}\underline{D}(1+\underline{r}) + \alpha\underline{W}\right).$$

Proof: we put

$$Z = Y_f S + \left(\frac{\beta}{2\overline{r}} + 1\right)\ x_b + x_r + \frac{Y_f\beta}{2K_0\overline{r}}D_o,$$

then

$$\begin{aligned}
\dot{Z} &= Y_f\dot{S} + \left(\frac{\beta}{2\overline{r}} + 1\right)\ \dot{x_b} + \dot{x_r} + \frac{Y_f\beta}{2K_0\overline{r}}\dot{D}_o,\\
&= Y_f\left(-\frac{\mu}{Y_f}x_b - D(1+r)S + DS_{in}\right) + \left(\frac{\beta}{2\overline{r}} + 1\right)\left(\mu x_b - D(1+r)x_b + rDx_r\right)\\
&+ \left(D(1+r)x_b - D(\beta+r)x_r\right)\\
&+ \frac{Y_f\beta}{2K_0\overline{r}}\left(-K_0\frac{\mu}{Y_f}x_b - D(1+r)D_o + DD_{o_{in}} + \alpha W[D_{o_{max}} - D_o]\right),\\
&= 0.\mu x_b - Y_f D(1+r)S + \left(-D(1+r)\left(\frac{\beta}{2\overline{r}} + 1\right) + D(1+r)\right)x_b\\
&+ \left(\left(\frac{\beta}{2\overline{r}} + 1\right)rD - D(\beta+r)\right)x_r - \left(D(1+r)\frac{Y_f\beta}{2K_0\overline{r}} + \alpha W\right)D_o\\
&+ Y_f DS_{in} + \frac{Y_f\beta}{2K_0\overline{r}}\left(DD_{o_{in}} + \alpha WD_{o_{max}}\right).
\end{aligned}$$

$$\begin{aligned}\dot{Z} &< -Y_f\underline{D}(1+\underline{r})S - \underline{D}(1+\underline{r})\frac{\beta}{2\overline{r}}x_b - \underline{D}\beta x_r + \\ &- \left(\frac{Y_f\beta}{2K_0\overline{r}}\underline{D}(1+\underline{r}) + \alpha\underline{W}\right)D_o + \\ &+ \overline{D}S_{in} + \frac{Y_f\beta}{2K_0\overline{r}}\left(\overline{D}D_{o_{in}} + \alpha\overline{W}D_{o_{max}}\right).\end{aligned}$$

We put

$$\gamma = Y_f\overline{D}S_{in} + \frac{Y_f\beta}{2K_0\overline{r}}\left[\overline{D}D_{o_{in}} + \alpha\overline{W}D_{o_{max}}\right],$$

and

$$0 < M < \min\left(Y_f\underline{D}(1+\underline{r}), \underline{D}(1+\underline{r})\frac{\beta}{2\overline{r}}, \underline{D}\beta, \frac{Y_f\beta}{2K_0\overline{r}}\underline{D}(1+\underline{r}) + \alpha\underline{W}\right).$$

We obtain

$$\dot{Z} < -MZ + \gamma.$$

Therefore, by applying the properties of differential inequalities, we can deduce that:

$$Z(t) < (Z(0) - \frac{\gamma}{M})e^{-Mt} + \frac{\gamma}{M}.$$

So, if $Z(0) \in \Omega$, then $Z(0) - \frac{\gamma}{M} \leq 0$, and

$$Z(t) \leq \frac{\gamma}{M}.$$

Which implies that the trajectory (S, x_b, x_r, D_o) starting in Ω remains in it thereafter. as required. □

2.3. Existence of Solution for Dynamical system

We begin by showing that, for every control u, the dynamical system admits a unique solution $x(\cdot)$.

Proposition 2.3 *The problem (2.3) is well-posed, such that $\forall u \in \mathcal{U}$ there is a unique solution $x(\cdot)$ to (2.3), defined along the interval* $[0, T]$.

Proof: It clear that the function $f(.,u)$ is locally Lipschitz for all $u \in U$, in the other hand, the function f satisfies a linear growth condition. More precisely, for all $(x,u) \in \Omega \times U$, we examine each component $f_i(x,u)$, $i = 1, \ldots, 4$, of the vector field f, and derive suitable upper bounds. For the first component, we have:

$$|f_1(x,u)| < \frac{\mu(x_1,x_4)}{Y_f}|x_2| + u_1(1+u_2)|x_1| + u_1 S_{in}.$$

For the second component, we obtain

$$|f_2(x,u)| < u_2u_1|x_3| + [\mu(x_1,x_4) + u_1(1+u_2)]|x_2.$$

The third component satisfies

$$|f_3(x,u)| < u_1(1+u_2)|x_2| + u1(\beta+u_2)|x_3|.$$

And the fourth component is bounded by

$$|f_4(x,u)| < K_0\frac{\mu(x_1,x_4)}{Y_f}|x_2| + [u_1(1+u_2) + \alpha u_3]|x_4| + u_1 D_{o_{in}} + \alpha u_3 D_{o_{max}}.$$

Each of these inequalities expresses a linear dependence on the state vector components $x = (x_1, x_2, x_3, x_4)$, with coefficients depending on the control u and bounded system parameters. Summing these componentwise bounds, and invoking norm equivalence in $\mathbb{R}^4$, we deduce the existence of constants $K > 0$ and $N > 0$ such that the full vector field satisfies the global estimate

$$\|f(x,u)\| \leq K\|x\| + N, \quad \forall (x,u) \in \Omega \times U,$$

which confirms that f has a linear growth criterion on $\Omega \times U$. for some constants $K > 0$, $N > 0$, which depend only on the bounds of the control and model parameters. Explicitly, we may choose

$$K = max\left\{\overline{D}(\beta + \overline{r}), \mu_{max} + \overline{D}(1+\overline{r}), \frac{\mu_{max}}{Y_f}, K_0\frac{\mu_{max}}{Y_f}, \overline{D}(1+\overline{r}) + \alpha\overline{W}\right\}$$

and

$$N = \overline{D}S_{in} + \overline{D}D_{o_{in}} + \alpha\overline{W}D_{o_{max}}$$

Since the function f is locally Lipschitz and satisfies a linear growth condition, we invoke standard results from the theory of differential equations (see, for instance, [6,21]) to conclude that, for any measurable control $u \in \mathcal{U}$, the system (2.3) admits a unique solution $x(\cdot)$, defined on the entire interval $[0,T]$. This concludes the proof of the proposition. □

3. Optimal Control

3.1. Presentation of an optimal control problem

The optimal control problem in wastewater treatment models has been previously investigated, for instance, see [3,7,9,11,14]. Various methods have been tackled, such as the method using the Pontryagin Maximum Principle. In their study [14], Raissi and Serhani analyzed a model for the continuous operation of a wastewater treatment reactor and proposed an optimal strategy to maintain pollutant and fungal effluent levels near zero, ensuring uninterrupted plant activity through controlled pollutant feeding. Likewise, Bouhafs et al. [3] investigated an optimal control approach for a biological batch reactor, emphasizing the optimization of both time and process trajectories. Furthermore, in 2010, Ellina et al. [7] explored an optimal control problem aimed at reducing pollutant concentrations in wastewater.

Other methods can be found in [11], J. Moreno, in his seminal work optimal control of the activated sludge process, applied Green's theorem to solve the problem, enabling the analytical derivation of a unique global solution, Similarly, Kabouris and Georgakakos [9], proposed a solution based on a Newton-type successive approximations method for optimal control. Our work here employs viscosity solutions of the HJB equation to rigorously determine the optimal control.
Our aim of this section is to find an optimal control process (D, r, W) that minimize the density of bacteria, the consumption of dissolved oxygen $D_o(t)$ and the substrate density at the exit of the aerator, together with the recycled rate r as well as the aeration rate W since there manipulation generate costs. Hence, the objective function will be stated as

$$J(u) := \int_0^T \beta_1 D_0(t) + \beta_2 x_b(t) + \beta_3 r(t) + \beta_4 W(t) dt + S(T). \tag{3.1}$$

for $u \in \mathcal{U}$.
Where the constants β_1 , β_3 , β_4, β_2 are weights associated to running cost. Consider, for $(x,u) \in \Omega \times U$, the Lagrangian function

$$L(x,u) = \beta_1 x_4 + \beta_2 x_2 + \beta_3 u_2 + \beta_4 u_3,$$

representing the running cost and

$$\varphi(x) = x_1,$$

representing the terminal cost.
The optimal control problem is then given by

$$\begin{cases} \inf_{u\in\mathcal{U}} J(u) = \displaystyle\int_0^T L(x(s),u(s))\,ds + \varphi(x(T)) \\ \dot{x}(t) = f(x(t),u(t)), \qquad a.e.\ s\in[0,T], \\ x(0) = x_0. \end{cases} \tag{3.2}$$

The objective function is directly linked to the value function typically denoted as $V(t,x)$ which is represents the minimum cost or maximum starting from a given state $x(t) = x$ at time t, it plays a crucial role in solving optimal control problems and it is given by:

3.2. Viscosity solutions of The Hamilton Jacobi Bellman equation

In optimal control theory, the Hamilton–Jacobi–Bellman (HJB) equation serves as a key tool for establishing the existence of an optimal solution and for identifying the corresponding optimal strategy. Nevertheless, solving the HJB equation is often difficult because its solutions may lack smoothness [2, 5,15,19,20]. The value function, which represents the optimal cost-to-go, is typically considered as a candidate solution for the HJB equation. However, in many practical applications, the value function may not meet the differentiability requirements necessary for classical solutions. This is where the notion of viscosity solutions becomes crucial [2,5,20]. The aim of this work is to demonstrate that the value function is the unique viscosity solution of the HJB equation associated with our optimal control problem, thereby also proving the existence of an optimal control.
Let us first, define the value function. We associated to the control problem (3.2) a value function $V(t,x)$ representing the perturbation in initial conditions $(0,x)$ as follows:

$$\begin{cases} V(t,x) = \inf_{u\in\mathcal{U}} \displaystyle\int_t^T L(x(s),u(s))\,ds + \varphi(x(T)) \\ \dot{x}(s) = f(x(s),u(s)) \qquad a.e.\ s\in[t,T] \\ x(t) = x; \end{cases} \tag{3.3}$$

It is well established that our optimal control problem (3.2) can be associated with the following Hamilton–Jacobi–Bellman (HJB) equation for every $(t,x)\in[0,T]\times\Omega$

$$\frac{\partial\psi}{\partial t}(t,x) + \min_{u\in U}\left\{\frac{\partial\psi}{\partial x}(t,x)\cdot f(x,u) + L(x,u)\right\} = 0, \tag{3.4}$$

where $\psi\in C^1([0,T]\times\Omega,\mathbb{R})$.

As discussed above, this equation fails to have a smooth solutions, the well solution concept devoted to this equation is the so-called viscosity solutions. We say that $\phi\in C([0,T]\times\Omega,\mathbb{R})$ is a viscosity solution of the equation (3.4) if it is both a super-solution and a sub-solution.
We say that $\phi\in C([0,T]\times\Omega,\mathbb{R})$ is a viscosity sub-solution of the (3.4), if for any $x_0\in\Omega$ and any C^1-function ψ such that $\psi(x_0) = \phi(x_0)$ and $\psi\geq\phi$ in a neighbourhood of x_0 we have

$$\frac{\partial\psi}{\partial t}(t,x) + \min_{u\in U}\left\{\frac{\partial\psi}{\partial x}(t,x)\cdot f(x,u) + L(x,u)\right\} \leq 0.$$

We say that $\phi\in C([0,T]\times\Omega,\mathbb{R})$ is a viscosity supersolution of the equation (3.4) if for any $x_0\in\Omega$ and any C^1 function ψ such that $\psi(x_0) = \phi(x_0)$ and $\psi\leq\phi$ in a neighbourhood of x_0 we have

$$\frac{\partial\psi}{\partial t}(t,x) + \min_{u\in U}\left\{\frac{\partial\psi}{\partial x}(t,x)\cdot f(x,u) + L(x,u)\right\} \geq 0.$$

The value function V given by (3.3) is a potential candidate to be the solution of the HJB equation. We check to prove that it is a continuous viscosity solution of the HJB equation. Let us prove that $V(t,\cdot)$ is Lipschtiz for all $t\in[0,T]$.

Lemma 3.1 *The value function V defined in (3.3) is Lipschitz continuous regarding x.*

Proof:

To provide the Lipschitz continuity of the function V, we need to ensure for two trajectories $x(t)$ and $y(t)$ starting from $x(t) = x$ and $y(t) = y$, respectively, under the same control $u(\cdot)$, that there is a constant $C > 0$ verifying:

$$|V(t,x) - V(t,y)| \leq C\|x-y\|, \quad \text{for all } (x,y) \in \Omega^2, \tag{3.5}$$

As signaled in the proposition (2.3), the vector field f is Lipschitz, that is, there exists a constant $C_3 > 0$ such that

$$\| f(x,u) - f(y,u) \| \leq C_3 \| x-y \| \text{ for all } (x,y) \in \Omega^2, \tag{3.6}$$

On the other hand, it's clear that the running cost L is Lipschitz, so there exists a constant $C_4 > 0$, such that

$$\| L(x,u) - L(y,u \| \leq C_4 \| x-y \| \text{ for all } (x,y) \in \Omega^2, \tag{3.7}$$

In addition, we have that the final cost φ satisfies

$$|\varphi(x) - \varphi(y)| \leq \|x-y\| \quad \forall x,y \in \Omega, \tag{3.8}$$

Let two trajectories $x(r)$ and $y(r)$, starting from initial states $x(t) = x$ and $y(t) = y$ and evolving under the same control $u(\cdot)$, the solutions $x(r)$ and $y(r)$ can be written in integral form:

$$x(r) = x + \int_t^r f(x(s),u(s))\,ds,$$

$$y(r) = y + \int_t^r f(y(s),u(s))\,ds,$$

Subtracting the two equations and taking norms

$$\|x(r)-y(r)\| \leq \|x-y\| + \left\| \int_t^r (f(x(s),u(s)) - f(y(s),u(s)))\,ds \right\|.$$

Using the triangle inequality for integrals and the Lipschitz condition for the function f

$$\|x(r)-y(r)\| \leq \|x-y\| + \int_t^r \|f(x(s),u(s)) - f(y(s),u(s))\|\,ds,$$

$$\|x(r)-y(r)\| \leq \|x-y\| + C_3 \int_t^r \|x(s)-y(s)\|\,ds.$$

Applying Grönwall's inequality, we derive

$$\|x(r)-y(r)\| \leq \|x-y\| e^{C_3(r-t)}. \tag{3.9}$$

For any fixed control $u(\cdot)$, the cost difference satisfies

$$|J(x,u) - J(y,u)| \leq \int_t^T |L(x(s),u(s)) - L(y(s),u(s))| dt + |x_1(T) - y_1(T)|.$$

Using the Lipschitz properties of L and φ, we get

$$|J(x,u) - J(y,u)| \leq C_4 \int_t^T \|x(s)-y(s)\| dt + \|x_1(T) - y_1(T)\|.$$

Substituting the trajectory bound from (3.9), we obtain

$$|J(x,u) - J(y,u)| \leq \left(C_4 \int_t^T e^{C_3(s-t)} ds + e^{C_3(T-t)} \right) \|x-y\|. \tag{3.10}$$

Define the constant

$$C = C_4 \frac{e^{C_3(T-t)} - 1}{C_3} + e^{C_3(T-t)} \tag{3.11}$$

then, we conclude

$$|J(x,u) - J(y,u)| \leq C\|x-y\|.$$

Now let u_1^* be a control that is "nearly optimal" for x, i.e., $J(x,u_1^*) \leq V(t,x) + \epsilon$ (for small $\epsilon > 0$), since $|J(y,u_1^*) - J(x,u_1^*)| \leq C\|x-y\|$, we have

$$J(y,u_1^*) \leq J(x,u_1^*) + C\|x-y\| \leq V(t,x) + \epsilon + C\|x-y\|.$$

But $V(t,y)$ is the infimum over all controls, so

$$V(t,y) \leq J(y,u_1^*) \leq V(t,x) + \epsilon + C\|x-y\|.$$

Taking $\epsilon \to 0$, we get

$$V(t,y) - V(t,x) \leq C\|x-y\|.$$

Similarly we prove that

$$V(t,x) - V(t,y) \leq C\|x-y\|.$$

Since the inequality holds for any control $u(\cdot)$

$$|V(t,x) - V(t,y)| \leq C\|x-y\|. \tag{3.12}$$

Thus, V is Lipschitz continuous with constant C.

□

Theorem 3.1 *The value function V given by (3.3) is the unique continuous viscosity solution of the Hamilton Jacobi Bellman equation (3.4).*

Proof:

First we put

$$V_1(t,x) := -V(t,x),$$

and

$$H_1 := \max_u \left\{ \frac{\partial V_1}{\partial x}(t,x) \cdot f(x,u) - L(x,u) \right\}.$$

Then solving equation

$$\frac{\partial V}{\partial t}(t,x) + \min_u \left\{ \frac{\partial V}{\partial x}(t,x) \cdot f(x,u) + L(x,u) \right\} = 0,$$

is equivalent to solving

$$\frac{\partial V_1}{\partial t}(t,x) + H_1(x, \frac{\partial V_1}{\partial x}(t,x)) = 0. \tag{3.13}$$

To prove this theorem, we invoke the theorem 8.21 in [21] for the equation (3.13) . It suffices to verify that its hypotheses are satisfied, which are the following

$$\| f(x,u) \| \leq C_1, \qquad \text{for all } (x,u) \in \Omega \times U,$$

$$\| L(x,u) \| \leq C_2, \qquad \text{for all } (x,u) \in \Omega \times U,$$

$$|\varphi(x)| \leq C_5, \qquad \text{for all } x \in \Omega,$$

and

$$|\varphi(x) - \varphi(y)| \leq \|x-y\|, \qquad \text{for all } (x,y) \in \Omega^2.$$

In addition, the required Lipschitz continuity of f and L is already established in inequalities (3.6) and (3.7).
Indeed, we have that Ω is bounded, then according to Lipschitz property of f, there exist $C_1 > 0$ such that

$$\| f(x,u) \| \leq C_1, \qquad \text{for all } (x,u) \in \Omega \times U.$$

Moreover, according to boundedness of the sets Ω and U, it follows that there exists a constant $C_2 > 0$ such that

$$\|L(x,u)\| \leq C_2, \qquad \text{for all } (x,u) \in \Omega \times U.$$

The boundedness of the terminal cost φ is immediate on the bounded set Ω.
Finlay, the Lipschitz inequality (3.8) gives the last required inequality.
Therefore, by applying Theorem 8.21 from [21], we conclude the existence of a unique viscosity solution V_1 that is continuous for the Hamilton–Jacobi–Bellman equation

$$\frac{\partial V_1}{\partial t}(t,x) + H_1(x, \frac{\partial V_1}{\partial x}(t,x)) = 0.$$

□

Corollary 3.1 *There exists an optimal control $u^\star \in U$ for the problem ((2.3))-((3.1)), such that the corresponding trajectory $x^\star$ satisfies the dynamical system (2.3).*

Proof: From the theorem (3.1) we get that V is the unique viscosity solution of the HJB equation (3.4).
Consider

$$u^* = \arg\min \left\{ \frac{\partial V}{\partial t}(t,x) \cdot f(x,u) + L(x,u) \right\},$$

where $\frac{\partial V}{\partial t}(t,x)$ is taken in the viscosity sense.
Hence, the bellman principle state that, for all $\tau \in [t,T]$

$$V(t,x) = \int_t^\tau L(x^*(s), u^*(s))ds + V(\tau, x^*(\tau)).$$

For $t = 0$, $\tau = T$ and $x = x_0$ it derives that

$$V(0,x_0) = \int_0^T L(x^*(s), u^*(s))ds + V(T, x^*(T)).$$

According to $V(T, x^*(T)) = \varphi(x^*(T))$, we obtain

$$V(0,x_0) = \int_0^T L(x^*(s), u^*(s))ds + \varphi(x(T),$$

but the value function in $(0, x_0)$

$$V(0,x_0)) = \inf_{u \in \mathcal{U}} \int_0^T L(x(s), u(s))ds + \varphi(x(T)).$$

We conclude that $\inf_{u\in\mathcal{U}} \displaystyle\int_0^T L(x(s), u(s))ds + \varphi(x(T)$, is reached at u^* and that

$$u^* = \arg\min \left\{ \frac{\partial V}{\partial t}(t,x) \cdot f(x,u) + L(x,u) \right\}.$$

is an optimal control for the problem (2.3)-(3.1). □

4. Numerical Simulation

In this section, we validate the theoretical result of our proposed optimal control strategy in a wastewater treatment system. A numerical simulation was performed using Matlab for which we implement the numerical optimization using a gradient descent algorithm with a learning rate of 0.01 and a numerical gradient perturbation of 10^{-4}. Starting with mid-range values, control inputs were updated iteratively while being projected to ensure viability within constraints. Additionally, the convergence was declared when the cost improvement fell below a precalculated threshold. The outcomes show the evolution of states over time according to the optimal control policy and show the interaction between effective pollutant removal and operational efficiency.

The simulation was conducted over a 150-day time horizon, and we initialize our system with following state values: $S_0 = 50$ mg/L, biomass $x_{b_0} = 300$ mg/L, $x_{r_0} = 100$ mg/L, and $D_{o_0} = 20$ mg/L. The influent concentrations were $S_{\text{in}} = 39\,\text{mg/L}$ and $D_{o,\text{in}} = 2\,\text{mg/L}$. The model paramter's values were : yield coefficient $Y_f = 0.65\,\text{mg}$ biomass/mg substrate, oxygen consumption rate $K_0 = 0.5\,\text{mg}\ O_2/\text{mg}$ substrate, oxygen transfer coefficient $\alpha = 0.02\,\text{L/mg}$, and decay rate for recycled biomass $\beta = 0.2\,\text{day}^{-1}$. The microbial growth dynamics were governed by Monod kinetics using a maximum specific growth rate $\mu_{\max} = 0.5\,\text{day}^{-1}$, a substrate half-saturation constant $K_s = 100\,\text{mg/L}$, and an oxygen half-saturation constant $K_o = 0.2\,\text{mg/L}$. The maximum dissolved oxygen concentration was set as $D_{o,\max} = 62\,\text{mg/L}$. Control variables included the dilution rate $D \in [0, 0.4]\,\text{day}^{-1}$, recycle ratio $r \in [0, 2]$, and aeration rate $W \in [0, 1]$.

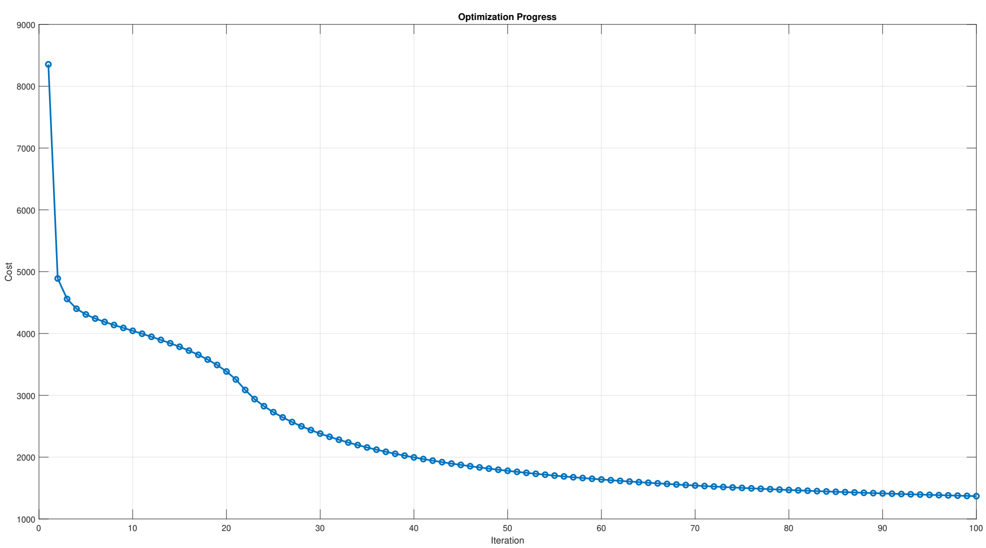

Figure 1: Cost Function

In the corresponding figure (1), we present the evolution of cost function value over the optimization iterations. As noted, the cost decreases significantly during early stages, indicating rapid convergence of the optimal control strategy. As the iterations progress, the curve flattens, confirming progression of the solution in reaching a minimum, and indicating the successful minimization of the control efforts alongside the concentrations of both biomass and substrate at the final time. This progressive decrease demonstrates the efficiency of gradient descent algorithm in converging toward an optimal control strategy. A detailed analysis and explanation of the resulting system dynamics and control profiles will be provided in the subsequent simulation results.

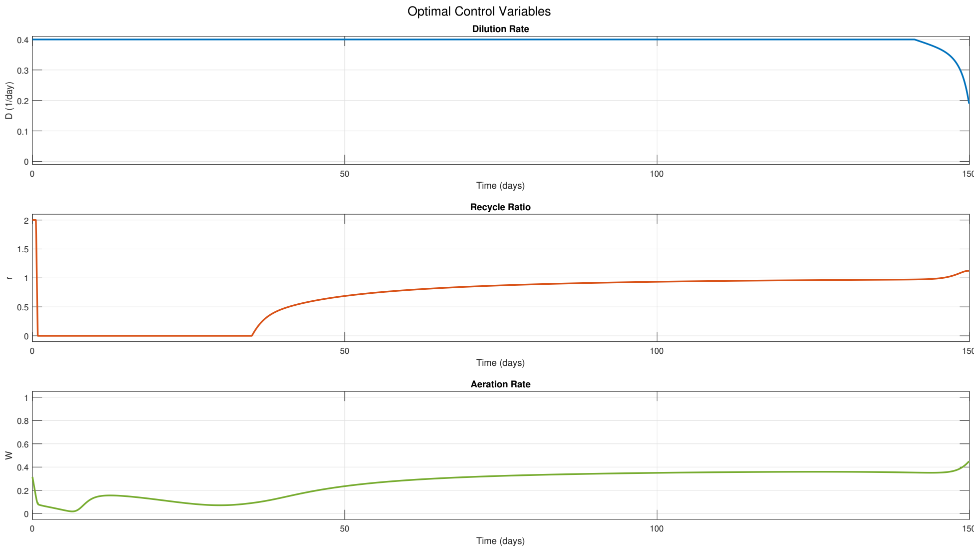

Figure 2: Control input $u(t)$ over the time T.

The simulation figure (2) reveal the evolution of the optimal control input efforts over time. Firstly, the dilution rate D remains near its upper bound during most of the process to ensure rapid treatment and flushing of pollutants. Then, a noticeable decrease occurs near the end of the time horizon, which suggest a reduction in inflow rate as the system approaches a stable clean state with low substrate levels. The recycle ratio r is initially very high, supporting biomass retention and rapid biomass buildup. However, it is subsequently reduced to zero and slowly increased again in the second half of the time horizon, indicating a shift from aggressive early recycling to a more cost-effective long-term regime that maintains biomass without overshooting. The aeration rate W exhibits moderate oscillations in response to oxygen demand and biomass activity that is high during initial stages to facilitate microbial growth. Later, it alters to maintain the dissolved oxygen at the desired level, avoiding excessive energy consumption.

The evolution of the optimal state variables demonstrates in figure (3) the direct impact of the control strategies on system performance and validates the effectiveness of the optimization process. The substrate concentration S, representing organic matter in the influent, initially decreased rapidly thereafter it rises and stabilizes around 40 mg/L before gradually declining as the biomass becomes active. This behavior reflects the microbial adaptation phase, followed by an efficient degradation of organic pollutants. Simultaneously, the biomass concentration x_b rapidly decreases next it increases slightly again, then declines significantly and get near zero, indicating effective biomass recycling and control. The recycled biomass x_r shows a sharp early increase due to high initial recycle rates, followed by a decay as recycle strategy r shifts to a phase where its value near 1. The dissolved oxygen concentration D_o initially decreases immediately after it increases as aeration intensifies maintaining values sufficient to support aerobic microbial activity. Fluctuations in D_o around days 40–50 correspond to changes in

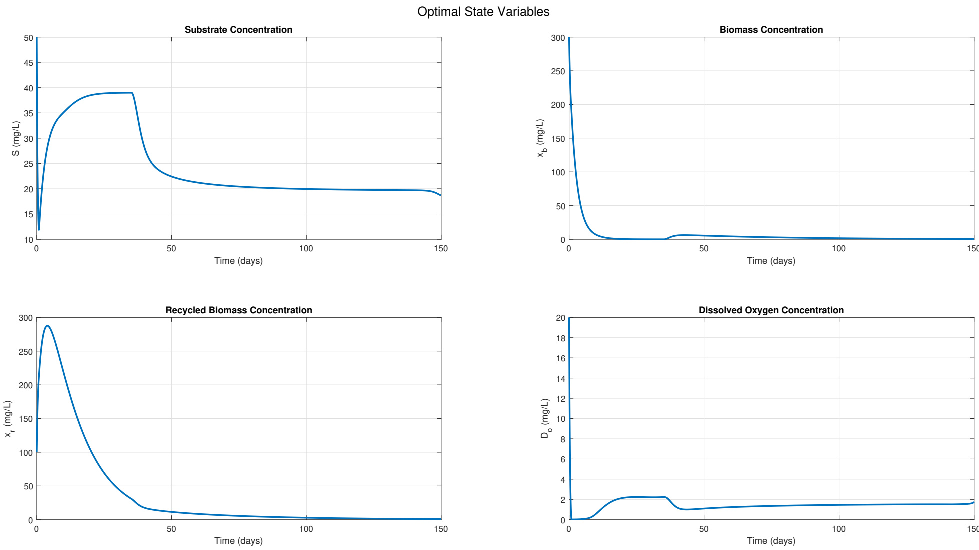

Figure 3: Control states $x(t)$ over the time T.

aeration rate and biomass demand, becoming nearly steady afterward due to optimized oxygen supply. The coupled behavior of the state and control variables highlights the importance of coordinated control actions. Initially, high recycle and aeration rates boost biomass growth and oxygen availability, which enhance pollutant degradation. Next, as the substrate level drops and the microbial population almost stagnates, the controls adapt to a more sustainable regime of lower dilution, moderate aeration, and minimal recycling, thus reducing energy and operational costs. As evidenced by the declining cost function, this dynamic optimization not only achieves cost-effectiveness but also guarantees treatment efficiency by lowering pollutant concentrations. The results confirm that time-dependent control strategies significantly outperform static control settings.

5. Conclusion

This study focused on a non-convex optimal control problem modeling a wastewater treatment process. The goal was to minimize the output substrate, the dissolved oxygen and density of bacteria together with minimizing the cost of treatment throughout minimizing the recycle rate and aeration rate. The viscosity solution framework was employed to address the challenges posed by non convexity and non-smoothness in the value function and to derive optimal control strategies. The effectiveness of the resulting strategy is reflected by its ability to control dilution, recycling, and aeration in such a way as to maintain substrate and oxygen levels within desirable operational ranges. The system exhibits rapid convergence toward optimal behavior, ensuring both efficient pollutant removal and effective biomass control. Furthermore, the relevance of this approach for real-time implementation under realistic operational conditions is clearly emphasized. Initially, we analyzed the system's invariance and dissipation properties, providing a physically consistent interpretation and establishing the existence and uniqueness of state trajectories for any admissible control. In the second part, since the problem is not convex with respect to controls variables, the viscosity solution approach to the HJB equation was developed and used to characterize the value function and determine the optimal control, Furthermore, the existence of an optimal control was demonstrated, confirming that the theoretical framework is not only well-posed but also practically attainable.

Finally, numerical simulations were presented to illustrate the system's behavior under the derived control strategy. The results validated the theoretical findings and demonstrated the capacity of the control policy to meet operational and environmental objectives effectively within the dynamic constraints of the treatment process.

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