Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–15.
ISSN-0037-8712
doi:10.5269/bspm.79212

On Integral Relations of Incomplete Aleph-functions and Applications

Dheerandra Shanker Sachan, Praveen Agarwal and Manish Kumar Bansal*

ABSTRACT: Aleph (ℵ)-functions are very general and advanced types of mathematical functions used in many areas. Over time, they have been extended in different ways, one of which is the incomplete Aleph-function. In this paper, we find integral relations for incomplete Aleph-functions and use them to solve double integrals that involve Bessel functions, hypergeometric functions, and incomplete Aleph-functions. We show that these double integrals can be solved more easily when certain known results are used. The results we present are quite general, and we also give some specific examples to show how they work in special cases.

Keywords: Incomplete Gamma functions, Bessel functions, Gauss hypergeometric functions, Mellin-Barnes contour integrals, Aleph-functions, incomplete Aleph-functions, integral relations.

Contents

1 Introduction **1**

2 An Integral Involving Incomplete Aleph-Function **5**

3 Integral Relations of Incomplete Aleph-Functions **5**

4 Applications Based on the Integral Relations of Incomplete Aleph-Functions **6**

5 Particular Cases **9**

6 Conclusion **13**

1. Introduction

The Aleph-function (written as ℵ) (see [1], [2]) is a type of special function in mathematics. It is broader than some other well-known functions, such as the I-function (see [3]) and the H-function (see [4]), because these functions can be obtained as particular cases of it. The Aleph-function is very useful and has been applied in many areas. For instance, one of its special cases has been used in the study of fractional driftless Fokker–Planck equations with power-law diffusion coefficients. It also has important applications in several other fields. For example, Saxena and Pogány [5] gave closed-form results for a family of Mathieu-type **a**-series that contain the Aleph-function in their terms. Chaurasia and Singh [6] formulated a representation of Lambert's law involving the Aleph-function. Prakash and Agnihotri [7] explored double integral transform of Aleph-function which leads to the formation of another important process of augmenting in Aleph-function. Chaurasia and Gill [8] obtained two integral formulas involving the Aleph-function, which generate a broad class of elliptic-type integrals. Dubey [9] represented a blood glucose equation with the help of the Aleph-function. Ayant and Kumar [10] solved a multiple integral involving the Aleph-function, which led to the derivation of a multiple exponential Fourier series. Using operational methods, Agarwal et al. [11] constructed a sequence of functions based on the Aleph-function (see also [12,13]). Moreover, the Aleph-function has significant uses in fractional calculus, which studies integrals and derivatives of non-integer order (see [14,15,16],).

The incomplete Aleph-function is a special mathematical function that generalizes the usual Aleph-function (ℵ-function). This function is important and has been applied in many areas of mathematics and science. For instance, Dinesh et al. [17] investigated certain unified infinite integrals involving incomplete Aleph-functions, (see also [18]). Sachan et al. [19] proposed solutions to the axisymmetric Dirichlet potential problem for a half-space using incomplete Aleph-functions. Tyagi et al. [20] used

* Corresponding author.
2020 *Mathematics Subject Classification:* 33B20, 33C60, 44A20.
Submitted September 28, 2025. Published April 01, 2026

Jacobi polynomials, multivariable Aleph-functions, incomplete Fox's Wright functions, and incomplete Aleph-functions to analyze the deflection and bending stresses of a clamped circular plate under uneven loading. Sachan and Ayant [21] evaluated finite integrals that included elliptic integrals of the first kind, extended Mittag-Leffler functions, and incomplete Aleph-functions. Kumar and Dassani [22] solved Fredholm integral equations with kernels that involved the S-function, extended Mittag-Leffler function, and incomplete ℵ-function. Sachan et al. [23] worked on finite integrals combining Fresnel integrals with incomplete Aleph-functions. Singh et al. [24] expressed Beer-Lambert's law using incomplete Aleph-functions. Singh et al. [25] also solved modified fractional kinetic equations with incomplete Aleph-functions through Elzaki and inverse Elzaki transforms. Moreover, Kumar et al. [26] studied finite integrals related to incomplete Aleph-functions.

The *incomplete Aleph–functions* ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ are defined as follows, (see, [27]);

$$
{}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z) = {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z \left| \begin{array}{c} (a_1,\mathcal{B}_1,\xi)\,,\,(a_j,\mathcal{B}_j)_{2,n}\,,\,[\tau_i\,(a_{ji},\mathcal{B}_{ji})]_{n+1,p_i} \\ \\ (g_j,\mathcal{G}_j)_{1,m}\,,\,[\tau_i\,(g_{ji},\mathcal{G}_{ji})]_{m+1,q_i} \end{array} \right. \right]
$$

$$
= \frac{1}{2\pi\omega}\int_L \phi_1(s,\xi) z^{-s}\mathrm{d}s, \tag{1.1}
$$

where, $\omega = \sqrt{-1}$,

$$
\phi_1(s,\xi) = \frac{\Gamma\,(1-a_1-\mathcal{B}_1 s,\xi)\,\prod\limits_{j=2}^{n}\Gamma\,(1-a_j-\mathcal{B}_j s)\,\prod\limits_{j=1}^{m}\Gamma\,(g_j+\mathcal{G}_j s)}{\sum\limits_{i=1}^{r}\tau_i\left[\prod\limits_{j=m+1}^{q_i}\Gamma\,(1-g_{ji}-\mathcal{G}_{ji}s)\,\prod\limits_{j=n+1}^{p_i}\Gamma\,(a_{ji}+\mathcal{B}_{ji}s)\right]} \tag{1.2}
$$

and

$$
{}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z) = {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z \left| \begin{array}{c} (a_1,\mathcal{B}_1,\xi)\,,\,(a_j,\mathcal{B}_j)_{2,n}\,,\,[\tau_i\,(a_{ji},\mathcal{B}_{ji})]_{n+1,p_i} \\ \\ (g_j,\mathcal{G}_j)_{1,m}\,,\,[\tau_i\,(g_{ji},\mathcal{G}_{ji})]_{m+1,q_i} \end{array} \right. \right]
$$

$$
= \frac{1}{2\pi\omega}\int_L \phi_2(s,\xi) z^{-s}\mathrm{d}s, \tag{1.3}
$$

where

$$
\phi_2(s,\xi) = \frac{\gamma\,(1-a_1-\mathcal{B}_1 s,\xi)\,\prod\limits_{j=2}^{n}\Gamma\,(1-a_j-\mathcal{B}_j s)\,\prod\limits_{j=1}^{m}\Gamma\,(g_j+\mathcal{G}_j s)}{\sum\limits_{i=1}^{r}\tau_i\left[\prod\limits_{j=m+1}^{q_i}\Gamma(1-g_{ji}-\mathcal{G}_{ji}s)\,\prod\limits_{j=n+1}^{p_i}\Gamma(a_{ji}+\mathcal{B}_{ji}s)\right]}. \tag{1.4}
$$

Here $\gamma\,(\vartheta,\xi)$ and $\Gamma\,(\vartheta,\xi)$ are the incomplete gamma functions defined as follows:

$$
\gamma\,(\vartheta,\xi) = \int_0^{\xi} u^{\vartheta-1}\,e^{-u}\,\mathrm{d}u \quad (\xi \geqslant 0;\, \Re\,(\vartheta) > 0) \tag{1.5}
$$

and

$$
\Gamma\,(\vartheta,\xi) = \int_{\xi}^{\infty} u^{\vartheta-1}\,e^{-u}\,\mathrm{d}u \quad (\xi \geqslant 0;\, \Re\,(\vartheta) > 0 \ \text{when} \ \xi = 0)\,. \tag{1.6}
$$

The sum of the functions given in (1.5) and (1.6) results in the familiar Gamma function (see, e.g., [28, Section 1.1]):

$$
\Gamma\,(\vartheta) = \gamma\,(\vartheta,\xi) + \Gamma\,(\vartheta,\xi) = \int_0^{\infty} u^{\vartheta-1}\,e^{-u}\,\mathrm{d}u \quad (\Re\,(\vartheta) > 0)\,. \tag{1.7}
$$

The incomplete $\aleph$-functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ in (1.1) and (1.3) exist for $\xi \geqslant 0$ under following conditions: The contour L is of Mellin-Barnes type and runs from $\varrho - i\infty$ to $\varrho + i\infty$ for some real ϱ with indentations, if necessary, so that the poles of $\Gamma\left(1 - a_j - \mathcal{B}_j s\right),\ j = 2, \ldots, n$, can be placed to the right of the contour L and the poles of $\Gamma\left(g_j + \mathcal{G}_j s\right),\ j = 1, \ldots, m$, to the left of the contour L. The quantities τ_i, m, n, p_i, q_i are non-negative integers satisfying $0 \leqslant n \leqslant p_i$, $0 \leqslant m \leqslant q_i$; a_j, g_j, a_{ji}, g_{ji} are complex number parameters and $\mathcal{B}_j$, $\mathcal{G}_j$, $\mathcal{B}_{ji}$, $\mathcal{G}_{ji}$ are positive real number parameters. Assume that all poles of (1.2) and (1.4) are simple and the empty product there (elsewhere) is (as usual) interpreted to be unity. The following conditions hold :

$$\Omega_i > 0,\ |\arg(z)| < \frac{\pi}{2}\Omega_i,\ i = 1, \ldots, r \tag{1.8}$$

$$\Omega_i \geq 0,\ |\arg(z)| < \frac{\pi}{2}\Omega_i \quad \text{and} \quad \Re(\zeta_i) + 1 < 0, \tag{1.9}$$

where

$$\Omega_i = \sum_{j=1}^{n} \mathcal{B}_j + \sum_{j=1}^{m} \mathcal{G}_j - \tau_i \max_{1\leq i\leq r}\left(\sum_{j=n+1}^{p_i} \mathcal{B}_{ji} + \sum_{j=m+1}^{q_i} \mathcal{G}_{ji}\right), \tag{1.10}$$

and

$$\zeta_i = \sum_{j=1}^{m} g_j - \sum_{j=1}^{n} a_j + \tau_i\left(\sum_{j=m+1}^{q_i} g_{ji} - \sum_{j=n+1}^{p_i} a_{ji}\right) + \frac{p_i - q_i}{2}, \quad i = 1, \ldots, r. \tag{1.11}$$

Remark 1.1 It can easily be seen that

$${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z) + {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z) = \aleph^{m,n}_{p_i,q_i,\tau_i;r}(z), \tag{1.12}$$

last of these functions is known as the Aleph-function (see, e.g., [5], [29]).

Letting $\xi \to 0$ in (1.1) (or $\xi \to \infty$ in (1.3)), the incomplete Aleph ($\aleph$)-functions reduce to the Aleph ($\aleph$)-functions.

If we set $\tau_i \to 1$, then the incomplete $\aleph$–functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ reduce to incomplete I-functions ${}^{(\Gamma)}I^{m,n}_{p_i,q_i;r}(z)$ and ${}^{(\gamma)}I^{m,n}_{p_i,q_i;r}(z)$, respectively, given as follows:

$$\begin{aligned} {}^{(\Gamma)}I^{m,n}_{p_i,q_i;r}(z) &:= {}^{(\Gamma)}I^{m,n}_{p_i,q_i;r}\left[z \,\middle|\, \begin{matrix} (a_1, \mathcal{B}_1, \xi), (a_j, \mathcal{B}_j)_{2,n}, (a_{ji}, \mathcal{B}_{ji})_{n+1,p_i} \\ (g_j, \mathcal{G}_j)_{1,m}, (g_{ji}, \mathcal{G}_{ji})_{m+1,q_i} \end{matrix}\right] \\ &= \frac{1}{2\pi\omega}\int_L \phi_3(s,\xi) z^{-s} \mathrm{d}s, \end{aligned} \tag{1.13}$$

where

$$\phi_3(s,\xi) = \frac{\Gamma\left(1 - a_1 - \mathcal{B}_1 s, \xi\right) \prod\limits_{j=2}^{n} \Gamma\left(1 - a_j - \mathcal{B}_j s\right) \prod\limits_{j=1}^{m} \Gamma\left(g_j + \mathcal{G}_j s\right)}{\sum\limits_{i=1}^{r}\left[\prod\limits_{j=m+1}^{q_i} \Gamma\left(1 - g_{ji} - \mathcal{G}_{ji} s\right) \prod\limits_{j=n+1}^{p_i} \Gamma\left(a_{ji} + \mathcal{B}_{ji} s\right)\right]}$$

and

$${}^{(\gamma)}I^{m,n}_{p_i,q_i;r}(z) := {}^{(\gamma)}I^{m,n}_{p_i,q_i;r}\left[z \,\middle|\, \begin{matrix} (a_1, \mathcal{B}_1, \xi), (a_j, \mathcal{B}_j)_{2,n}, (a_{ji}, \mathcal{B}_{ji})_{n+1,p_i} \\ (g_j, \mathcal{G}_j)_{1,m}, (g_{ji}, \mathcal{G}_{ji})_{m+1,q_i} \end{matrix}\right]$$

$$= \frac{1}{2\pi\omega}\int_L \phi_4(s,\xi)z^{-s}\mathrm{d}s, \tag{1.14}$$

where

$$\phi_4(s,\xi) = \frac{\gamma\left(1-a_1-\mathcal{B}_1 s,\xi\right)\prod_{j=2}^{n}\Gamma\left(1-a_j-\mathcal{B}_j s\right)\prod_{j=1}^{m}\Gamma\left(g_j+\mathcal{G}_j s\right)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma\left(1-g_{ji}-\mathcal{G}_{ji}s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+\mathcal{B}_{ji}s\right)\right]}.$$

The existence and other conditions can be obtained from the incomplete $\aleph$-functions in (1.1) and (1.3) when $\tau_i \to 1$.

Letting $r = 1$ in the incomplete I-functions ${}^{(\Gamma)}I^{m,n}_{p_i,q_i;r}(z)$ and ${}^{(\gamma)}I^{m,n}_{p_i,q_i;r}(z)$ reduce to incomplete H-functions ${}^{(\Gamma)}H^{m,n}_{p,q}(z)$ and ${}^{(\gamma)}H^{m,n}_{p,q}(z)$, respectively, given as follows:

$${}^{(\Gamma)}H^{m,n}_{p,q}(z) = {}^{(\Gamma)}H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix}(a_1,\mathcal{B}_1,\xi),\,(a_j,\mathcal{B}_j)_{2,p}\\ \\ (g_j,\mathcal{G}_j)_{1,q}\end{matrix}\right]$$

$$= \frac{1}{2\pi\omega}\int_L \phi_5(s,\xi)z^{-s}\mathrm{d}s, \tag{1.15}$$

where

$$\phi_5(s,\xi) = \frac{\Gamma\left(1-a_1-\mathcal{B}_1 s,\xi\right)\prod_{j=2}^{n}\Gamma\left(1-a_j-\mathcal{B}_j s\right)\prod_{j=1}^{m}\Gamma\left(g_j+\mathcal{G}_j s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-g_j-\mathcal{G}_j s\right)\prod_{j=n+1}^{p}\Gamma\left(a_j+\mathcal{B}_j s\right)}$$

and

$${}^{(\gamma)}H^{m,n}_{p,q}(z) = {}^{(\gamma)}H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix}(a_1,\mathcal{B}_1,\xi),\,(a_j,\mathcal{B}_j)_{2,p}\\ \\ (g_j,\mathcal{G}_j)_{1,q}\end{matrix}\right]$$

$$= \frac{1}{2\pi\omega}\int_L \phi_6(s,\xi)z^{-s}\mathrm{d}s, \tag{1.16}$$

where

$$\phi_6(s,\xi) = \frac{\gamma\left(1-a_1-\mathcal{B}_1 s,\xi\right)\prod_{j=2}^{n}\Gamma\left(1-a_j-\mathcal{B}_j s\right)\prod_{j=1}^{m}\Gamma\left(g_j+\mathcal{G}_j s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-g_j-\mathcal{G}_j s\right)\prod_{j=n+1}^{p}\Gamma\left(a_j+\mathcal{B}_j s\right)}.$$

The existence and other conditions can be obtained from the incomplete I-functions in (1.13) and (1.14) when $r = 1$.

Just as in the case of relation (1.12), the following relations hold:

$${}^{(\Gamma)}I^{m,n}_{p_i,q_i;r}(z) + {}^{(\gamma)}I^{m,n}_{p_i,q_i;r}(z) = I^{m,n}_{p_i,q_i;r}(z), \tag{1.17}$$

last of these functions is known as the I-function (see, e.g., [30]);

$${}^{(\Gamma)}H^{m,n}_{p,q}(z) + {}^{(\gamma)}H^{m,n}_{p,q}(z) = H^{m,n}_{p,q}(z), \tag{1.18}$$

last of these functions is known as the H-function (see, e.g., [31], [4]).

2. An Integral Involving Incomplete Aleph-Function

In this section, we present an important integral that involves the incomplete Aleph-function given in equation (1.1). This integral will play a key role in the next part of the paper, where we will use it as a foundation to prove Theorem 3.1.

Theorem 2.1 *Let* $\Re(\alpha) > 0$, $\Re(\beta) > 0$ *and* h *be a positive integer, then the following integral holds true*

$$\int_0^{\frac{\pi}{2}} \sin^{2\alpha-1}\theta \cos^{2\beta-1}\theta \; {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;\kappa}\left[vz(\sin\theta)^{-2h}\,\middle|\, \begin{matrix} T_1 \\ \\ T_2 \end{matrix}\right] \mathrm{d}\theta$$
$$= \frac{\Gamma(\beta)}{2}\, {}^{(\Gamma)}\aleph^{m+1,n}_{p_i+1,q_i+1,\tau_i;\kappa}\left[vz\,\middle|\, \begin{matrix} T_1, (\alpha+\beta, h) \\ \\ (\alpha, h), T_2 \end{matrix}\right] \tag{2.1}$$

where $T_1 = (a_1, \mathcal{B}_1, \xi)\,, (a_j, \mathcal{B}_j)_{2,n}\,, [\tau_i(a_{ji}, \mathcal{B}_{ji})]_{n+1,p_i}$
and $T_2 = (g_j, \mathcal{G}_j)_{1,m}\,, [\tau_i(g_{ji}, \mathcal{G}_{ji})]_{m+1,q_i}$
provided that $\Re\left\{\alpha - h\left(\frac{a_j-1}{\mathcal{B}_j}\right)\right\} > 0$, $j = 2\ldots n$ *and other restrictions would be deducible from those in* (1.1).

Proof: Let $\mathcal{L}_1$ be the left-handed member of (2.1). To establish (2.1), we need following formula

$$\int_0^{\frac{\pi}{2}} \sin^{2\alpha-1}\theta \cos^{2\beta-1}\theta \,\mathrm{d}\theta = \frac{\Gamma(\alpha)\Gamma(\beta)}{2\Gamma(\alpha+\beta)}, \quad \Re(\alpha) > 0, \Re(\beta) > 0, \tag{2.2}$$

expressing incomplete Aleph-function in its contour integral from (1.1), interchanging the order of integration, which is permissible due to absolute convergence of the integrals involved in the process and evaluate inner integral by means of the integral (2.2), we have

$$\mathcal{L}_1 = \int_L \phi_1(s,\xi)(vz)^{-s}\left[\frac{\Gamma(\beta)\Gamma(\alpha+hs)}{2\Gamma(\alpha+\beta+hs)}\right]\mathrm{d}s$$

which, upon expressing in terms of the incomplete Aleph-function (1.1), leads to the result in (2.1). This completes the proof. □

3. Integral Relations of Incomplete Aleph-Functions

The object of the present paper is to study the following integral relations for incomplete Aleph-functions.

Theorem 3.1 *Let* $\Re(\alpha) > 0$, $\Re(\beta) > 0$ *and* h *be a positive integer, then the following integral relation holds true*

$$\int_0^\infty\int_0^\infty x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^{1-\alpha-\beta}\, {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;\kappa}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}}\,\middle|\, \begin{matrix} T_1 \\ \\ T_2 \end{matrix}\right] f(x^2+y^2)\mathrm{d}y\mathrm{d}x$$
$$= \frac{\Gamma(\beta)}{4}\int_0^\infty f(z)\, {}^{(\Gamma)}\aleph^{m+1,n}_{p_i+1,q_i+1,\tau_i;\kappa}\left[vz\,\middle|\, \begin{matrix} T_1, (\alpha+\beta, h) \\ \\ (\alpha, h), T_2 \end{matrix}\right]\mathrm{d}z \tag{3.1}$$

where T_1 *and* T_2 *are same as those in* Theorem 2.1, *provided that* $\Re\left\{\alpha - h\left(\frac{a_j-1}{\mathcal{B}_j}\right)\right\} > 0$, $j = 2\ldots n$ *and other restrictions would be deducible from those in* (1.1).

Proof: Substituting $z = r^2$ in (2.1) and then multiplying both sides by $rf(r^2)$ and integrating between the limits $(0, \infty)$ with respect to r, we have

$$\int_0^\infty rf(r^2)\mathrm{d}r \int_0^{\frac{\pi}{2}} \sin^{2\alpha-1}\theta \cos^{2\beta-1}\theta \; {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;\kappa}\left[vr^2(\sin\theta)^{-2h} \,\middle|\, \begin{matrix} T_1 \\ T_2 \end{matrix} \right] \mathrm{d}\theta$$
$$= \frac{\Gamma(\beta)}{2} \int_0^\infty rf(r^2)\, {}^{(\Gamma)}\aleph^{m+1,n}_{p_i+1,q_i+1,\tau_i;\kappa}\left[vr^2 \,\middle|\, \begin{matrix} T_1, (\alpha+\beta, h) \\ (\alpha, h), T_2 \end{matrix} \right] \mathrm{d}r.$$

Now on setting $x = r\cos\theta$, $y = r\sin\theta$ and making some simplifications, we get the required result (3.1). □

Using the same method as in Theorem 3.1, we may reach the conclusion for the incomplete Aleph-function (1.3) as stated in the subsequent theorem.

Theorem 3.2 *Let $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and h be a positive integer, then the following integral relation holds true*

$$\int_0^\infty \int_0^\infty x^{2\beta-1} y^{2\alpha-1} (x^2+y^2)^{1-\alpha-\beta} \; {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;\kappa}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}} \,\middle|\, \begin{matrix} T_1 \\ T_2 \end{matrix} \right] f(x^2+y^2)\mathrm{d}y\mathrm{d}x$$
$$= \frac{\Gamma(\beta)}{4} \int_0^\infty f(z)\, {}^{(\gamma)}\aleph^{m+1,n}_{p_i+1,q_i+1,\tau_i;\kappa}\left[vz \,\middle|\, \begin{matrix} T_1, (\alpha+\beta, h) \\ (\alpha, h), T_2 \end{matrix} \right] \mathrm{d}z \tag{3.2}$$

where T_1 and T_2 are same as those in Theorem 2.1*, provided that $\Re\left\{\alpha - h\left(\frac{a_j-1}{\mathcal{B}_j}\right)\right\} > 0$, $j = 2 \ldots n$ and other restrictions would be deducible from those in* (1.1).

4. Applications Based on the Integral Relations of Incomplete Aleph-Functions

For practical use, double integrals can be solved more easily by choosing a suitable form for $f(z)$. This means that the integral on the right side of equation (3.1) is either already known or becomes easier to evaluate. Below, we give some examples to show how this can be applied.

Theorem 4.1 *Let $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and h be a positive integer, then the following result holds true*

$$\int_0^\infty \int_0^\infty x^{2\beta-1} y^{2\alpha-1} (x^2+y^2)^{-\alpha-\beta} \sin(x^2+y^2)^{\frac{1}{\sigma}} J_\nu (x^2+y^2)^{\frac{1}{\sigma}}$$
$$\times {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;\kappa}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}} \,\middle|\, \begin{matrix} T_1 \\ T_2 \end{matrix} \right] \mathrm{d}y\mathrm{d}x$$
$$= 2^{\nu-3}\sigma\Gamma(\beta)\, {}^{(\Gamma)}\aleph^{m+2,n+1}_{p_i+4,q_i+2,\tau_i;\kappa}\left[v \,\middle|\, \begin{matrix} \omega_1, T_1, \omega_2 \\ \omega_3, T_2 \end{matrix} \right] \tag{4.1}$$

where $\omega_1 = \left(\frac{1}{2} + \frac{\nu}{2} + \sigma, \frac{\sigma}{2}\right)$, $\omega_2 = (\alpha+\beta, h), (1+\nu-2\sigma, \sigma), \left(1 - \frac{\nu}{2} - \sigma, \frac{\sigma}{2}\right)$
and $\omega_3 = (\alpha, h), \left(\frac{1}{2} - 2\sigma, \sigma\right)$, *$T_1$ and T_2 are same as those in* Theorem 2.1, *provided that*

(i) $\Re\left\{\nu + 2\sigma + \sigma\left(\frac{g_j}{\mathcal{G}_{ji}}\right)\right\} > -1$, $j = 1 \ldots m$.

(ii) $\Re\left\{\alpha - h\left(\frac{a_j-1}{\mathcal{B}_j}\right)\right\} > 0$, $j = 2 \ldots n$.

(iii) $\Re\left\{4\sigma + 2\sigma\left(\frac{a_j-1}{\mathcal{B}_j}\right)\right\} < 1$, $j = 2 \ldots n$.

and other restrictions would be deducible from those in (1.1).

Proof: If we take $f(z) = z\,\sin\left(z^{\frac{1}{\sigma}}\right) J_\nu\left(z^{\frac{1}{\sigma}}\right)$, therefore, we have from relation (3.1).

$$\begin{aligned}\int_0^\infty\int_0^\infty & x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^{-\alpha-\beta}\sin(x^2+y^2)^{\frac{1}{\sigma}}J_\nu(x^2+y^2)^{\frac{1}{\sigma}}\\ &\times {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;\kappa}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}}\,\middle|\,\begin{matrix}T_1\\ \\ T_2\end{matrix}\right]\mathrm{d}y\mathrm{d}x\\ &=\frac{\Gamma(\beta)}{4}\int_0^\infty z\,\sin\left(z^{\frac{1}{\sigma}}\right)J_\nu\left(z^{\frac{1}{\sigma}}\right){}^{(\Gamma)}\aleph^{m+1,n}_{p_i+1,q_i+1,\tau_i;\kappa}\left[vz\,\middle|\,\begin{matrix}T_1,(\alpha+\beta,h)\\ \\ (\alpha,h),T_2\end{matrix}\right]\mathrm{d}z.\end{aligned}\tag{4.2}$$

Let $\mathcal{L}_2$ be the left-handed member of (4.2). Expressing incomplete Aleph-function in its contour integral from (1.1) in right-handed member of (4.2), interchanging the order of integration and setting $z=(ax)^\sigma$, we have

$$\begin{aligned}\mathcal{L}_2 &= \frac{\Gamma(\beta)}{4}\int_L \phi_1(s,\xi)v^{-s}\frac{\Gamma(\alpha+hs)}{\Gamma(\alpha+\beta+hs)}\\ &\times\sigma\,a^{2\sigma-\sigma s}\left\{\int_0^\infty x^{2\sigma-\sigma s-1}\sin(ax)\,J_\nu(ax)\,\mathrm{d}x\right\}\mathrm{d}s.\end{aligned}\tag{4.3}$$

Recall the following Mellin transform (see, e.g., [32, p. 328, Entry (10)]):

$$\begin{aligned}\mathscr{M}\{\sin(ax)\,J_\nu(ax)\} &= \int\limits_0^\infty x^{\rho-1}\sin(ax)\,J_\nu(ax)\,\mathrm{d}x\\ &=\frac{2^{\nu-1}\Gamma(\frac{1}{2}-\rho)\Gamma(\frac{1}{2}+\frac{\nu}{2}+\frac{\rho}{2})}{a^\rho\Gamma(1+\nu-\rho)\Gamma(1-\frac{\nu}{2}-\frac{\rho}{2})},\\ &\left(a>0,\ -1<\Re(\nu)<\Re(\rho)<\frac{1}{2}\right),\end{aligned}\tag{4.4}$$

using (4.4) to evaluate the inner integral in (4.3), then, upon expressing in terms of the incomplete Aleph-function (1.1), leads to the result in (4.1). This completes the proof.

□

Theorem 4.2 *Let $\Re(\alpha)>0$, $\Re(\beta)>0$ and h be a positive integer, then the following result holds true*

$$\begin{aligned}\int_0^\infty\int_0^\infty & x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^{\frac{1}{2h}(1-u)-\alpha-\beta}J_\mu(x^2+y^2)^{\frac{1}{2h}}J_\nu(x^2+y^2)^{\frac{1}{2h}}\\ &\times {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;\kappa}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}}\,\middle|\,\begin{matrix}T_1\\ \\ T_2\end{matrix}\right]\mathrm{d}y\mathrm{d}x\\ &=\frac{h\,\Gamma(\beta)}{2^{u+1}}\,{}^{(\Gamma)}\aleph^{m+2,n+1}_{p_i+5,q_i+2,\tau_i;\kappa}\left[2^hv\,\middle|\,\begin{matrix}\omega_4,T_1,\omega_5\\ \\ \omega_6,T_2\end{matrix}\right]\end{aligned}\tag{4.5}$$

where $\omega_4=\left(\frac{1+u-\nu-\mu}{2},h\right)$, $\omega_5=(\alpha+\beta,h),\left(\frac{1+u+\mu+\nu}{2},h\right),\left(\frac{1+u-\mu+\nu}{2},h\right),\left(\frac{1+u+\mu-\nu}{2},h\right)$ *and* $\omega_6=(\alpha,h),(u,2h)$, T_1 *and* T_2 *are same as those in* Theorem 2.1, *provided that*

(i) $\Re\left\{\mu+\nu-u+2h\left(\frac{g_j}{\mathcal{G}_{ji}}\right)\right\}>-1,\ j=1\ldots m.$

(ii) $\Re\left\{u-2h\left(\frac{a_j-1}{\mathcal{B}_j}\right)\right\}>0,\ j=2\ldots n.$

(iii) $\Re\left\{\alpha - h\left(\frac{a_j - 1}{\mathcal{B}_j}\right)\right\} > 0,\ j = 2 \ldots n.$

and other restrictions would be deducible from those in (1.1).

Proof: On taking $f(z) = z^{\frac{1}{2h}(1-u)-1} J_\mu(z^{\frac{1}{2h}})\, J_\nu(z^{\frac{1}{2h}})$ and using the following formula (see, e.g., [33, p. 342, Entry (24)]):

$$\int_0^\infty x^{-\rho} J_\mu(ax)\, J_\nu(ax)\, \mathrm{d}x = \frac{2^{-\rho}\,\Gamma(\rho)\,\Gamma\left(\frac{\mu+\nu-\rho+1}{2}\right)}{\Gamma\left(\frac{\mu+\nu+\rho+1}{2}\right)\,\Gamma\left(\frac{\rho-\mu+\nu+1}{2}\right)\,\Gamma\left(\frac{\mu-\nu+\rho+1}{2}\right)}, \tag{4.6}$$
$$\left(0 < \Re(\rho) < \Re(\mu+\nu)+1\right),$$

result in (4.5) can easily be established. □

Theorem 4.3 *Let* $\Re(\alpha) > 0$, $\Re(\beta) > 0$ *and* h *be a positive integer, then the following result holds true*

$$\begin{aligned} &\int_0^\infty \int_0^\infty x^{2\beta-1} y^{2\alpha-1} (x^2+y^2)^{\frac{\lambda+1}{\sigma}-\alpha-\beta}\, {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \middle| -\delta(x^2+y^2)^{\frac{1}{\sigma}}\right] \\ &\quad \times {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;\kappa}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}} \middle| \begin{matrix} T_1 \\ T_2 \end{matrix}\right] \mathrm{d}y\mathrm{d}x \\ &\quad = \frac{\sigma\,\Gamma(\beta)\,\Gamma(c)}{4\,\Gamma(a)\,\Gamma(b)\,\delta^{\lambda+1}}\, {}^{(\Gamma)}\aleph^{m+3,n+1}_{p_i+3,q_i+3,\tau_i;\kappa}\left[\frac{v}{\delta^\sigma} \middle| \begin{matrix} (-\lambda,\sigma), T_1, (\alpha+\beta, h), (c-\lambda-1,\sigma) \\ (a-\lambda-1,\sigma), (b-\lambda-1,\sigma), (\alpha,h), T_2 \end{matrix}\right] \end{aligned} \tag{4.7}$$

where T_1 *and* T_2 *are same as those in* Theorem 2.1, *provided that*

(i) $\Re\left\{\lambda + \sigma\left(\frac{g_j}{\mathcal{G}_{ji}}\right)\right\} > -1,\ j = 1 \ldots m.$

(ii) $\Re\left\{\lambda + 1 + \sigma\left(\frac{a_j - 1}{\mathcal{B}_j}\right)\right\} < a,\ j = 2 \ldots n.$

(iii) $\Re\left\{\lambda + 1 + \sigma\left(\frac{a_j - 1}{\mathcal{B}_j}\right)\right\} < b,\ j = 2 \ldots n.$

and other restrictions would be deducible from those in (1.1).

Proof: On taking $f(z) = z^{\frac{\lambda+1}{\sigma}-1}{}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \middle| -\delta z^{\frac{1}{\sigma}}\right]$ and using the following Mellin transform formula of Gauss hypergeometric function with a negative argument (see, e.g., [32, p. 336, Entry (3)], for new research related to transforms, see [34], [35]):

$$\begin{aligned} \mathscr{M}\left\{{}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \middle| -x\right]\right\} &= \int_0^\infty x^{\rho-1}{}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \middle| -x\right] \mathrm{d}x \\ &= \frac{\mathrm{B}(\rho, a-\rho)\,\mathrm{B}(\rho, b-\rho)}{\mathrm{B}(\rho, c-\rho)}, \end{aligned} \tag{4.8}$$
$$\left(0 < \Re(\rho) < \min(\Re(a), \Re(b))\right),$$

result in (4.7) can easily be established. □

Remark 4.1

By following the same method used in Theorems 4.1, 4.2, and 4.3, we can also derive similar types of results for the incomplete Aleph-function given in equation (1.3).

5. Particular Cases

The incomplete Aleph-function is considered a broad and unified function since many well-known special functions can be obtained from it by adjusting its parameters. Because of this, it is highly flexible and useful. In this section, we demonstrate that by selecting suitable parameter values, how the results of this study lead to several significant outcomes as particular cases.

Letting $\tau_i \to 1$, in Theorem 3.1, we find an integral relation for incomplete I-function in (1.13) as in the following corollary.

Corollary 5.1 *Let $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and h be a positive integer, then the following integral relation holds true*

$$\begin{aligned}\int_0^\infty \int_0^\infty x^{2\beta-1} y^{2\alpha-1} (x^2+y^2)^{1-\alpha-\beta}\, {}^{(\Gamma)}I^{m,n}_{p_i,q_i;\kappa}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}} \,\middle|\, \begin{matrix} T_3 \\ \\ T_4 \end{matrix}\right] f(x^2+y^2)\mathrm{d}y\mathrm{d}x \\ = \frac{\Gamma(\beta)}{4}\int_0^\infty f(z)\, {}^{(\Gamma)}I^{m+1,n}_{p_i+1,q_i+1;\kappa}\left[vz \,\middle|\, \begin{matrix} T_3, (\alpha+\beta, h) \\ \\ (\alpha, h), T_4 \end{matrix}\right]\mathrm{d}z\end{aligned} \tag{5.1}$$

where $T_3 = (a_1, \mathcal{B}_1, \xi),\, (a_j, \mathcal{B}_j)_{2,n},\, (a_{ji}, \mathcal{B}_{ji})_{n+1,p_i}$
and $T_4 = (g_j, \mathcal{G}_j)_{1,m},\, (g_{ji}, \mathcal{G}_{ji})_{m+1,q_i}$,
provided that $\Re\left\{\alpha - h\left(\frac{a_j-1}{\mathcal{B}_j}\right)\right\} > 0$, $j = 2\ldots n$ *and other restrictions would be deducible from those in* (1.1).

Letting $\tau_i \to 1$ and $\kappa \to 1$ in Theorem 3.1, we find an integral relation for incomplete H-function in (1.15) as in the following corollary.

Corollary 5.2 *Let $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and h be a positive integer, then the following integral relation holds true*

$$\begin{aligned}\int_0^\infty \int_0^\infty x^{2\beta-1} y^{2\alpha-1} (x^2+y^2)^{1-\alpha-\beta}\, {}^{(\Gamma)}H^{m,n}_{p,q}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}} \,\middle|\, \begin{matrix} T_5 \\ \\ T_6 \end{matrix}\right] f(x^2+y^2)\mathrm{d}y\mathrm{d}x \\ = \frac{\Gamma(\beta)}{4}\int_0^\infty f(z)\, {}^{(\Gamma)}H^{m+1,n}_{p+1,q+1}\left[vz \,\middle|\, \begin{matrix} T_5, (\alpha+\beta, h) \\ \\ (\alpha, h), T_6 \end{matrix}\right]\mathrm{d}z\end{aligned} \tag{5.2}$$

where $T_5 = (a_1, \mathcal{B}_1, \xi),\, (a_j, \mathcal{B}_j)_{2,p}$
and $T_6 = (g_j, \mathcal{G}_j)_{1,q}$
provided that $\Re\left\{\alpha - h\left(\frac{a_j-1}{\mathcal{B}_j}\right)\right\} > 0$, $j = 2\ldots n$ *and other restrictions would be deducible from those in* (1.1).

Letting $\tau_i \to 1$, in Theorems 4.1, 4.2, and 4.3, the results turn out to be that for the incomplete I–function in (1.13) as in the following corollaries.

Corollary 5.3 *Let $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and h be a positive integer, then the following result holds true*

$$\begin{aligned}\int_0^\infty \int_0^\infty x^{2\beta-1} y^{2\alpha-1} (x^2+y^2)^{-\alpha-\beta} \sin(x^2+y^2)^{\frac{1}{\sigma}} J_\nu (x^2+y^2)^{\frac{1}{\sigma}} \\ \times\, {}^{(\Gamma)}I^{m,n}_{p_i,q_i;\kappa}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}} \,\middle|\, \begin{matrix} T_3 \\ \\ T_4 \end{matrix}\right]\mathrm{d}y\mathrm{d}x \\ = 2^{\nu-3}\sigma\Gamma(\beta)\, {}^{(\Gamma)}I^{m+2,n+1}_{p_i+4,q_i+2,;\kappa}\left[v \,\middle|\, \begin{matrix} \omega_1, T_3, \omega_2 \\ \\ \omega_3, T_4 \end{matrix}\right]\end{aligned} \tag{5.3}$$

where $\omega_1 = \left(\frac{1}{2}+\frac{\nu}{2}+\sigma, \frac{\sigma}{2}\right)$, $\omega_2 = (\alpha+\beta, h), (1+\nu-2\sigma, \sigma), \left(1-\frac{\nu}{2}-\sigma, \frac{\sigma}{2}\right)$ *and* $\omega_3 = (\alpha, h), \left(\frac{1}{2}-2\sigma, \sigma\right)$, T_3 *and* T_4 *are same as those in* Corollary 5.1. *The convergence conditions would follow from those in* Theorem 4.1 *and other restrictions would be deducible from those in* (1.1).

Corollary 5.4 *Let* $\Re(\alpha) > 0$, $\Re(\beta) > 0$ *and* h *be a positive integer, then the following result holds true*

$$\int_0^\infty \int_0^\infty x^{2\beta-1} y^{2\alpha-1} (x^2+y^2)^{\frac{1}{2h}(1-u)-\alpha-\beta} J_\mu (x^2+y^2)^{\frac{1}{2h}} J_\nu (x^2+y^2)^{\frac{1}{2h}} \times {}^{(\Gamma)}I^{m,n}_{p_i,q_i;\kappa}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}}\,\middle|\,\begin{matrix} T_3 \\ T_4 \end{matrix}\right] \mathrm{d}y\mathrm{d}x = \frac{h\,\Gamma(\beta)}{2^{u+1}}\,{}^{(\Gamma)}I^{m+2,n+1}_{p_i+5,q_i+2;\kappa}\left[2^h v\,\middle|\,\begin{matrix} \omega_4, T_3, \omega_5 \\ \omega_6, T_4 \end{matrix}\right] \tag{5.4}$$

where $\omega_4 = \left(\frac{1+u-\nu-\mu}{2}, h\right)$, $\omega_5 = (\alpha+\beta, h), \left(\frac{1+u+\mu+\nu}{2}, h\right), \left(\frac{1+u-\mu+\nu}{2}, h\right), \left(\frac{1+u+\mu-\nu}{2}, h\right)$ *and* $\omega_6 = (\alpha, h), (u, 2h)$, T_3 *and* T_4 *are same as those in* Corollary 5.1. *The convergence conditions would follow from those in* Theorem 4.2 *and other restrictions would be deducible from those in* (1.1).

Corollary 5.5 *Let* $\Re(\alpha) > 0$, $\Re(\beta) > 0$ *and* h *be a positive integer, then the following result holds true*

$$\int_0^\infty \int_0^\infty x^{2\beta-1} y^{2\alpha-1} (x^2+y^2)^{\frac{\lambda+1}{\sigma}-\alpha-\beta} {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, -\delta(x^2+y^2)^{\frac{1}{\sigma}}\right] \times {}^{(\Gamma)}I^{m,n}_{p_i,q_i;\kappa}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}}\,\middle|\,\begin{matrix} T_3 \\ T_4 \end{matrix}\right] \mathrm{d}y\mathrm{d}x = \frac{\sigma\,\Gamma(\beta)\,\Gamma(c)}{4\,\Gamma(a)\,\Gamma(b)\,\delta^{\lambda+1}}\,{}^{(\Gamma)}I^{m+3,n+1}_{p_i+3,q_i+3;\kappa}\left[\frac{v}{\delta^\sigma}\,\middle|\,\begin{matrix} (-\lambda, \sigma), T_3, (\alpha+\beta, h), (c-\lambda-1, \sigma) \\ (a-\lambda-1, \sigma), (b-\lambda-1, \sigma), (\alpha, h), T_4 \end{matrix}\right] \tag{5.5}$$

where T_3 *and* T_4 *are same as those in* Corollary 5.1. *The convergence conditions would follow from those in* Theorem 4.3 *and other restrictions would be deducible from those in* (1.1).

Letting $\tau_i \to 1$, and $\kappa \to 1$ in Theorems 4.1, 4.2, and 4.3, the results turn out to be that for the incomplete H–function in (1.15) as in the following corollaries.

Corollary 5.6 *Let* $\Re(\alpha) > 0$, $\Re(\beta) > 0$ *and* h *be a positive integer, then the following result holds true*

$$\int_0^\infty \int_0^\infty x^{2\beta-1} y^{2\alpha-1} (x^2+y^2)^{-\alpha-\beta} \sin(x^2+y^2)^{\frac{1}{\sigma}} J_\nu (x^2+y^2)^{\frac{1}{\sigma}} \times {}^{(\Gamma)}H^{m,n}_{p,q}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}}\,\middle|\,\begin{matrix} T_5 \\ T_6 \end{matrix}\right] \mathrm{d}y\mathrm{d}x = 2^{\nu-3}\sigma\Gamma(\beta)\,{}^{(\Gamma)}H^{m+2,n+1}_{p+4,q+2}\left[v\,\middle|\,\begin{matrix} \omega_1, T_5, \omega_2 \\ \omega_3, T_6 \end{matrix}\right] \tag{5.6}$$

where $\omega_1 = \left(\frac{1}{2}+\frac{\nu}{2}+\sigma, \frac{\sigma}{2}\right)$, $\omega_2 = (\alpha+\beta, h), (1+\nu-2\sigma, \sigma), \left(1-\frac{\nu}{2}-\sigma, \frac{\sigma}{2}\right)$ *and* $\omega_3 = (\alpha, h), \left(\frac{1}{2}-2\sigma, \sigma\right)$, T_5 *and* T_6 *are same as those in* Corollary 5.2. *The convergence conditions would follow from those in* Theorem 4.1 *and other restrictions would be deducible from those in* (1.1).

Corollary 5.7 *Let $\Re(\alpha)>0$, $\Re(\beta)>0$ and h be a positive integer, then the following result holds true*

$$\begin{aligned}\int_0^\infty\int_0^\infty & x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^{\frac{1}{2h}(1-u)-\alpha-\beta}J_\mu(x^2+y^2)^{\frac{1}{2h}}J_\nu(x^2+y^2)^{\frac{1}{2h}}\\ &\times {}^{(\Gamma)}H^{m,n}_{p,q}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}}\,\middle|\,\begin{matrix}T_5\\ T_6\end{matrix}\right]\mathrm{d}y\mathrm{d}x\\ &=\frac{h\,\Gamma(\beta)}{2^{u+1}}\,{}^{(\Gamma)}H^{m+2,n+1}_{p+5,q+2}\left[2^h v\,\middle|\,\begin{matrix}\omega_4,T_5,\omega_5\\ \omega_6,T_6\end{matrix}\right]\end{aligned}\tag{5.7}$$

where $\omega_4=\left(\frac{1+u-\nu-\mu}{2},h\right)$, $\omega_5=(\alpha+\beta,h),\left(\frac{1+u+\mu+\nu}{2},h\right),\left(\frac{1+u-\mu+\nu}{2},h\right),\left(\frac{1+u+\mu-\nu}{2},h\right)$ *and* $\omega_6=(\alpha,h),(u,2h)$, T_5 *and* T_6 *are same as those in* Corollary 5.2. *The convergence conditions would follow from those in* Theorem 4.2 *and other restrictions would be deducible from those in* (1.1).

Corollary 5.8 *Let $\Re(\alpha)>0$, $\Re(\beta)>0$ and h be a positive integer, then the following result holds true*

$$\begin{aligned}\int_0^\infty\int_0^\infty & x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^{\frac{\lambda+1}{\sigma}-\alpha-\beta}{}_2F_1\left[\begin{matrix}a,b\\ c\end{matrix}\,\middle|-\delta(x^2+y^2)^{\frac{1}{\sigma}}\right]\\ &\times {}^{(\Gamma)}H^{m,n}_{p,q}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}}\,\middle|\,\begin{matrix}T_5\\ T_6\end{matrix}\right]\mathrm{d}y\mathrm{d}x\\ &=\frac{\sigma\,\Gamma(\beta)\,\Gamma(c)}{4\,\Gamma(a)\,\Gamma(b)\,\delta^{\lambda+1}}\,{}^{(\Gamma)}H^{m+3,n+1}_{p+3,q+3}\left[\frac{v}{\delta^\sigma}\,\middle|\,\begin{matrix}(-\lambda,\sigma),T_5,(\alpha+\beta,h),(c-\lambda-1,\sigma)\\ (a-\lambda-1,\sigma),(b-\lambda-1,\sigma),(\alpha,h),T_6\end{matrix}\right]\end{aligned}\tag{5.8}$$

where T_5 and T_6 are same as those in Corollary 5.2. *The convergence conditions would follow from those in* Theorem 4.3 *and other restrictions would be deducible from those in* (1.1).

Letting $\xi\to 0$, in Theorem 3.1, we find an integral relation for Aleph ($\aleph$)-function (see, e.g., [5], [29]) as in the following corollary.

Corollary 5.9 *Let $\Re(\alpha)>0$, $\Re(\beta)>0$ and h be a positive integer, then the following integral relation holds true*

$$\begin{aligned}\int_0^\infty\int_0^\infty & x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^{1-\alpha-\beta}\,\aleph^{m,n}_{p_i,q_i,\tau_i;\kappa}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}}\,\middle|\,\begin{matrix}T_7\\ T_8\end{matrix}\right]f(x^2+y^2)\mathrm{d}y\mathrm{d}x\\ &=\frac{\Gamma(\beta)}{4}\int_0^\infty f(z)\,\aleph^{m+1,n}_{p_i+1,q_i+1,\tau_i;\kappa}\left[vz\,\middle|\,\begin{matrix}T_7,(\alpha+\beta,h)\\ (\alpha,h),T_8\end{matrix}\right]\mathrm{d}z\end{aligned}\tag{5.9}$$

where $T_7=(a_j,\mathcal{B}_j)_{1,n},\,[\tau_i\,(a_{ji},\mathcal{B}_{ji})]_{n+1,p_i}$
and $T_8=(g_j,\mathcal{G}_j)_{1,m},\,[\tau_i\,(g_{ji},\mathcal{G}_{ji})]_{m+1,q_i}$
provided that $\Re\left\{\alpha-h\left(\frac{a_j-1}{\mathcal{B}_j}\right)\right\}>0$, $j=1\ldots n$ *and other restrictions would be deducible from those in* (1.1).

Letting $\xi\to 0$, in Theorems 4.1, 4.2, and 4.3, the results turn out to be that for Aleph ($\aleph$)-function (see, e.g., [5], [29]) as in the following corollaries.

Corollary 5.10 *Let $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and h be a positive integer, then the following result holds true*

$$\begin{aligned}\int_0^\infty \int_0^\infty & x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^{-\alpha-\beta}\sin(x^2+y^2)^{\frac{1}{\sigma}} J_\nu(x^2+y^2)^{\frac{1}{\sigma}} \\ & \times \aleph^{m,n}_{p_i,q_i,\tau_i;\kappa}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}} \,\middle|\, \begin{matrix} T_7 \\ \\ T_8 \end{matrix}\right] \mathrm{d}y\mathrm{d}x \\ & = 2^{\nu-3}\sigma\Gamma(\beta)\,\aleph^{m+2,n+1}_{p_i+4,q_i+2,\tau_i;\kappa}\left[v \,\middle|\, \begin{matrix} \omega_1, T_7, \omega_2 \\ \\ \omega_3, T_8 \end{matrix}\right]\end{aligned} \tag{5.10}$$

where $\omega_1 = \left(\frac{1}{2}+\frac{\nu}{2}+\sigma, \frac{\sigma}{2}\right)$, $\omega_2 = (\alpha+\beta, h), (1+\nu-2\sigma, \sigma), \left(1-\frac{\nu}{2}-\sigma, \frac{\sigma}{2}\right)$
and $\omega_3 = (\alpha, h), \left(\frac{1}{2}-2\sigma, \sigma\right)$, *$T_7$ and T_8 are same as those in* Corollary 5.9, *provided that*

(i) $\Re\left\{\nu + 2\sigma + \sigma\left(\frac{g_j}{\mathcal{G}_{ji}}\right)\right\} > -1,\ j = 1 \ldots m.$

(ii) $\Re\left\{\alpha - h\left(\frac{a_j - 1}{\mathcal{B}_j}\right)\right\} > 0,\ j = 1 \ldots n.$

(iii) $\Re\left\{4\sigma + 2\sigma\left(\frac{a_j - 1}{\mathcal{B}_j}\right)\right\} < 1,\ j = 1 \ldots n.$

and other restrictions would be deducible from those in (1.1).

Corollary 5.11 *Let $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and h be a positive integer, then the following result holds true*

$$\begin{aligned}\int_0^\infty \int_0^\infty & x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^{\frac{1}{2h}(1-u)-\alpha-\beta} J_\mu(x^2+y^2)^{\frac{1}{2h}} J_\nu(x^2+y^2)^{\frac{1}{2h}} \\ & \times \aleph^{m,n}_{p_i,q_i,\tau_i;\kappa}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}} \,\middle|\, \begin{matrix} T_7 \\ \\ T_8 \end{matrix}\right] \mathrm{d}y\mathrm{d}x \\ & = \frac{h\,\Gamma(\beta)}{2^{u+1}}\,\aleph^{m+2,n+1}_{p_i+5,q_i+2,\tau_i;\kappa}\left[2^h v \,\middle|\, \begin{matrix} \omega_4, T_7, \omega_5 \\ \\ \omega_6, T_8 \end{matrix}\right]\end{aligned} \tag{5.11}$$

where $\omega_4 = \left(\frac{1+u-\nu-\mu}{2}, h\right)$, $\omega_5 = (\alpha+\beta, h), \left(\frac{1+u+\mu+\nu}{2}, h\right), \left(\frac{1+u-\mu+\nu}{2}, h\right), \left(\frac{1+u+\mu-\nu}{2}, h\right)$
and $\omega_6 = (\alpha, h), (u, 2h)$, *$T_7$ and T_8 are same as those in* Corollary 5.9, *provided that*

(i) $\Re\left\{\mu + \nu - u + 2h\left(\frac{g_j}{\mathcal{G}_{ji}}\right)\right\} > -1,\ j = 1 \ldots m.$

(ii) $\Re\left\{u - 2h\left(\frac{a_j - 1}{\mathcal{B}_j}\right)\right\} > 0,\ j = 1 \ldots n.$

(iii) $\Re\left\{\alpha - h\left(\frac{a_j - 1}{\mathcal{B}_j}\right)\right\} > 0,\ j = 1 \ldots n.$

and other restrictions would be deducible from those in (1.1).

Corollary 5.12 *Let $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and h be a positive integer, then the following result holds true*

$$\begin{aligned}\int_0^\infty \int_0^\infty & x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^{\frac{\lambda+1}{\sigma}-\alpha-\beta}\,{}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, -\delta(x^2+y^2)^{\frac{1}{\sigma}}\right] \\ & \times \aleph^{m,n}_{p_i,q_i,\tau_i;\kappa}\left[\frac{v(x^2+y^2)^{h+1}}{y^{2h}} \,\middle|\, \begin{matrix} T_7 \\ \\ T_8 \end{matrix}\right] \mathrm{d}y\mathrm{d}x \\ & = \frac{\sigma\,\Gamma(\beta)\,\Gamma(c)}{4\,\Gamma(a)\,\Gamma(b)\,\delta^{\lambda+1}}\,\aleph^{m+3,n+1}_{p_i+3,q_i+3,\tau_i;\kappa}\left[\frac{v}{\delta^\sigma} \,\middle|\, \begin{matrix} (-\lambda, \sigma), T_7, (\alpha+\beta, h), (c-\lambda-1, \sigma) \\ \\ (a-\lambda-1, \sigma), (b-\lambda-1, \sigma), (\alpha, h), T_8 \end{matrix}\right]\end{aligned} \tag{5.12}$$

where T_7 and T_8 are same as those in Corollary 5.9, *provided that* *(i)* $\Re\left\{\lambda+\sigma\left(\frac{g_j}{\mathcal{G}_{ji}}\right)\right\}>-1,\ j=1\ldots m.$

(ii) $\Re\left\{\lambda+1+\sigma\left(\frac{a_j-1}{\mathcal{B}_j}\right)\right\}<a,\ j=1\ldots n.$

(iii) $\Re\left\{\lambda+1+\sigma\left(\frac{a_j-1}{\mathcal{B}_j}\right)\right\}<b,\ j=1\ldots n.$

and other restrictions would be deducible from those in (1.1).

Remark 5.1

By appropriately specializing the parameters in Theorems 3.1, 4.1, 4.2, and 4.3, analogous results can also be obtained for Saxena's I-function and Fox's H-function.

6. Conclusion

In this work, we explored new integral relations for incomplete Aleph-functions and showed how these relations can be used to solve certain double integrals more easily. By applying known mathematical results, we simplified integrals involving Bessel functions, hypergeometric functions, and incomplete Aleph-functions. The formulas we developed are broad and flexible, and the special examples we provided demonstrate how these results can be applied in practical situations. Overall, this work contributes to the ongoing development of Aleph-function theory and offers practical tools for future analytical investigations.

References

1. N. Südland, B. Baumann and T.F. Nonnenmacher, Open problem: Who knows about the Aleph-functions, *Fract. Calc. Appl. Anal.* **1**(4) (1998), 401–402.

2. N. Südland, B. Baumann and T. F. Nannenmacher, *Fractional driftless Fokker-Planck equation with power law diffusion coefficients* in V. G. Gangha, E. W. Mayr, W. G. Vorozhtsov (Eds.), Computer Algebra in Scientific Computing (CASC Konstanz 2001), Springer, Berlin, 2001, 513-525.

3. V.P. Saxena, Formal solution of certain new pair of dual integral equations involving H-functions, *Proc. Nat. Acad. Sci. India* **52**(A) (1982), III 366-375.

4. A.M. Mathai, R.K. Saxena and H.J. Haubold, *The H-function: Theory and Applications*, Springer, New York, NY, USA, 2010.

5. R.K. Saxena and T.K. Pogány, Mathieu-type Series for the ℵ-function occurring in Fokker-Planck equation, *Eur. J. Pure Appl. Math.* **3**(6) (2010), 980–988.

6. V.B.L. Chaurasia and Y. Singh, Lambert's Law and Aleph function, *Int. J. Modern Math. Sci.* **4**(2) (2012), 84–88.

7. B. Prakash and S. Agnihotri, Aleph function with transformation of double integral, *Journal of Pharmaceutical Negative Results* **13**(8) (2022), 1580–1583. 10.47750/pnr.2022.13.S08.191

8. V.B.L. Chaurasia and V. Gill, Generalized elliptic-type integrals and generating function with Aleph-function, *Gen. Math. Notes* **14**(1) (2013), 21–34.

9. R.S. Dubey, An application of the Aleph (ℵ)-function for detecting glucose supply in human blood, *Int. J. Modern Math. Sci.* **14**(3) (2016), 221–226.

10. F. Ayant and D. Kumar, A unified study of Fourier series involving the Aleph-function and the Kampé de Fériet's function, *Int. J. Math. Trends Technol.* 35(1) (2016), 40–48. 10.14445/22315373/IJMTT-V35P507

11. P. Agarwal, M. Chand, I.O. Kıymaz and A. Çetinkaya, A certain sequence of functions involving the Aleph function, *Open Phys.* **14**(1) (2016), 187-191. https://doi.org/10.1515/phys-2016-0018

12. D.L. Suthar, S. Agarwal and D. Kumar, Certain integrals involving the product of Gaussian hypergeometric function and Aleph function, *Honam Math. J.* **41**(1) (2019), 1-17. http://dx.doi.org/10.5831/HMJ.2019.41.1.1

13. D. Kumar, F. Ayant and F. Uçar, Integral involving Aleph-function and the generalized incomplete hypergeometric function, *TWMS J. Appl. and Eng. Math.* **11**(2) (2021), 408–414.

14. R.K. Saxena and D. Kumar, Generalized fractional calculus of the Aleph-function involving a general class of polynomials, *Acta Math. Sci.* **35B**(5) (2015), 1095-1110.

15. J. Choi and D. Kumar, Certain unified fractional integrals and derivatives for a product of Aleph function and a general class of multivariable polynomials, *J. Inequal. Appl.* **2014** (2014), 15 pages. https://doi.org/10.1186/1029-242X-2014-499

16. S.D. Kharabe, B. Bundela, M.K. Bansal and G.S. Paliwal Pathway Fractional Integral Formulas Associated with the Incomplete ℵ-Functions, *Zhongguo Kuangye Daxue Xuebao(Journal of China University of Mining and Technology)*, **29(3)**, 218-223, 2024.

17. D. Kumar, F. Ayant and D.S. Sachan, Certain unified infinite integrals involving incomplete special functions, *Int. J. App. Comput. Math.*, **11**, 173, (2025). https://doi.org/10.1007/s40819-025-02013-6

18. D. Kumar, F. Ayant and M.S. Bajrolia, Improper integrals involving the incomplete Aleph–functions, *Sohag Journal of Mathematics*, **11**(2) (2025), 23–27.

19. D.S. Sachan, J. Choi and D. Kumar, On the certain properties of Aleph-functions with an application to axisymmetric Dirichlet potential problem, (Communicated).

20. S. Tyagi, M. Jain and J. Singh, Large deflection of a circular plate with incomplete Aleph functions under non-uniform Load, *Int. J. Appl. Comput. Math.* **8** (267) (2022). https://doi.org/10.1007/s40819-022-01450-x

21. D.S. Sachan and F. Ayant, Finite integrals involving incomplete Aleph-functions, extension of the Mittag-Leffler function and elliptic integral of first species, *Jour. of Frac. Calc. and Appl.* **14**(2) (2023), 1–12.

22. D. Kumar and R. Dassani, Solution of integral equations of Fredholm kind involving incomplete ℵ-function, generalized extended Mittag-Leffler function and S-function, *Journal of Computational Analysis and Applications* (JoCAAA), **33**(1A) (2024), 48–58. Retrieved from https://eudoxuspress.com/index.php/pub/article/view/13

23. D.S. Sachan, B. Rajak and F. Ayant, Finite integral involving incomplete Aleph-functions and Fresnel integral, *Jour. of Frac. Calc. and Appl.* **15**(2) (2024), 1–9.

24. Y. Singh, R. Sharma and R. Maanju , Beer-Lambert's law as an application of incomplete Aleph (ℵ) functions, *AIP Conf. Proc.* 2768, 020038 (2023), https://doi.org/10.1063/5.0149300

25. Y. Singh , V. Gill , J. Singh , D. Kumar and I. Khan, Computable generalization of fractional kinetic equation with special functions, *Journal of King Saud University – Science*, **33** (2021) 101221

26. S. Kumar, M.N. Mishra, R.S. Dubey and R. Sharma, Investigation of some finite integrals incorporating incomplete Aleph functions, *International Journal of Mathematics for Industry*, **16**, Supp. 1 (2024) 2450007 (15 pages). https://doi.org/10.1142/S2661335224500072

27. M.K. Bansal, D. Kumar, K.S. Nisar and J. Singh, Certain fractional calculus and integral transform results of incomplete ℵ-functions with applications, *Math. Meth. Appl. Sci.* **43**(8) (2020), 5602–5614. https://doi.org/10.1002/mma.6299

28. H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers: Amsterdam, The Netherland; London, UK; New York, NY, USA, 2012.

29. R.K. Saxena and T.K. Pogány, On fractional integration formulae for Aleph functions, *Appl. Math. Comput.* **218**(3) (2011), 985–990. https://doi.org/10.1016/j.amc.2011.03.026

30. V.P. Saxena, *The I-function*, Anamaya Publishers, New Delhi, 2008.

31. C. Fox, The G and H-functions as symmetric Fourier kernels, *Trans. Amer. Math. Soc.* **98**(3) (1961), 395–429. https://doi.org/10.2307/1993339

32. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Tables of Integral Transforms*, Vol. I, McGraw-Hill, New York, Toronto, London, 1954.

33. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Tables of Integral Transforms*, Vol. II, Mc Graw Hill, New York, Toronto, London, 1954.

34. J. Maan and E.R. Negrín, Parseval–Goldstein Type Theorems for the Index ${}_2F_1$ -Transform. Int. J. Appl. Comput. Math 10, 69 (2024). https://doi.org/10.1007/s40819-024-01713-9

35. J. Maan and E.R. Negrín, Parseval-Goldstein type theorems for the index Whittaker transform. Integral Transforms and Special Functions, 36(1) (2025), 53–63. https://doi.org/10.1080/10652469.2024.2386664

Dheerandra Shanker Sachan,

Department of Mathematics,

St. Mary's PG College, Vidisha-464001, (M.P.), India.

E-mail address: sachan.dheerandra17@gmail.com

and

P. Agarwal,

Department of Mathematic,

Anand International College of Engineering,

Jaipur-303012(Raj.), India

E-mail address: goyal.praveen2011@gmail.com *and*

M. K. Bansal,
Department of Mathematics,
Jaypee Institute of Information Technology,
Noida-201309 (U.P.), India
E-mail address: bansalmanish443@gmail.com