Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 3** : 1–8.
ISSN-0037-8712
doi:10.5269/bspm.79220

On the Existence of Entropy Solutions for Some Nonlinear Elliptic Problem in the Setting of Generalized Sobolev Spaces

Ouidad AZRAIBI, Mohamed Badr Benboubker, Badr EL HAJI and Jalal EL HAJOUJI

ABSTRACT: The novelty of this note is to establish existence result for the following anisotropic elliptic equation

$$-\operatorname{div} B(x,\vartheta,\nabla\vartheta)=f$$

where the datum $f\in L^1(\Omega)$. Furthermore only the large monotonicity conditions will be assumed on $B(x,s,\xi)$. To overcome this difficulty we will use the approach of Minty's lemma in the anisotropic weighted Sobolev spaces.

Keywords: Anisotropic weighted Sobolev spaces, elliptic problem, entropy solutions, measure data, truncation.

Contents

1. Introduction

The anisotropic weighted Sobolev spaces introduce a novel framework incorporating directional derivatives with varying weights. Initially, consider Ω as an open bounded subset of $\mathbb{R}^N$ $(N\geq 2)$, and let p_i denote $N+1$ exponents, where $1<p_i<N$ for $i\in 0,\ldots,N$. In this works, we investigate the existence of entropy solutions for the given problem.

$$\begin{cases} -\operatorname{div}(B(x,\vartheta,\nabla\vartheta))=f \text{ in } \Omega\\ \vartheta=0 \quad \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

Here, $w=\{w_i, 0\leq i\leq N\}$ is the set of weight functions on Ω, and $w^*=\left\{w_i^{1-p_i'}, 0\leq i\leq N\right\}$, The second term term f belongs to $L^1(\Omega)$.

In recent works, existence result with some qualitative properties and regularity of nonlinear anisotropic elliptic equations where the data belonns to L^1- have been proved, for example we may refer the reader to [8] When L. Boccardo examined the problem (1.1) within the conventional Sobolev space $W_0^{1,p}(\Omega)$.

Furthermore we can mention the paper [16] when B. El haji et al have been shown the existence result of entropy solution in weighted-Orlicz spaces, other works found by Y. Akdim et al. in their paper [1] devoted to study a degenerated problem (1.1) via Minty's Lemma in weigthed Orlicz-Sobolev space, in the similar direction faria et al (see [24]) have been treated the similar problem as (1.1) where the solution ϑ of the elliptic problem studied depend on the gradient.

2020 *Mathematics Subject Classification:* 35J70, 35J62, 46E35.
Submitted September 28,2025. Published February 22, 2026

The goal of this note is to examine several elliptic problems for which the classical monotone operators. (introduced by Visik [11], Minty [10], Browder [9] and by Akdim-Azroul-Benkirane [2] do not apply. We employ a tool to address the challenge posed by the absence of strict monotonicity (which doesn't ensure the almost everywhere convergence of the gradient). This tool involves exploring certain techniques derived from Minty's lemma. The pseudo-monotonicity approach cannot be applied because of the presence of $f \in L^1(\Omega)$. To verify the almost everywhere convergence of $\nabla\vartheta_n$, the authors in [7] demonstrated the boundedness of ϑ_n in the Marcinkiewicz space. Nonetheless, in this investigation, we establish the local convergence in measure of ϑ_n (as outlined in Section 4.2.2).

The mathematical literature dealing existence of solutions to some parabolic and elliptic problems is massive; we refer the reader to [4,5,6,14,15,16,17,18,20,21,22,23].

The outline of this note is as follows. After introducing our work , we recall in Section 2 the new anisotropic weighted sobolev spaces, and the section 3 is devoted to some essential assumptions which are necessary to have existence solution, finally section 4 will be devoted to give our Major Conclusions.

2. Preliminaries

Here, we present the enlargement of Sobolev spaces to a novel class of anisotropic weighted Sobolev spaces. Let Ω be a bounded open set of $\mathbf{R}^N, p_0, p_1, \ldots, p_N$ be $N+1$ exponents with $1 < p_i < \infty$ for $i = 0, 1, \ldots, N$ and $w = \{w_i(x), 0 \le i \le N\}$ be a vector of weight functions which is measurable and strictly positive a.e. in Ω. Furthermore, we assume the following hypothesis that, there exists

$$w_i \in L^1_{\text{loc}}(\Omega), \tag{2.1}$$

$$w_i^{\frac{-1}{p_i-1}} \in L^1_{\text{loc}}(\Omega), \tag{2.2}$$

for any $0 \le i \le N$. The anisotropic weighted orlicz space $L^{p_i}(\Omega, \gamma)$, where γ is a weight function on Ω will be formulated by the following expression,

$$L^{p_i}(\Omega, \gamma) = \left\{ \vartheta = \vartheta(x), \vartheta\gamma^{\frac{1}{p_i}} \in L^{p_i}(\Omega) \right\}$$

and endowed by the norm

$$\|\vartheta\|_{L^{p_i}(\Omega,\gamma)} = \|\vartheta\|_{p_i,\gamma} = \left(\int_\Omega |\vartheta(x)|^{p_i} \gamma(x) dx \right)^{\frac{1}{p_i}}.$$

We put

$$(p_i) = (p_0, \ldots, p_N), \quad D^0\vartheta = \vartheta \quad \text{and} \quad D^i\vartheta = \frac{\partial\vartheta}{\partial x_i} \quad \text{for } i = 1, \ldots, N,$$

and we consider that

$$\underline{p_i} = \min\{p_0, p_1, \ldots, p_N\} \quad \text{then} \quad \underline{p_i} > 1. \tag{2.3}$$

and

$$\|\vartheta\|_{1,(p_i),w} = \|\vartheta\|_{p_0,w_0} + \sum_{i=1}^{N} \left\| \frac{\partial\vartheta}{\partial x_i} \right\|_{p_i,w_i}. \tag{2.4}$$

The anisotropic weighted Sobolev space $W^{1,(p_i)}(\Omega, w)$ is defined as $\vartheta \in L^{p_0}(\Omega, w_0) = \left\{ u(x), uw_0^{\frac{1}{p_0}} \in L^{p_0}(\Omega) \right\}$ such that $D^i\vartheta \in L^{p_i}(\Omega, w_i)$ for $i = 1, \ldots, N$.

Remark 2.1 We use the technical lemmas introduced in [3] and in [2].

Lemma 2.1 *[3] Let $(\vartheta_n)_n$ be a bounded sequence in $W_0^{1,(p_i)}(\Omega, w)$. If $\vartheta_n \to \vartheta$ in $W_0^{1,(p_i)}(\Omega, w)$, then $T_k(\vartheta_n) \to T_k(\vartheta)$ in $W_0^{1,(p_i)}(\Omega, w)$ where $T_k(.)$ is the truncation function.*

3. Basic Assumptions and Existence Results

We suppose that the norm :

$$\|\vartheta\| = \left(\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial\vartheta}{\partial x_i}\right|^{p_i} w_i(x)dx\right)^{\frac{1}{p_i}} \tag{3.1}$$

is equivalent to (2.4), and there exists $\sigma(x)$(a weight function) on Ω and, $1 < q_i < \infty$ such that the Hardy inequality,

$$\left(\int_{\Omega}|\vartheta(x)|^{q_i}\sigma(x)dx\right)^{\frac{1}{q_i}} \leq c\left(\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial\vartheta}{\partial x_i}\right|^{p_i} w_i(x)dx\right)^{\frac{1}{p_i}} \tag{3.2}$$

is verified for every $\vartheta \in W_0^{1,(p_i)}(\Omega, w)$ with a constant $c > 0$ independent of ϑ. Furthermore, the imbedding,

$$W_0^{1,(p_i)}(\Omega, w) \hookrightarrow\hookrightarrow L^{q_i}(\Omega, \sigma) \text{ is compact .} \tag{3.3}$$

Let $\mathbb{B}$ denote a nonlinear operator mapping from $W_0^{1,(p_i)}(\Omega, w)$ to its dual space $W^{-1,p_i'}(\Omega, w^*)$. This operator is defined as:

$$\mathbb{B}(\vartheta) = -\operatorname{div}(B(x, \vartheta, \nabla\vartheta))$$

where $B(x, s, \xi) : \Omega \times \boldsymbol{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function fulfills the hypothesises listed bellow.

For $i = 1, \ldots, N$

$$|B_i(x, s, \xi)| \leq \beta w_i^{\frac{1}{p_i}}(x)\left[k(x) + \sigma^{\frac{1}{p_i}}|s|^{\frac{q_i}{p_i'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p_i'}}(x)\,|\xi_j|^{p_i-1}\right], \tag{3.4}$$

for a.e., $x \in \Omega$, all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $k(x) \in L^{p_i'}(\Omega)$ $\left(\frac{1}{p_i} + \frac{1}{p_i'} = 1\right)$ and $\beta > 0$. Here σ and q_i are as in (3.2).

$$\text{for all } (\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N, \langle B(x, s, \xi) - B(x, s, \eta), \xi - \eta\rangle \geq 0, \tag{3.5}$$

$$\langle B(x, s, \xi), \xi\rangle \geq \alpha\sum_{i=1}^{N} w_i\,|\xi_i|^{p_i}, \quad \alpha > 0. \tag{3.6}$$

Let us consider, the truncation function :

$$T_k(s) = \begin{cases} s \text{ if } |s| \leq k \\ k\dfrac{s}{|s|} \text{ if } |s| > k \end{cases} \qquad \text{for } k > 1 \text{ and } s \text{ in } \mathbb{R}.$$

Our goal is to study the problem:

$$\begin{cases} \mathbb{B}(\vartheta) = f \text{ in } \Omega \\ \vartheta = 0 \qquad \text{on } \partial\Omega \end{cases} \tag{3.7}$$

Where, $\mathbb{B}(\vartheta) = -\operatorname{div}(B(x, \vartheta, \nabla\vartheta)),$ the datum f belongs to $L^1(\Omega)$.

Definition 3.1 A measurable function ϑ is named an entropy solution of (3.7) if $T_k(\vartheta)$ belongs to $W_0^{1,(p_i)}(\Omega, w)$ for every $k > 0$ and

$$\int_{\Omega}\langle B(x, \vartheta, \nabla\vartheta), \nabla T_k[\vartheta - \Psi]\rangle\, dx = \int_{\Omega} f T_k[\vartheta - \Psi]dx$$

for every $\Psi \in W_0^{1,(p_i)}(\Omega, w) \cap L^\infty(\Omega)$.

Theorem 3.1 *Let the assumptions (3.1)-(3.6) holds, then ϑ is a solution of (3.7) in the sense of the definition* **??** *such that.*

$$\int_\Omega \langle B(x,\vartheta,\nabla\Psi),\nabla T_k[\vartheta-\Psi]\rangle\,dx=\int_\Omega fT_k[\vartheta-\Psi]dx$$

for every $\Psi\in W_0^{1,(p_i)}(\Omega,w)\cap L^\infty(\Omega)$, for every $k>0$.

4. Proof of Existence Theorem

4.1. Main Lemma

Lemma 4.1 *Let ϑ be a measurable function such that $T_k(\vartheta)\in W_0^{1,(p_i)}(\Omega,w)$ for every $k>0$. Then*

$$\int_\Omega \langle B(x,\vartheta,\nabla\vartheta),\nabla T_k[\vartheta-\Psi]\rangle\,dx\leq\int_\Omega fT_k[\vartheta-\Psi]dx.$$

$\Leftrightarrow$

$$\int_\Omega \langle B(x,\vartheta,\nabla\vartheta),\nabla T_k[\vartheta-\Psi]\rangle\,dx=\int_\Omega fT_k[\vartheta-\Psi]dx.$$

for every Ψ in $W^{1,p_i}(\Omega,w)\cap L^\infty(\Omega)$, and for every $k>0$.

Proof The proof is done in the same way as the proof in [19]

4.2. Proof of Existence theorem 3.1

4.2.1. A Priori estimate.

Let $f_n\in L^\infty(\Omega)\longrightarrow f$ strongly in $L^1(\Omega)$ such that $\|f_n\|_{L^1}\leq\|f\|_{L^1}$, and let ϑ_n be a solution in $W_0^{1,(p_i)}(\Omega,w)$ of the problem

$$\begin{cases}-\operatorname{div}B(x,\vartheta_n,\nabla\vartheta_n)=f_n \text{ in } \Omega\\ \vartheta_n=0 \quad\quad\quad\quad \text{on } \partial\Omega\end{cases}\tag{4.1}$$

which exists thanks to [13].

Now let us taking $T_k(\vartheta_n)$ as test function in (4.1), we may obtain

$$\int_\Omega \langle B(x,\vartheta_n,\nabla\vartheta_n),\nabla T_k(\vartheta_n)\rangle\,dx=\int_\Omega f_nT_k(\vartheta_n)\,dx$$

using $\nabla T_k(\vartheta_n)=\nabla\vartheta_n\chi_{\{|\vartheta_n|\leq k\}}$ and according to hypothesis (3.6), we may get

$$\int_\Omega \langle B(x,\vartheta_n,\nabla\vartheta_n),\nabla T_k(\vartheta_n)\rangle\,dx\geq\alpha\sum_{i=1}^N\int_\Omega w_i\left|\frac{\partial T_k(\vartheta_n)}{\partial x_i}\right|^{p_i}dx$$

then we have,

$$\alpha\sum_{i=1}^N\int_\Omega w_i\left|\frac{\partial T_k(\vartheta_n)}{\partial x_i}\right|^{p_i}dx\leq k\|f\|_{L^1}$$

$$\leq k\|f\|_{L^1}.$$

Then Young's inequality yields

$$\alpha\sum_{i=1}^N\int_\Omega w_i\left|\frac{\partial T_k(\vartheta_n)}{\partial x_i}\right|^{p_i}dx\leq k\|f\|_{L^1}$$

Then,

$$\alpha\sum_{i=1}^N\int_\Omega w_i\left|\frac{\partial T_k(\vartheta_n)}{\partial x_i}\right|^{p_i}dx\leq k\|f\|_{L^1}.$$

for $k>1$, which implies that

$$\left(\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial T_k(\vartheta_n)}{\partial x_i}\right|^{p_i} w_i(x)dx\right)^{\frac{1}{p_i}} \le ck^{\frac{1}{p_i}} \quad \forall k>1. \tag{4.2}$$

4.2.2. Locally convergence of ϑ_n in measure.

Let $k>0$ large enough, by using (3.3), we may obtain

$$\begin{aligned} k \text{ meas } (\{|\vartheta_n|>k\}\cap B_R) &= \int_{\{|\vartheta_n|>k\}\cap B_R} |T_k(\vartheta_n)|\,dx \le \int_{B_R} |T_k(\vartheta_n)|\,dx \\ &\le \left(\int_{\Omega} |T_k(\vartheta_n)|^{p_i} w_0 dx\right)^{\frac{1}{p_i}} \cdot \left(\int_{B_R} w_0^{1-p_i'} dx\right)^{\frac{1}{q_i}} \\ &\le c\left(\int_{\Omega}\sum_{i=1}^{N}\left|\frac{\partial T_k(\vartheta_n)}{\partial x_i}\right|^{p_i} w_i(x)dx\right)^{\frac{1}{p_i}} \\ &\le c_1 k^{\frac{1}{p_i}}. \end{aligned}$$

which gives

$$\text{meas}\,(\{|\vartheta_n|>k\}\cap B_R) \le \frac{c_1}{k^{1-\frac{1}{p_i}}} \quad \forall k>1. \tag{4.3}$$

We obtain, for every $\delta>0$,

$$\begin{aligned} \text{meas}\,(\{|\vartheta_n-\vartheta_m|>\delta\}\cap B_R) \le \text{meas}\,(\{|\vartheta_n|>k\}\cap B_R) &+ \text{ meas } (\{|\vartheta_m|>k\}\cap B_R) \\ &+ \text{ meas } \{|T_k(\vartheta_n)-T_k(\vartheta_m)|>\delta\}. \end{aligned} \tag{4.4}$$

Since $T_k(\vartheta_n)$ is bounded in $W_0^{1,(p_i)}(\Omega,w)$, there exists some $v_k\in W_0^{1,(p_i)}(\Omega,w)$, such that

$$\begin{cases} T_k(\vartheta_n) \rightharpoonup v_k & \text{weakly in } W_0^{1,(p_i)}(\Omega,w), \\ T_k(\vartheta_n) \to v_k & \text{strongly in } L^{q_i}(\Omega,\sigma) \text{ and a.e. in } \Omega. \end{cases}$$

Consequently, it can be deduced that $T_k(\vartheta_n)$ forms a Cauchy sequence in measure in Ω. Let $\varepsilon>0$. By utilizing (4.3) and (4.4), there exists a specific $k(\varepsilon)>0$ such that meas $(\{|\vartheta_n-\vartheta_m|> \ \delta\}\cap B_R)<\varepsilon$ for all $n,m\ge n_0(k(\varepsilon),\delta,R)$. This implies that (ϑ_n) is a Cauchy sequence in measure within B_R, thus converging almost everywhere to a measurable function ϑ. Then,

$$\begin{cases} T_k(\vartheta_n) \rightharpoonup T_k(\vartheta) \text{ weakly in } W_0^{1,(p_i)}(\Omega,w), \\ T_k(\vartheta_n) \to T_k(\vartheta) \quad \text{strongly in } L^{q_i}(\Omega,\sigma) \text{ and a.e in } \Omega. \end{cases} \tag{4.5}$$

4.2.3. An intermediate Inequality.

Here, we may show for $\Psi\in W_0^{1,(p_i)}(\Omega,w)\cap L^\infty(\Omega)$, that

$$\int_{\Omega}\langle B(x,\vartheta_n,\nabla\Psi),\nabla T_k[\vartheta_n-\Psi]\rangle\,dx \le \int_{\Omega} f_n T_k[\vartheta_n-\Psi]\,dx. \tag{4.6}$$

Furthermore, by taking $T_k(\vartheta_n-\Psi)$ as test in (4.1), with Ψ in $W_0^{1,(p_i)}(\Omega,w)\cap L^\infty(\Omega)$, we get

$$\int_{\Omega}\langle B(x,\vartheta_n,\nabla\vartheta_n),\nabla T_k[\vartheta_n-\Psi]\rangle\,dx = \int_{\Omega} f_n T_k[\vartheta_n-\Psi]\,dx.$$

Then,

$$\begin{aligned}&\int_\Omega \langle B(x,\vartheta_n,\nabla\vartheta_n),\nabla T_k[\vartheta_n-\Psi]\rangle\,dx+\int_\Omega \langle B(x,\vartheta_n,\nabla\Psi),\nabla T_k[\vartheta_n-\Psi]\rangle\,dx\\&-\int_\Omega \langle B(x,\vartheta_n,\nabla\Psi),\nabla T_k[\vartheta_n-\Psi]\rangle\,dx\\&=\int_\Omega f_nT_k[\vartheta_n-\Psi]\,dx\end{aligned}\tag{4.7}$$

According to (3.5) and using the truncation function, we may get

$$\int_\Omega \langle[B(x,\vartheta_n,\nabla\vartheta_n)-B(x,\vartheta_n,\nabla\Psi)],\nabla T_k[\vartheta_n-\Psi]\rangle\,dx\geq 0.\tag{4.8}$$

According to (4.7) and (4.8), we get (4.6).

4.2.4. Passing to the limit.

Here we prove that for $\Psi\in W_0^{1,(p_i)}(\Omega,w)\cap L^\infty(\Omega)$, we have

$$\int_\Omega \langle B(x,\vartheta,\nabla\Psi),\nabla T_k[\vartheta-\Psi]\rangle\,dx\leq\int_\Omega fT_k[\vartheta-\Psi]dx.$$

Firstly, we verify that

$$\int_\Omega \langle B(x,\vartheta_n,\nabla\Psi),\nabla T_k[\vartheta_n-\Psi]\rangle\,dx\to\int_\Omega \langle B(x,\vartheta(\nabla\Psi),\nabla T_k[\vartheta-\Psi]\rangle\,dx \text{ as } n\to+\infty.$$

As $T_M(\vartheta_n)\rightharpoonup T_M(\vartheta)$ weakly in $W_0^{1,(p_i)}(\Omega,w)$, with $M=k+\|\Psi\|_\infty$, so by using Lemma 2.1, we may get

$$T_k(\vartheta_n-\Psi)\rightharpoonup T_k(\vartheta-\Psi)\text{ in }W_0^{1,(p_i)}(\Omega,w),\tag{4.9}$$

which gives

$$\frac{\partial T_k}{\partial x_i}(\vartheta_n-\Psi)\rightharpoonup\frac{\partial T_k}{\partial x_i}(\vartheta-\Psi)\text{ weakly in }L^{p_i}(\Omega,w_i)\,\forall i=1,..,N.\tag{4.10}$$

We now show that

$$B_i(x,T_M(\vartheta_n),\nabla\Psi)\to B_i(x,T_M(\vartheta),\nabla\Psi)\ \text{ strongly in }L^{p_i'}(\Omega,w_i^*)$$

By the hypothesise (3.4), we may have

$$\begin{aligned}|B_i(x,T_M(\vartheta_n),\nabla\Psi)|^{p_i'}w_i^{\frac{-p_i'}{p_i}}&\leq\beta\left[k(x)+|T_M(\vartheta_n)|^{\frac{q^i}{p_i'}}\sigma^{\frac{1}{p_i'}}+\sum_{j=1}^N\left|\frac{\partial\Psi}{\partial x_i}\right|^{p_i-1}w_i^{\frac{1}{p_i'}}\right]^{p_i'}\\&\leq\gamma\left[k(x)^{p_i'}+|T_M(\vartheta_n)|^{q_i}\sigma+\sum_{j=1}^N\left|\frac{\partial\Psi}{\partial x_i}\right|^{p_i}w_i\right],\end{aligned}\tag{4.11}$$

with $\beta,\gamma>0$, Since $T_M(\vartheta_n)\rightharpoonup T_M(\vartheta)$ weakly in $W_0^{1,(p_i)}(\Omega,w)$ and $W_0^{1,(p_i)}(\Omega,w)\hookrightarrow\hookrightarrow L^q(\Omega,\sigma)$, then $T_M(\vartheta_n)\to T_M(\vartheta)$ strongly in $L^q(\Omega,\sigma)$ and a.e. in Ω, consequently,

$$|B_i(x,T_M(\vartheta_n),\nabla\Psi)|^{p_i'}w_i^*\to|B_i(x,T_M(\vartheta),\nabla\Psi)|^{p_i'}w_i^*\text{ a.e.in }\Omega.$$

and

$$\gamma\left[k(x)^{p_i'}+|T_M(\vartheta_n)|^{q_i}\sigma+\sum_{j=1}^N\left|\frac{\partial\Psi}{\partial x_i}\right|^{p_i}w_i\right]\to\gamma\left[k(x)^{p_i'}+|T_M(\vartheta)|^{q_i}\sigma+\sum_{j=1}^N\left|\frac{\partial\Psi}{\partial x_i}\right|^{p_i}w_i\right]\text{ a.e. in }\Omega.$$

We now using Vitali's theorem in order to get

$$B_i\left(x, T_M\left(\vartheta_n\right), \nabla \Psi\right) \rightarrow B_i\left(x, T_M(\vartheta), \nabla \Psi\right) \text{ strongly in } L^{p_i'}\left(\Omega, w_i^*\right), \text{ as } n \rightarrow +\infty. \tag{4.12}$$

According to (4.9) and (4.10), we get

$$\int_\Omega \left\langle B\left(x, \vartheta_n, \nabla \Psi\right), \nabla T_k\left[\vartheta_n-\Psi\right]\right\rangle dx \rightarrow \int_\Omega \langle B(x, \eta, \nabla \Psi), \nabla T_k[\vartheta-\Psi]\rangle\, dx, \text{ as } n \rightarrow +\infty \tag{4.13}$$

Finally, we prove that

$$\int_\Omega f_n T_k\left[\vartheta_n-\Psi\right] dx \rightarrow \int_\Omega f T_k[\vartheta-\Psi] dx. \tag{4.14}$$

We have $f_n T_k\left[\vartheta_n-\Psi\right] \rightarrow f T_k[\vartheta-\Psi]$ a.e. in Ω and $\left|f_n T_k\left[\vartheta_n-\Psi\right]\right| \leq k\left|f_n\right|$ and $k\left|f_n\right| \rightarrow k|f|$ in $L^1(\Omega)$, then by using Vitali's theorem, we may get (4.14).

According to (4.13) and (4.14) we may pass to the limit in (4.6), so that $\forall \Psi \in W_0^{1,(p_i)}(\Omega, w) \cap L^{\infty}(\Omega)$, we deduce

$$\int_\Omega \langle B(x, \vartheta(\nabla \Psi), \nabla T_k[\vartheta-\Psi]\rangle\, dx \leq \int_\Omega f T_k[\vartheta-\Psi] dx.$$

Based on the Lemma 4.1, we conclude that ϑ be a solution to the problem (3.7) in the sense of the definition 3.1 .

References

1. Y. Akdim, ,E. Azroul, M.Rhoudaf; *Existence of T-solution for degenerated problem via Minty's Lemma.* Acta Math. sinica (English Ser.) $24, 431-438(2008)$.

2. Y. Akdim, E. Azroul, A. Benkirane : *Pseudo-monotonicity and degenerated elliptic operator of second order.* Electron. J. Differ. Equations 71, 9-24 (2001) (conference 09, 2003, N).

3. E. Azroul, M. Bouziani, A. Barbara *Existence of entropy solutions for anisotropic quasilinear degenerated elliptic problems with Hardy potential* SeMA Journal https://doi.org/10.1007/s40324-021-00247-0

4. A. Benkirane, B. El Haji and M. El Moumni, On the existence of solution for degenerate parabolic equations with singular terms, Pure and Applied Mathematics Quarterly Volume 14, Number 3-4, 591-606(2018).

5. A. Benkirane, B. El Haji and M. El Moumni; *Strongly nonlinear elliptic problem with measure data in Musielak-Orlicz spaces.* Complex Variables and Elliptic Equations, 1-23, https://doi.org/10.1080/17476933.2021.1882434 .

6. A. Benkirane, B. El Haji and M. El Moumni; On the existence solutions for some Nonlinear elliptic problem, Boletim da Sociedade Paranaense de Matemática,(3s.) v. 2022 (40) : 1-8, DOI: https://doi.org/10.5269/bspm.53111

7. L. Boccardo, L. Orsina, Existence Results for Dirichlet Problem in L^1 via Minty's lemma, Applicable Ana (1999) pp 309-313.

8. Boccardo, L. :*A remark on some nonlinear elliptic problems.* Electronic J. Diff. Equ. Conference, 08, $47-52(2002)$.

9. F. E. Browder, Existence theorems for nonlinear partial differential equations, Global Analysis (Berkeley, 1968), Proc. Sympos. Pure Math., no. XVI, AMS, Providence, 1970, pp. 1-60, MR $42 \neq 4855$.

10. G. J. Minty, Monotone (nonlinear) operators in Hilbert space, Duke Math. J. 29 (1962).

11. M.L. Visik, Solvability of the first boundary value problem for quasilinear equations with rapidly increasing coefficients in Orlicz classes, Dok1. Akad. Nauk SSSR 151, 1963, pp. 758-761=Sovier math. Dok1. 4 (1963), 1060-1064. MR $27 \neq 5032$.

12. P. Drabek, A. Kufner, F. Nicolosi, : Quasilinear elliptic equations with degenerations and singularities. De Gruyter Series in Nonlinear Analysis and Applications. New York (1997)

13. P. Drabek, A. KuFNER and V. Mustonen, Pseudo-monotonicity and degenerated or singular elliptic operators, Bull. Austral. Math. Soc. Vol. 58 (1998), 213-221.

14. B. El Haji and M. El Moumni and A. Talha, *Entropy solutions for nonlinear parabolic equations in Musielak Orlicz spaces without $Delta_2$-conditions*, Gulf Journal of Mathematics Vol 9, Issue 1 (2020) 1-26.

15. B. El Haji and M. El Moumni; Entropy solutions of nonlinear elliptic equations with L^1-data and without strict monotonocity conditions in weighted Orlicz-Sobolev spaces, Journal of Nonlinear Functional Analysis, Vol. 2021 (2021), Article ID 8, pp. 1-17

16. B. El Haji, M. El Moumni, K. Kouhaila; *On a nonlinear elliptic problems having large monotonocity with L^1-data in weighted Orlicz-Sobolev spaces* ,Moroccan J. of Pure and Appl. Anal. (MJPAA) Volume 5(1), 2019, Pages 104-116, DOI 10.2478/mjpaa-2019-0008

17. B. El Haji and M. El Moumni and K. Kouhaila; Existence of entropy solutions for nonlinear elliptic problem having large monotonicity in weighted Orlicz-Sobolev spaces , LE MATEMATICHE Vol. LXXVI (2021) - Issue I, pp. 37-61, https://doi.org/10.4418/2021.76.1.3.

18. N. El Amarti, B. El Haji and M. El Moumni. Entropy solutions for unilateral parabolic problems with L^1-data in Musielak-Orlicz-Sobolev spaces Palestine Journal of Mathematics, Vol. 11(1)(2022) , 504-523.

19. B. El Haji and M. Mabdaoui, Entropy Solutions for Some Nonlinear Elliptic Problem via Minty's Lemma in Musielak-Orlicz-Sobolev Spaces, Asia Pac. J. Math., 8 (2021), 18.doi:10.28924/APJM/8-18.

20. B. El Haji, M. El Moumni, A. Talha; *Entropy Solutions of Nonlinear Parabolic Equations in Musielak Framework Without Sign Condition and L^1-Data* Asian Journal of Mathematics and Applications 2021.

21. O. Azraibi, B.EL haji, M. Mekkour; Nonlinear parabolic problem with lower order terms in Musielak-Sobolev spaces without sign condition and with Measure data, Palestine Journal of Mathematics,Vol. 11(3)(2022) , 474-503.

22. O. Azraibi, B.EL haji, M. Mekkour; On Some Nonlinear Elliptic Problems with Large Monotonocity in Musielak–Orlicz–Sobolev Spaces, Journal of Mathematical Physics, Analysis, Geometry 2022, Vol. 18, No. 3, pp. 1–18.

23. N. El Amarty, B. El Haji and M. El Moumni, *Existence of renomalized solution for nonlinear Elliptic boundary value problem without Δ_2 -condition* SeMA 77, 389-414 (2020). https://doi.org/10.1007/s40324-020-00224-z.

24. L.F.O. Faria a, O.H. Miyagaki a, D. Motreanub, M. Tanakac, *Existence results for nonlinear elliptic equations with Leray-Lions operator and dependence on the gradient*,Nonlinear Analysis Vol 96, (2014) 154-166.

25. E. Hewitt, K. Stromberg: *Real and Abstract Analysis. Springer, Berlin, Heidelberg, New York (1965)* .

Ouidad AZRAIBI,
Department of Mathematics,
Laboratory LAMA,
Fez, Morocco.
E-mail address: ouidadazraibi@gmail.com

and

Mohamed Badr Benboubker,
Higher School of Technology,
Sidi Mohamed Ben Abdellah University,
Fez, Morocco.
E-mail address: simo.ben@hotmail.com

and

Badr EL HAJI,
Department of Mathematics,
Laboratory LaR2A,
Tetouan, Morocco.
E-mail address: badr.elhaji@gmail.com

and

Jalal EL HAJOUJI,
Department of Mathematics,
Laboratory LaR2A,
Tetouan, Morocco.
E-mail address: jalal.elhajouji1@etu.uae.ac.ma