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Optimality Conditions for Nonsmooth Interval-Valued Multiobjective Semi-Infinite Programming Problem Subject to Switching Constraints via Tangential Subdifferentials

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ABSTRACT: This paper explores optimality conditions for a nonsmooth interval-valued multiobjective semi-infinite programming problem with switching constraints. Specifically, we use an appropriate constraint qualification to establish necessary M-stationary conditions utilizing tangential subdifferentials. Furthermore, sufficient optimality conditions are derived based on generalized convexity. Results are well illustrated by example.

Key Words: Interval-valued multiobjective semi-infinite optimization, switching constraints, stationary conditions, LU-efficient solutions, generalized convexity.

Contents

1. Introduction

Interval-valued multiobjective semi-infinite programming (IVMOSIP) extends the classical interval-valued programming (IVP) framework to accommodate multiple objective functions along with semi-infinite constraints. This extension provides a robust approach for identifying Pareto optimal solutions under conditions of uncertainty, allowing decision-makers to strike a balance between conflicting objectives despite the presence of infinite constraint sets and imprecise data. Several notable contributions have advanced the field of IVMOSIP. Gadhi and El Idrissi [1] analyzed IVMOSIP using limiting subdifferentials, providing a deeper understanding of optimality in this context. Huy Hung et al. [2] established optimality criteria as well as duality theorems for approximate quasi-Pareto solutions of IVMOSIP, employing limiting subdifferentials while Dwivedi et al. [3] addressed the same class of problems by employing Clarke subdifferentials to derive both optimality and duality results. Jennane et al. [4] developed Karush-Kuhn-Tucker (KKT)-type optimality conditions by utilizing Abadie's constraint qualification and convexificators for semi-infinite programs, where both the multiobjective function and constraints are interval-valued but may not be locally Lipschitz. Tung [5] also contributed to this area by establishing KKT optimality conditions and investigating duality for semi-infinite programming problems involving multiple interval-valued objective functions. Antczak and Farajzadeh [6] studied nondifferentiable semi-infinite vector optimization problems, where the objective and constraint functions are both expressed using interval values. Their analysis was conducted under suitable invexity assumptions. IVMOSIP represents a promising and evolving research area that addresses the challenges of IVP problems with multiple objectives and semi-infinite constraints. The advancements made by these researchers have significantly contributed to the theoretical development of this field.

Mathematical programming problems with switching constraints (MPSC) have become a prominent research area in optimization in recent years (see [7,8] and references therein). MPSC is characterized

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by optimization problems that include equality constraints formed by the product of two functions. The term "switching constraints" reflects that when this product equals zero, then at least one of the involved functions must vanish. MPSC has valuable applications across various research fields, particularly in optimal control (see [9,10,11]), where one control function may need to be vanish at any point in time (see [12,13,14]) when multiple control functions are present. MPSC are closely connected to mathematical programs with vanishing constraints (MPVC) (see, [15,16,17,18]) and mathematical programs with equilibrium constraints (MPEC) (see [19,20,21]).

Researchers have explored various aspects of MPSC. Liang and Ye [22] studied constraint qualifications, optimality conditions, and exact penalization. Shikhman [23] approached MPSC from a topological perspective, establishing important theorems in Morse theory. Mehlitz [7] investigated stationary conditions and constraint qualifications. Li and Guo [8] examined Mordukhovich stationary conditions under weak constraint qualifications. Kanzow et al. [24] explored relaxation schemes, and Shabankareh et al. [25] investigated Abadie constraint qualification and stationary conditions for locally Lipschitz functions using Mordukhovich subdifferentials while Pandey and Singh [26] focused on constraint qualifications for multiobjective MPSC. Jennane et al. [27] and Jennane and Kalmoun [28] examined the conditions for optimal solutions in a nonsmooth multiobjective semi-infinite programming problem (MOSIP) with switching constraints by using Clarke subdifferentials and tangential subdifferentials, respectively, while Upadhyay and Ghosh [29] derived the Optimality conditions and duality for MOSIP with switching constraints on Hadamard manifolds.

Tangentially convex functions were introduced by Pshenichnyi [30] and the term itself was coined earlier by Lemaréchal [31]. This class includes Clarke regular functions, Gâteaux differentiable functions, and functions associated with Michel–Penot regular subdifferentials [31,32]. Optimality conditions based on tangential subdifferentials offer broader generalizations than those using traditional subdifferentials. Numerous researchers have employed tangential subdifferentials to derive optimality conditions and formulate duality theorems of Wolfe and Mond–Weir types in multiobjective optimization see [33,34,35,36] and [37]. Tung [38] utilized this framework to establish strong KKT optimality conditions for both efficient and weakly efficient solutions in MOSIP. Later, Tung [39] introduced suitable generalized constraint qualifications to derive optimality conditions for efficient and generalized efficient solutions in nonsmooth MOSIP. Liu et al. [40]) further extended this approach to formulate approximate optimality conditions and mixed-type duality results for MOSIP. More recently, Long et al. [41] explored approximate quasi-efficient solutions for nonsmooth MOSIP using tangential subdifferentials. Also Jennane and Kalmoun [28] investigate optimality conditions for a nonsmooth MOSIP subject to switching constraints by using tangential subdifferentials.

Many practical optimization problems feature several conflicting objectives, uncertain data, and complex constraints, such as switching conditions and infinitely many inequality constraints. Nonsmooth interval-valued multiobjective semi-infinite programming problems with switching constraints (IVNMPSC) naturally model such situations by capturing both data uncertainty (via intervals), discontinuous behaviour (via switching constraints). Despite their relevance, this class of problems remains underexplored especially in the context of nonsmooth analysis. Motivated by this gap, and building upon the limitations of existing methods, we aim to establish optimality conditions for this broader and more realistic class of problems. To achieve this, we employ tangential subdifferentials, a powerful tool well suited for handling nonsmoothness, to derive optimality conditions that generalize and strengthen existing results. This paper is organized as follows: Section 2 outlines the key concepts and preliminary results that form the basis of this study. Section 3 focuses on developing necessary M-stationary optimality conditions for local LU-efficient solutions of IVNMPSC using tangential subdifferentials. Section 4 presents sufficient optimality conditions for weakly LU-efficient solutions of IVNMPSC under the assumptions of generalized convexity through tangential subdifferentials. The paper concludes in Section 5 with a summary of the main results and suggestions for potential directions that future research could explore.

2. Preliminaries

From this point onward, we adopt the following ordering on the Euclidean space $\mathbb{R}^n$:
For vectors $\gamma, \theta \in \mathbb{R}^n$:

- $\gamma \leq \theta$ iff $\gamma_i \leq \theta_i$ for all $i = 1, \ldots, n$ and the inequality is strict for at least one index i.
- $\gamma < \theta$ iff $\gamma_i < \theta_i$ for all $i = 1, \ldots, n$.

Given a non empty subset $A \subseteq \mathbb{R}^n$, we define:

- coA: the convex hull of A.
- clA: the closure of A.
- The polar cone of A, denoted A^o, is $\{\xi \in \mathbb{R}^n : \langle \xi, d \rangle \leq 0, \ \forall \ d \in A\}$.
- The strictly negative polar cone of A, denoted A^s, is $\{\xi \in \mathbb{R}^n : \langle \xi, d \rangle < 0, \ \forall \ d \in A \setminus \{0\}\}$.
- The orthogonal complement of A, denoted $A^\perp$, is $\{\xi \in \mathbb{R}^n : \langle \xi, d \rangle = 0, \ \forall \ d \in A\}$.

It is easy to see that $A^\perp = A^o \cap (-A)^o$. Furthermore, for any point $\zeta \in clA$, the tangent cone, the convex cone generated by A and the linear hull of A are given respectively by

$$T(A, \zeta) = \{u \in \mathbb{R}^n : \exists \tau_n \downarrow 0, \ \exists u_n \to u, \ \zeta + \tau_n u_n \in A\},$$
$$cone(A) = \{\omega = \sum_{i=1}^{n} \lambda_i \omega_i : n \in \mathbb{N}, \ \lambda_i \geq 0, \ \omega_i \in A, \ i = 1, \ldots, n\},$$
$$lin(A) = \{\omega = \sum_{i=1}^{n} \lambda_i \omega_i : n \in \mathbb{N}, \ \lambda_i \in \mathbb{R}, \ \omega_i \in A, \ i = 1, \ldots, n\}.$$

Pshenichnyi [30] first introduced a class of functions now known as tangentially convex functions, a term later formalized by Lemaréchal [31]. This broad class includes many important types of functions, such as Gâteaux differentiable functions, convex functions with open domains, and regular functions, making it highly relevant in various optimization contexts.

Definition 2.1 *[30,31] A function $\eta : \mathbb{R}^n \to \mathbb{R}$ is said to be*

(a) directionally differentiable at a point $\zeta \in \mathbb{R}^n$ in any direction $d \in \mathbb{R}^n$ iff the limit

$$\eta'(\zeta; d) := \lim_{\tau \downarrow 0} \frac{\eta(\zeta + \tau d) - \eta(\zeta)}{\tau}$$

exists and is finite;

(b) tangentially convex at $\zeta \in \mathbb{R}^n$, if for every $d \in \mathbb{R}^n$, $\eta'(\zeta; d)$ exists, is finite, and defines a convex function with respect to d;

(c) Hadamard directionally differentiable at $\zeta \in \mathbb{R}^n$, if its Hadamard directional derivative

$$\eta^H(\zeta, d) := \lim_{\tau \downarrow 0, d' \to d} \frac{\eta(\zeta + \tau d') - \eta(\zeta)}{\tau},$$

exists for all directions d.

The tangentially subdifferential of η at ζ is

$$\partial_T \eta(\zeta) := \{\zeta^* \in \mathbb{R}^n : \langle \zeta^*, d \rangle \leq \eta'(\zeta; d), \ \forall d \in \mathbb{R}^n\}.$$

Remark 2.1

- *Since the directional (Dini) derivative of a tangentially convex function is positively homogeneous, it behaves as a sublinear function with respect to the direction.*
- *Whenever $\eta^H(\zeta, d)$ exists, $\eta'(\zeta, d)$ exists as well, and both are equal.*
- *If η is locally Lipschitz and directionally differentiable at ζ, then η is Hadamard directionally differentiable at ζ in d.*

Now, consider the following nonsmooth multiobjective semi-infinite programming problem with switching constraints:

$$\begin{aligned} &\min f(\xi) = (f_1(\xi), \ldots, f_m(\xi)), \\ &\text{s.t.} \quad \phi_s(\xi) \leq 0, \quad \forall s \in S, \\ &\qquad \psi_k(\xi) = 0, \quad \forall k \in K := \{1, \ldots, q\}, \\ &\qquad \mathfrak{M}_i(\xi)\mathfrak{N}_i(\xi) = 0, \quad \forall i \in I := \{1, \ldots, l\}, \end{aligned} \tag{NMPSC}$$

where S denotes a nonempty index set, which may be infinite. The functions f_j, $j \in J := \{1, \ldots, m\}$, ϕ_s, $s \in S$, ψ_k, $k \in K$, and $\mathfrak{M}_i, \mathfrak{N}_i$, $i \in I$ are real-valued functions defined on $\mathbb{R}^n$ and need not be convex or differentiable..

The feasible region of NMPSC is given by

$$\mathcal{F} := \{\xi \in \mathbb{R}^n : \phi_s(\xi) \leq 0, \forall s \in S; \ \psi_k(\xi) = 0, \forall k \in K; \ \mathfrak{M}_i(\xi)\mathfrak{N}_i(\xi) = 0, \forall i \in I\}.$$

Let us assume the following conditions for the functions:

(i) $\zeta \in \mathcal{F}$.

(ii) $f_j, j \in J$, is Hadamard directionally differentiable at ζ.

(iii) $\phi_s, s \in S, \psi_k, k \in K, \mathfrak{M}_i$, and $\mathfrak{N}_i, i \in I$, are tangentially convex at ζ.

A point ζ is called a *local (weak) efficient solution* to the problem NMPSC if there exists a neighborhood U of ζ such that, for every $y \in U \cap \mathcal{F}$, the inequality $f(y) \leq (\text{or} <) f(\zeta)$ does not hold.

It is easy to verify that every *local efficient solution* to NMPSC is also a *local weak efficient solution*. When $U = \mathbb{R}^n$, we will simply drop the term *local*.

We define $\mathbb{R}_+^{|S|}$ as the set of all functions $\lambda : S \to \mathbb{R}$ such that $\lambda_s > 0$ only for finitely many indices $s \in S$, and $\lambda_s = 0$ for all other s. For any $\zeta \in \mathcal{F}$, let $S(\zeta) := \{s \in S \mid \phi_s(\zeta) = 0\}$ be the index set of all active constraints at ζ. The set of active constraint multipliers at ζ as:

$$A(\zeta) := \left\{\lambda \in \mathbb{R}_+^{|S|} \;\middle|\; \lambda_s \phi_s(\zeta) = 0 \text{ for all } s \in S\right\}.$$

In other words, a multiplier λ belongs to $A(\zeta)$ if there exists a finite subset $R \subset S(\zeta)$ such that $\lambda_s > 0$ for all $s \in R$, and $\lambda_s = 0$ for all $s \in S \setminus R$. Now, we define the index sets as follows:

$$I_{\mathfrak{M}} = I_{\mathfrak{M}}(\zeta) := \{i \in I \mid \mathfrak{M}_i(\zeta) = 0, \mathfrak{N}_i(\zeta) \neq 0\},$$

$$I_{\mathfrak{N}} = I_{\mathfrak{N}}(\zeta) := \{i \in I \mid \mathfrak{M}_i(\zeta) \neq 0, \mathfrak{N}_i(\zeta) = 0\},$$

and

$$I_{\mathfrak{M}\mathfrak{N}} = I_{\mathfrak{M}\mathfrak{N}}(\zeta) := \{i \in I \mid \mathfrak{M}_i(\zeta) = 0, \mathfrak{N}_i(\zeta) = 0\}.$$

We assume that $I_{\mathfrak{M}\mathfrak{N}} \neq \emptyset$, and $\mathcal{P}(I_{\mathfrak{M}\mathfrak{N}})$ denotes the collection of all disjoint bipartitions of $I_{\mathfrak{M}\mathfrak{N}}$; i.e.,

$$\mathcal{P}(I_{\mathfrak{M}\mathfrak{N}}) = \{(\Omega_1, \Omega_2) : \Omega_1 \cup \Omega_2 = I_{\mathfrak{M}\mathfrak{N}}, \ \Omega_1 \cap \Omega_2 = \emptyset\}.$$

A point ζ is called weakly stationary (or simply W-stationary) if there exist multipliers that satisfy the following system of conditions

$$\begin{aligned} 0 \in \sum_{j \in J} \lambda_j \, \partial_T f_j(\zeta) + \sum_{s \in S(\zeta)} \lambda_s^{\phi} \, \partial_T \phi_s(\zeta) + \sum_{k \in K} \lambda_k^{\psi} \, \partial_T \psi_k(\zeta) \\ + \sum_{i \in I} \lambda_i^{\mathfrak{M}} \, \partial_T \mathfrak{M}_i(\zeta) + \sum_{i \in I} \lambda_i^{\mathfrak{N}} \, \partial_T \mathfrak{N}_i(\zeta), \end{aligned} \tag{2.1}$$

$$\forall s \in S(\zeta) : \lambda_s^{\phi} \geq 0, \quad \forall i \in I_{\mathfrak{N}}(\zeta) : \lambda_i^{\mathfrak{M}} = 0, \quad \forall i \in I_{\mathfrak{M}}(\zeta) : \lambda_i^{\mathfrak{N}} = 0.$$

It is called *Mordukhovich-stationary* (or simply *M-stationary*) if, along with condition (2.1), $\lambda_i^{\mathfrak{M}} \lambda_i^{\mathfrak{N}} = 0 \ \forall i \in I_{\mathfrak{M}\mathfrak{N}}(\zeta)$.

Lastly, the point is called *strongly stationary* (or simply *S-stationary*), if in addition to (2.1),

$$\lambda_{\mathfrak{M}_i} = 0 \quad \text{and} \quad \lambda_{\mathfrak{N}_i} = 0 \quad \forall i \in I_{\mathfrak{M}\mathfrak{N}}(\zeta).$$

Note 1 *It is clear that S-stationarity leads to M-stationarity, which subsequently implies W-stationarity.*

To move forward, consider the following nonlinear programming problem corresponding to a partition (Ω_1, Ω_2) of $I_{\mathfrak{M}\mathfrak{N}}$:

$$\begin{aligned} \min\ & f(\xi) = (f_1(\xi), \ldots, f_m(\xi)), \\ \text{s.t. } & \phi_s(\xi) \leq 0, \quad \forall s \in S, \\ & \psi_k(\xi) = 0, \quad \forall k \in K, \\ & \mathfrak{M}_i(\xi) = 0, \quad \forall i \in I_{\mathfrak{M}} \cup \Omega_1, \\ & \mathfrak{N}_i(\xi) = 0, \quad \forall i \in I_{\mathfrak{N}} \cup \Omega_2. \end{aligned} \tag{2.2}$$

The feasible set of (2.2) is given by

$$\begin{aligned} \mathcal{F}_{\Omega_1,\Omega_2} :=& \big\{\xi \in \mathbb{R}^n : \phi_s(\xi) \leq 0,\ s \in S;\ \psi_k(\xi) = 0,\ k \in K;\ \mathfrak{M}_i(\xi) = 0,\ i \in I_{\mathfrak{M}} \cup \Omega_1; \\ & \mathfrak{N}_i(\xi) = 0,\ i \in I_{\mathfrak{N}} \cup \Omega_2\big\}. \end{aligned}$$

It can be easily verified that $\mathcal{F}_{\Omega_1,\Omega_2} \subseteq \mathcal{F}$. Next, we define the Abadie-type constraint qualification as follows:

$$\partial_T\text{-ACQ}(\Omega_1, \Omega_2): \quad \mathcal{L}(\Omega_1, \Omega_2)(\zeta) \subseteq T(\mathcal{F}_{\Omega_1,\Omega_2}, \zeta),$$

where

$$\begin{aligned} \mathcal{L}(\Omega_1, \Omega_2)(\zeta) =& \left(\bigcup_{s \in S} \partial_T \phi_s(\zeta)\right)^o \cap \left(\bigcap_{k \in K} \partial_T \psi_k(\zeta)\right)^\perp \\ & \cap \left(\bigcup_{i \in I_{\mathfrak{M}} \cup \Omega_1} \partial_T \mathfrak{M}_i(\zeta)\right)^\perp \cap \left(\bigcup_{i \in I_{\mathfrak{N}} \cup \Omega_2} \partial_T \mathfrak{N}_i(\zeta)\right)^\perp. \end{aligned}$$

Theorem 2.1 *[28] Let ζ be a local efficient solution of NMPSC. Suppose that there exists a partition $(\Omega_1, \Omega_2) \in \mathcal{P}(I_{\mathfrak{M}\mathfrak{N}})$ such that ∂_T-ACQ(Ω_1, Ω_2) holds for ζ and*

$$\begin{aligned} D =& \, cone\left(\bigcup_{s \in S} \partial_T \phi_s(\zeta)\right) \\ & + lin\left(\bigcup_{k \in K} \partial_T \psi_k(\zeta) \cup \bigcup_{i \in I_{\mathfrak{M}} \cup I_{\Omega_1}} \partial_T \mathfrak{M}_i(\zeta) \cup \bigcup_{i \in I_{\mathfrak{N}} \cup I_{\Omega_2}} \partial_T \mathfrak{N}_i(\zeta)\right) \end{aligned} \tag{2.3}$$

is closed, then ζ is an M-stationary point of NMPSC.

Let us revisit some key notations from interval-valued analysis, as discussed in [42,43,44].

Let $\mathcal{K}_c := \{[\gamma^L, \gamma^U] : \gamma^L, \gamma^U \in \mathbb{R}, \gamma^L \leq \gamma^U\}$ be the class of all closed intervals in $\mathbb{R}$. Let $\Gamma := [\gamma^L, \gamma^U]$ and $\Theta := [\theta^L, \theta^U]$ be two intervals in $\mathcal{K}_c$. Then,

(a) Addition of intervals: $\Gamma + \Theta := \{\gamma + \theta : \gamma \in \Gamma, \theta \in \Theta\} = [\gamma^L + \theta^L, \gamma^U + \theta^U]$;

(b) Subtraction of intervals: $\Gamma - \Theta := \{\gamma - \theta : \gamma \in \Gamma, \theta \in \Theta\} = [\gamma^L - \theta^U, \gamma^U - \theta^L]$;

(c) Scalar multiplication: For each $k \in \mathbb{R}$,

$$k\Gamma := \{k\gamma : \gamma \in \Gamma\} = \begin{cases} [k\gamma^L, k\gamma^U], & \text{if } k \geq 0 \\ [k\gamma^U, k\gamma^L], & \text{if } k < 0. \end{cases}$$

If $\gamma^L = \gamma^U$, then $\Gamma = [\gamma, \gamma] = \{\gamma\}$.

Definition 2.2 *[45, Definition 3] Consider two intervals $\Gamma = [\gamma^L, \gamma^U], \Theta = [\theta^L, \theta^U] \in \mathcal{K}_c$. We say that:*

(i) $\Gamma \leq_{LU} \Theta$ iff $\gamma^L \leq \theta^L$ and $\gamma^U \leq \theta^U$.

(ii) $\Gamma <_{LU} \Theta$ *iff* $\Gamma \leq_{LU} \Theta$ *and* $\Gamma \neq \Theta$,
or, equivalently,
$\Gamma <_{LU} \Theta$ *iff*

$$\begin{cases} \gamma^L < \theta^L \\ \gamma^U \leq \theta^U \end{cases} \quad or \quad \begin{cases} \gamma^L \leq \theta^L \\ \gamma^U < \theta^U \end{cases} \quad or \quad \begin{cases} \gamma^L < \theta^L \\ \gamma^U < \theta^U. \end{cases}$$

(iii) $\Gamma <^s_{LU} \Theta$ *iff* $\gamma^L < \theta^L$ *and* $\gamma^U < \theta^U$.

Let $\Gamma := (\Gamma_1, \ldots, \Gamma_n)$ be a vector consisting of interval values, where each component $\Gamma_i = [\gamma_i^L, \gamma_i^U]$, $i = 1, \ldots, n$, is a compact interval. Let Γ and Θ be two interval-valued vectors. If Γ_i and Θ_i are comparable for each $i = 1, \ldots, n$, then:

(i) $\Gamma \preceq_{LU} \Theta$ iff $\Gamma_i \leq_{LU} \Theta_i$ for all $i = 1, \ldots, n$;

(ii) $\Gamma \prec_{LU} \Theta$ iff $\Gamma_i \leq_{LU} \Theta_i$ for all $i = 1, \ldots, n$, $i \neq r$, and $\Gamma_r <_{LU} \Theta_r$ for some r.

Let us now examine the following nonsmooth interval-valued multiobjective semi-infinite programming problem with switching constraints:

$$\begin{aligned} &\min F(\xi) := (F_1(\xi), \ldots, F_m(\xi)) \\ &\text{s.t. } \phi_s(\xi) \leq 0, \quad \forall s \in S, \\ &\quad \psi_k(\xi) = 0, \quad \forall k \in K := \{1, \ldots, q\}, \\ &\quad \mathfrak{M}_i(\xi)\mathfrak{N}_i(\xi) = 0, \quad \forall i \in I := \{1, \ldots, l\}, \end{aligned} \tag{IVNMPSC}$$

where S denotes a nonempty index set, which may be infinite and $F_j : \mathbb{R}^n \to \mathcal{K}_c$, $j \in J$ are interval-valued functions defined by $F_j(\xi) := [f_j^L(\xi), f_j^U(\xi)]$. The functions f_j^L, f_j^U, $j \in J := \{1, \ldots, m\}$, ϕ_s, $s \in S$, ψ_k, $k \in K$, $\mathfrak{M}_i, \mathfrak{N}_i$, $i \in I$ are real-valued functions defined on $\mathbb{R}^n$ and are not necessarily convex nor differentiable such that $f_j^L(\xi) \leq f_j^U(\xi)$.

The feasible region of the IVNMPSC is same as the feasible region of NMPSC, which is denoted by $\mathcal{F}$.

The notion of LU-efficient solutions and weakly LU-efficient solutions for IVNMPSC are given as follows:

Definition 2.3 ***[LU-efficient solution]*** *[46] A point $\zeta \in \mathcal{F}$ is said to be an LU-efficient solution (or local LU-efficient solution) of IVNMPSC, if there does not exist any $\xi \in \mathcal{F}$ (respectively, $\xi \in B(\zeta, \delta) \cap \mathcal{F}$, for some $\delta > 0$) such that $F(\xi) \prec_{LU} F(\zeta)$.*

Definition 2.4 ***[Weakly LU-efficient solution]*** *[46] A point $\zeta \in \mathcal{F}$ is said to be a weakly LU-efficient solution (or local weakly LU-efficient solution) of IVNMPSC, if there does not exist any $\xi \in \mathcal{F}$ (respectively, $\xi \in B(\zeta, \delta) \cap \mathcal{F}$, for some $\delta > 0$) such that $F_j(\xi) <_{LU} F_j(\zeta)$ for all $j \in J$.*

Based on the results in [45], we can link the weakly LU-efficient solutions of IVNMPSC and the weakly efficient solutions of MOP1, as follows:

$$\min f(\xi) := (f_i^L(\xi), \ldots, f_m^L(\xi), f_1^U(\xi), ..., f_m^U(\xi)) \text{ s. t. } \xi \in \mathcal{F}. \tag{MOP1}$$

Theorem 2.2 *[45, Lemma 4] A point $\zeta \in \mathcal{F}$ is a weakly LU-efficient of the IVNMPSC iff ζ is a weakly efficient solution of the MOP1.*

3. Stationary Conditions for Local Efficient Solutions of IVNMPSC

In this section, we establish the M-stationary conditions for a locally efficient solution to the IVN-MPSC. We begin by defining the concepts of W-stationary, M-stationary, and S-stationary points for this problem IVNMPSC.

Definition 3.1 *A point ζ is called weakly stationary (or simply W-stationary) point for the problem IVNMPSC if there exist multipliers $\lambda = (\lambda_1^L, \ldots, \lambda_m^L, \lambda_1^U, \ldots, \lambda_m^U) \geq 0_{\mathbb{R}^{2m}}$ with $\sum_{j=1}^m (\lambda_j^L + \lambda_j^U) = 1$, $\lambda^\phi \in A(\zeta)$, $\lambda^\psi = (\lambda_1^\psi, \ldots, \lambda_q^\psi) \in \mathbb{R}^q$, $\lambda^{\mathfrak{M}} = (\lambda_1^{\mathfrak{M}}, \ldots, \lambda_l^{\mathfrak{M}}) \in \mathbb{R}^l$, and $\lambda^{\mathfrak{N}} = (\lambda_1^{\mathfrak{N}}, \ldots, \lambda_l^{\mathfrak{N}}) \in \mathbb{R}^l$, that satisfy the following system of conditions*

$$
\begin{aligned}
0 \in \sum_{j \in J} \lambda_j^L \, \partial_T f_j^L(\zeta) + \sum_{j \in J} \lambda_j^U \, \partial_T f_j^U(\zeta) + \sum_{s \in S(\zeta)} \lambda_s^\phi \, \partial_T \phi_s(\zeta) + \sum_{k \in K} \lambda_k^\psi \, \partial_T \psi_k(\zeta) \\
+ \sum_{i \in I} \lambda_i^{\mathfrak{M}} \, \partial_T \mathfrak{M}_i(\zeta) + \sum_{i \in I} \lambda_i^{\mathfrak{N}} \, \partial_T \mathfrak{N}_i(\zeta),
\end{aligned}
\tag{3.1}
$$

$$
\forall s \in S(\zeta) : \lambda_s^\phi \geq 0, \quad \forall i \in I_{\mathfrak{N}}(\zeta) : \lambda_i^{\mathfrak{M}} = 0, \quad \forall i \in I_{\mathfrak{M}}(\zeta) : \lambda_i^{\mathfrak{N}} = 0.
$$

It is called Mordukhovich-stationary (or simply M-stationary) if, along with condition (3.1), $\lambda_i^{\mathfrak{M}} \lambda_i^{\mathfrak{N}} = 0 \; \forall i \in I_{\mathfrak{M}\mathfrak{N}}(\zeta)$.

Lastly, the point is called strongly stationary (or simply S-stationary), if in addition to (3.1),

$$
\lambda_{\mathfrak{M}_i} = 0 \quad \text{and} \quad \lambda_{\mathfrak{N}_i} = 0 \quad \forall i \in I_{\mathfrak{M}\mathfrak{N}}(\zeta).
$$

To find the M-stationary conditions for the local efficient solution of IVNMPSC, we analyze a nonlinear programming problem based on a partition (Ω_1, Ω_2) of the index set $I_{\mathfrak{M}\mathfrak{N}}$.

$$
\begin{aligned}
\min \; & F(\xi) = (F_1(\xi), \ldots, F_m(\xi)), \\
\text{s.t. } & \phi_s(\xi) \leq 0, \quad \forall s \in S, \\
& \psi_k(\xi) = 0, \quad \forall k \in K, \\
& \mathfrak{M}_i(\xi) = 0, \quad \forall i \in I_{\mathfrak{M}} \cup \Omega_1, \\
& \mathfrak{N}_i(\xi) = 0, \quad \forall i \in I_{\mathfrak{N}} \cup \Omega_2.
\end{aligned}
\tag{3.2}
$$

The feasible set of (3.2) is given by $\mathcal{F}_{\Omega_1, \Omega_2}$. We are now prepared to establish the necessary optimality conditions for local LU-efficient solutions of IVNMPSC.

Theorem 3.1 *Let ζ be a local LU-efficient solution of IVNMPSC. Assume that there exists a partition $(\Omega_1, \Omega_2) \in \mathcal{P}(I_{\mathfrak{M}\mathfrak{N}})$ such that ∂_T-ACQ(Ω_1, Ω_2) holds for ζ and*

$$
\begin{aligned}
D = \mathit{cone} \left(\bigcup_{s \in S} \partial_T \phi_s(\zeta) \right) \\
+ \mathit{lin} \left(\bigcup_{k \in K} \partial_T \psi_k(\zeta) \cup \bigcup_{i \in I_{\mathfrak{M}} \cup I_{\Omega_1}} \partial_T \mathfrak{M}_i(\zeta) \cup \bigcup_{i \in I_{\mathfrak{N}} \cup I_{\Omega_2}} \partial_T \mathfrak{N}_i(\zeta) \right)
\end{aligned}
\tag{3.3}
$$

is closed, then ζ is an M-stationary point of IVNMPSC.

Proof: Since $\zeta \in \mathcal{F}$ is a local LU-efficient solution of IVNMPSC problem, Theorem 2.2 implies that ζ is also a local efficient solution of problem NMPSC-1 given by

$$
\begin{aligned}
\min \; & (f_1^L(\xi), \ldots, f_m^L(\xi), f_1^U(\xi), \ldots, f_m^U(\xi)) \\
\text{s.t. } & \phi_s(\xi) \leq 0, \quad \forall s \in S, \\
& \psi_k(\xi) = 0, \quad \forall k \in K := \{1, \ldots, q\}, \\
& \mathfrak{M}_i(\xi) \mathfrak{N}_i(\xi) = 0, \quad \forall i \in I := \{1, \ldots, l\}.
\end{aligned}
$$

Since ∂_T-ACQ(Ω_1, Ω_2) holds for ζ and D is closed, therefore by Theorem 2.1 there exists multipliers $\lambda = (\lambda_1^L, \ldots, \lambda_m^L, \lambda_1^U, \ldots, \lambda_m^U) \geq 0_{\mathbb{R}^{2m}}$ with $\sum_{j=1}^m (\lambda_j^L + \lambda_j^U) = 1$, $\lambda^\phi \in A(\zeta)$, $\lambda^\psi = (\lambda_1^\psi, \ldots, \lambda_q^\psi) \in \mathbb{R}^q$,

$\lambda^{\mathfrak{M}} = (\lambda_1^{\mathfrak{M}}, \ldots, \lambda_l^{\mathfrak{M}}) \in \mathbb{R}^l$, and $\lambda^{\mathfrak{N}} = (\lambda_1^{\mathfrak{N}}, \ldots, \lambda_l^{\mathfrak{N}}) \in \mathbb{R}^l$, solving the system

$$\begin{aligned} 0 \in \sum_{j\in J} \lambda_j^L \, \partial_T f_j^L(\zeta) + \sum_{j\in J} \lambda_j^U \, \partial_T f_j^U(\zeta) + \sum_{s\in S(\zeta)} \lambda_s^\phi \, \partial_T \phi_s(\zeta) + \sum_{k\in K} \lambda_k^\psi \, \partial_T \psi_k(\zeta) \\ + \sum_{i\in I} \lambda_i^{\mathfrak{M}} \, \partial_T \mathfrak{M}_i(\zeta) + \sum_{i\in I} \lambda_i^{\mathfrak{N}} \, \partial_T \mathfrak{N}_i(\zeta), \end{aligned} \tag{3.4}$$

$$\forall s \in S(\zeta) : \lambda_s^\phi \geq 0, \forall i \in I_{\mathfrak{N}}(\zeta) : \lambda_i^{\mathfrak{M}} = 0, \forall i \in I_{\mathfrak{M}}(\zeta) : \lambda_i^{\mathfrak{N}} = 0 \, \text{and} \, \forall i \in I_{\mathfrak{M}\mathfrak{N}}(\zeta) : \lambda_i^{\mathfrak{M}} \lambda_i^{\mathfrak{N}} = 0.$$

Therefore, ζ satisfies the M-stationarity conditions for the IVNMPSC problem, thus completing our proof. □

A remark of the Theorem 3.1, $f_j^L = f_j^U = f_j$ for every $j \in J$ is given as follows:

Remark 3.1 *If $f_j^L = f_j^U = f_j$ for every $j \in J$, then Theorem 3.1 simplifies to Theorem 2.1 as given by Jennane and Kalmoun [28].*

We present the following example to demonstrate Theorem 3.1.

Example 3.1 *Consider an IVNMPSC in $\mathbb{R}^2$ as follows:*

$$\begin{aligned} &\min F(\xi) := [f_1^L(\xi), f_1^U(\xi)] \\ &s.t. \ \phi_s(\xi) \leq 0, \quad \forall s \in S, \\ &\quad \psi_1(\xi) = 0, \\ &\quad \mathfrak{M}_i(\xi)\mathfrak{N}_i(\xi) = 0, \quad \forall i \in I := \{1, \ldots, 3\}, \end{aligned} \tag{P}$$

where

$$f_1^L(\xi_1, \xi_2) = \begin{cases} \frac{\xi_1^3}{\xi_2} - \xi_1, & \xi_2 \neq 0 \\ -\xi_1, & \xi_2 = 0 \end{cases}, \quad f_1^U(\xi_1, \xi_2) = |\xi_1| + |\xi_2|$$

$$\phi_s(\xi_1, \xi_2) = -\xi_1 - s, \quad \forall s \in S = [0, +\infty)$$

$$\psi_1(\xi_1, \xi_2) = \begin{cases} 0, & \xi_2 \geq 0 \\ -\xi_2, & \xi_2 < 0 \end{cases}$$

$$\mathfrak{M}_1(\xi_1, \xi_2) = \begin{cases} \xi_2, & \xi_2 \geq 0 \\ 0, & \xi_2 < 0 \end{cases}, \quad \mathfrak{N}_1(\xi_1, \xi_2) = \begin{cases} \xi_2, & \xi_2 \geq 0 \\ -\xi_2^2, & \xi_2 < 0 \end{cases}$$

$$\mathfrak{M}_2(\xi_1, \xi_2) = \begin{cases} \xi_2, & \xi_2 \geq 0 \\ -\xi_2^2, & \xi_2 < 0 \end{cases}, \quad \mathfrak{N}_2(\xi_1, \xi_2) = \begin{cases} 1, & \xi_2 \geq 0 \\ 1 - \xi_2, & \xi_2 < 0 \end{cases}$$

$$\mathfrak{M}_3(\xi_1, \xi_2) = \begin{cases} 1, & \xi_2 \geq 0 \\ 1 - \xi_2, & \xi_2 < 0 \end{cases}, \quad \mathfrak{N}_3(\xi_1, \xi_2) = \begin{cases} \xi_2, & \xi_2 \geq 0 \\ 0, & \xi_2 < 0. \end{cases}$$

We have the feasible set $\mathcal{F} = \mathbb{R}_+ \times \{0\}$ as shown in the figure 1, and $\zeta = (0, 0) \in \mathcal{F}$ is a local LU-efficient solution of (P).

The tangential subdifferentials at ζ are

$$\begin{aligned} &\partial^T f_1^L(\zeta) = \{(-1, 0)\}, \quad \partial^T f_1^U(\zeta) = [-1, 1] \times [-1, 1], \quad \partial^T \phi_s(\zeta) = \{(-1, 0)\} \quad \forall s \in S, \\ &\partial^T \psi_1(\zeta) = \partial^T \mathfrak{M}_3(\zeta) = \partial^T \mathfrak{N}_2(\zeta) = \{0\} \times [-1, 0], \\ &\partial^T \mathfrak{M}_1(\zeta) = \partial^T \mathfrak{M}_2(\zeta) = \partial^T \mathfrak{N}_1(\zeta) = \partial^T \mathfrak{N}_3(\zeta) = \{0\} \times [0, 1]. \end{aligned}$$

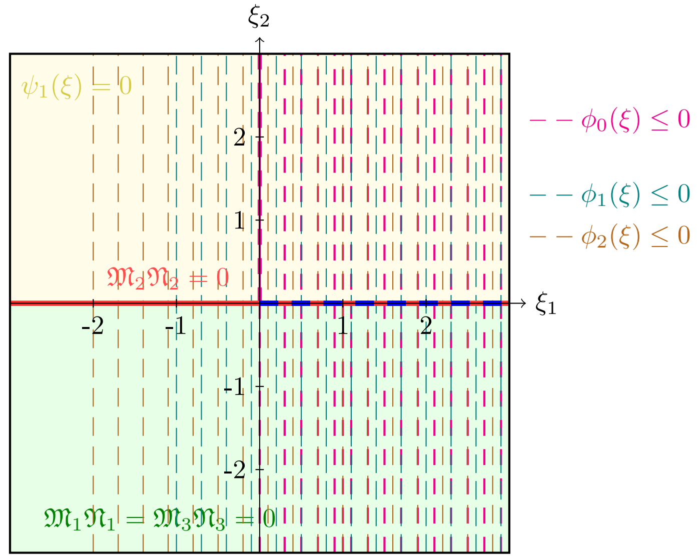

Figure 1: Blue dashed line shows the feasible region of the problem (P).

The tangent cone to $\mathcal{F}$ and the active constraint index sets at ζ are

$$T(\mathcal{F},\zeta) = \mathbb{R}_+ \times \{0\}, \quad I_{\mathfrak{M}\mathfrak{N}}(\zeta) = \{1\}, \quad I_{\mathfrak{M}}(\zeta) = \{2\}, \quad I_{\mathfrak{N}}(\zeta) = \{3\}.$$

In choosing $\Omega_1 = \emptyset$ and $\Omega_2 = I_{\mathfrak{M}\mathfrak{N}}(\zeta)$, we can easily verify that ∂^T-$ACQ(\Omega_1, \Omega_2)$ is satisfied at ζ, and the set D, as defined in equation (3.3), is closed. Thus, ζ satisfies all the assumptions of Theorem 3.1.

Now, by selecting the following multipliers:

$$\lambda_1^L = \lambda_s^\phi = \tfrac{1}{2}, \quad \lambda_1^U = \lambda_1^\psi = 1, \quad \lambda_{\mathfrak{M}_1} = \lambda_{\mathfrak{M}_2} = \lambda_{\mathfrak{N}_3} = \tfrac{1}{3}, \quad \lambda_{\mathfrak{M}_3} = \lambda_{\mathfrak{N}_1} = \lambda_{\mathfrak{N}_2} = 0$$

the condition (3.1) is satisfied with $\lambda_{\mathfrak{M}_i}\lambda_{\mathfrak{N}_i} = 0$, which implies that ζ is an M-stationary point of (P).

Remark 3.2 *The use of tangential subdifferentials offers certain advantages compared to other subdifferentials like Clarke subdifferentials. This is particularly relevant for our problem because the involved functions do not necessarily possess local Lipschitz properties at the local LU-efficient solution of IVN-MPSC, as demonstrated by the function f_1^L in the previous example.*

4. Sufficient M-Stationary Conditions

The following definitions are instrumental in establishing the sufficient M-stationarity conditions for problem IVNMPSC.

Definition 4.1 *([47]) Let $A \subset \mathbb{R}^n$ be a convex set, $\eta : \mathbb{R}^n \to \mathbb{R}$, and $\zeta \in A$. Then η is said to be*

(i) quasiconvex at ζ if $\forall\, \xi \in A,\ \eta(\xi) \leq \eta(\zeta) \Rightarrow \eta(\lambda\xi + (1-\lambda)\zeta) \leq \eta(\zeta),\ \forall\lambda \in (0,1)$.

(ii) Dini-convex at ζ if $\forall\, \xi \in A,\ \eta(\xi) \geq \eta(\zeta) + \eta'(\zeta, \xi - \zeta)$.

(iii) strictly Dini-convex at ζ if $\forall\, \xi \in A \setminus \{\zeta\},\ \eta(\xi) > \eta(\zeta) + \eta'(\zeta, \xi - \zeta)$.

(iv) Dini-pseudoconvex at ζ if $\forall\, \xi \in A,\ \eta(\xi) < \eta(\zeta) \Rightarrow \eta'(\zeta, \xi - \zeta) < 0$.

(v) strictly Dini-pseudoconvex at ζ if $\forall\, \xi \in A \setminus \{\zeta\},\ \eta(\xi) \leq \eta(\zeta) \Rightarrow \eta'(\zeta, \xi - \zeta) < 0$.

(vi) Dini-quasiconvex at ζ if $\forall\, \xi \in A,\ \eta(\xi) \leq \eta(\zeta) \Rightarrow \eta'(\zeta, \xi - \zeta) \leq 0$.

(vii) Dini-linearlike at ζ if $\forall\, \xi \in A,\ \eta(\xi) = \eta(\zeta) + \eta'(\zeta, \xi - \zeta)$.

(viii) *quasilinear, Dini-quasilinear, or Dini-pseudolinear at ζ if both η and $-\eta$ are quasiconvex, Dini-quasiconvex, or Dini-pseudoconvex at ζ, respectively.*

(ix) *quasiconvex on A if it is quasiconvex at every point of A. The other properties (Dini-convex, Dini-pseudoconvex, etc.) can similarly be defined pointwise and then extended to the whole set A.*

Remark 4.1 *([47,48]) Let $A \subset \mathbb{R}^n$ be convex, $\eta : \mathbb{R}^n \to \mathbb{R}$, and $\zeta \in A$. Then η is*

(i) *Dini-quasiconvex at ζ, if it is directionally differentiable and quasiconvex at ζ.*

(ii) *quasiconvex at ζ, if it is Dini-pseudoconvex at ζ and continuous on A.*

(iii) *quasiconvex on A, if it is Dini-quasiconvex and continuous on A.*

(iv) *both Dini-pseudolinear and Dini-quasilinear at ζ, if it is Dini-linearlike at ζ.*

(v) *Dini-quasilinear at ζ, if it is quasilinear and directionally differentiable at ζ.*

(vi) *both Dini-pseudoconvex and Dini-quasiconvex at ζ, if it is Dini-convex at ζ.*

Remark 4.2 *[38] Let $A \subset \mathbb{R}^n$ be a convex set, $\eta : \mathbb{R}^n \to \mathbb{R}$, and $\zeta \in A$. Assume η is tangentially convex at ζ. If η is*

(i) *Dini-convex at ζ and $\xi \in A$, then*

$$\eta(\xi) \geq \eta(\zeta) + \langle (\zeta)^*, \xi - \zeta \rangle, \quad \forall \zeta^* \in \partial_T \eta(\zeta).$$

(ii) *strictly Dini-convex at ζ and $\xi \in A \setminus \{\zeta\}$, then*

$$\eta(\xi) > \eta(\zeta) + \langle \zeta^*, \xi - \zeta \rangle, \quad \forall \zeta^* \in \partial_T \eta(\zeta).$$

(iii) *Dini-pseudoconvex at ζ and $\xi \in A$ with $\eta(\xi) < \eta(\zeta)$, then*

$$\langle \zeta^*, \xi - \zeta \rangle < 0, \quad \forall \zeta^* \in \partial_T \eta(\zeta).$$

(iv) *strictly Dini-pseudoconvex at ζ and $\xi \in A \setminus \{\zeta\}$ with $\eta(\xi) \leq \eta(\zeta)$, then*

$$\langle \zeta^*, \xi - \zeta \rangle < 0, \quad \forall \zeta^* \in \partial_T \eta(\zeta).$$

(v) *Dini-quasiconvex at ζ and $\xi \in A$ with $\eta(\xi) \leq \eta(\zeta)$, then*

$$\langle \zeta^*, \xi - \zeta \rangle \leq 0, \quad \forall \zeta^* \in \partial_T \eta(\zeta).$$

We divide the index set as follows:

$$\begin{aligned}
&I_\psi^+ := \{k \in K \mid \lambda_i^\psi > 0\}, \ I_\psi^- := \{k \in K \mid \lambda_i^\psi < 0\}.\\
&I_{\mathfrak{M}\mathfrak{N}}^{++} := \{i \in I_{\mathfrak{M}\mathfrak{N}} \mid \lambda_i^{\mathfrak{M}} > 0, \lambda_i^{\mathfrak{N}} > 0\}.\\
&I_{\mathfrak{M}\mathfrak{N}}^{0+} := \{i \in I_{\mathfrak{M}\mathfrak{N}} \mid \lambda_i^{\mathfrak{M}} = 0, \lambda_i^{\mathfrak{N}} > 0\}.\\
&I_{\mathfrak{M}\mathfrak{N}}^{+0} := \{i \in I_{\mathfrak{M}\mathfrak{N}} \mid \lambda_i^{\mathfrak{M}} > 0, \lambda_i^{\mathfrak{N}} = 0\}.\\
&I_{\mathfrak{M}\mathfrak{N}}^{0-} := \{i \in I_{\mathfrak{M}\mathfrak{N}} \mid \lambda_i^{\mathfrak{M}} = 0, \lambda_i^{\mathfrak{N}} < 0\}.\\
&I_{\mathfrak{M}\mathfrak{N}}^{-0} := \{i \in I_{\mathfrak{M}\mathfrak{N}} \mid \lambda_i^{\mathfrak{M}} < 0, \lambda_i^{\mathfrak{N}} = 0\}.\\
&I_{\mathfrak{M}}^+ := \{i \in I_{\mathfrak{M}} \,|\, \lambda_i^{\mathfrak{M}} > 0\}, \ I_{\mathfrak{M}}^- := \{i \in I_{\mathfrak{M}} \,|\, \lambda_i^{\mathfrak{M}} < 0\}.\\
&I_{\mathfrak{N}}^+ := \{i \in I_{\mathfrak{N}} \,|\, \lambda_i^{\mathfrak{N}} > 0\}, \ I_{\mathfrak{N}}^- := \{i \in I_{\mathfrak{N}} \,|\, \lambda_i^{\mathfrak{N}} < 0\}.
\end{aligned}$$

The next theorem demonstrates that the M-stationarity conditions for IVNMPSC serve as sufficient optimality criteria for identifying weakly LU-efficient solutions of the problem.

Theorem 4.1 *Let $\mathcal{F}$ be a convex set and $\zeta \in \mathcal{F}$. Assume that*

(i) there exist $\lambda_j^L, \lambda_j^U > 0$ $(j \in J)$ such that M-stationary conditions holds at ζ for IVNMPSC.

(ii) $f_j^L, f_j^U (j \in J)$ are tangentially convex and Dini-convex at ζ.

(iii) $\phi_s(s \in S(\zeta)), \psi_k(k \in I_\psi^+), -\psi_k(k \in I_\psi^-), \mathfrak{M}_i(i \in I_{\mathfrak{M}}^+ \cup I_{\mathfrak{M}\mathfrak{N}}^{+0} \cup I_{\mathfrak{M}\mathfrak{N}}^{++}), -\mathfrak{M}_i(i \in I_{\mathfrak{M}}^- \cup I_{\mathfrak{M}\mathfrak{N}}^{-0}), \mathfrak{N}_i(i \in I_{\mathfrak{N}}^+ \cup I_{\mathfrak{M}\mathfrak{N}}^{0+} \cup I_{\mathfrak{M}\mathfrak{N}}^{++}), -\mathfrak{N}_i(i \in I_{\mathfrak{N}}^- \cup I_{\mathfrak{M}\mathfrak{N}}^{0-})$ are tangentially convex and Dini-quasiconvex at ζ.

If $I_{\mathfrak{M}}^- \cup I_{\mathfrak{N}}^- \cup I_{\mathfrak{M}\mathfrak{N}}^{0-} \cup I_{\mathfrak{M}\mathfrak{N}}^{-0} = \emptyset$, then ζ is a weakly LU-efficient solution of IVNMPSC.

Proof: Assume that ζ is not a weakly LU-efficient solution of IVNMPSC. Then there exists $\xi \in \mathcal{F}$ such that

$$F_j(\xi) <_{LU} F_j(\zeta),\ \forall j \in J,$$

or equivalently $\forall j \in J$,

(i) $f_j^L(\xi) < f_j^L(\zeta)$ and $f_j^U(\xi) < f_j^U(\zeta)$; or

(ii) $f_j^L(\xi) < f_j^L(\zeta)$ and $f_j^U(\xi) \leq f_j^U(\zeta)$; or

(iii) $f_j^L(\xi) \leq f_j^L(\zeta)$ and $f_j^U(\xi) < f_j^U(\zeta)$.

Using the inequalities stated above, together with the tangential convexity and Dini-convexity of f_j^L, f_j^U at ζ, for each $j \in J$ one has

(i) $\left\langle \zeta^*_{f_j^L}, \xi - \zeta \right\rangle < 0,\ \forall \zeta^*_{f_j^L} \in \partial_T f_j^L(\zeta)$ and $\left\langle \zeta^*_{f_j^U}, \xi - \zeta \right\rangle < 0,\ \forall \zeta^*_{f_j^U} \in \partial_T f_j^U(\zeta)$; or

(ii) $\left\langle \zeta^*_{f_j^L}, \xi - \zeta \right\rangle < 0,\ \forall \zeta^*_{f_j^L} \in \partial_T f_j^L(\zeta)$ and $\left\langle \zeta^*_{f_j^U}, \xi - \zeta \right\rangle \leq 0,\ \forall \zeta^*_{f_j^U} \in \partial_T f_j^U(\zeta)$; or

(iii) $\left\langle \zeta^*_{f_j^L}, \xi - \zeta \right\rangle \leq 0,\ \forall \zeta^*_{f_j^L} \in \partial_T f_j^L(\zeta)$ and $\left\langle \zeta^*_{f_j^U}, \xi - \zeta \right\rangle < 0,\ \forall \zeta^*_{f_j^U} \in \partial_T f_j^U(\zeta)$;

Also $\lambda_j^L, \lambda_j^U > 0$ $(j \in J)$, therefore from above inequalities, we get

$$\left\langle \sum_{j \in J} \lambda_j^L \zeta^*_{f_j^L} + \sum_{j \in J} \lambda_j^U \zeta^*_{f_j^U}, \xi - \zeta \right\rangle < 0,\ \forall \zeta^*_{f_j^L} \in \partial_T f_j^L(\zeta),\ \forall \zeta^*_{f_j^U} \in \partial_T f_j^U(\zeta),\ \forall j \in J. \tag{4.1}$$

Again for each $s \in S(\zeta)$, $\phi_s(\xi) \leq 0 = \phi_s(\zeta)$. Hence by tangential convexity and Dini-quasiconvexity of ϕ_s, we have

$$\left\langle \zeta^*_{\phi_s}, \xi - \zeta \right\rangle \leq 0, \quad \forall \zeta^*_{\phi_s} \in \partial_T \phi_s(\zeta). \tag{4.2}$$

For any feasible point ξ of IVNMPSC and for each $k \in I_\psi^-$, $0 = -\psi_k(\zeta) = \psi_k(\xi)$. Hence by tangential convexity and Dini-quasiconvexity of ψ_k, we have

$$\left\langle \zeta^*_{\psi_k}, \xi - \zeta \right\rangle \geq 0, \quad \forall \zeta^*_{\psi_k} \in \partial_T \psi_k(\zeta), \quad \forall\ k \in I_\psi^-. \tag{4.3}$$

Similarly, we have

$$\left\langle \zeta^*_{\psi_k}, \xi - \zeta \right\rangle \leq 0, \quad \forall\ \zeta^*_{\psi_k} \in \partial_T \psi_k(\zeta), \quad \forall\ k \in I_\psi^+. \tag{4.4}$$

Also $\mathfrak{M}_i(\xi) \leq \mathfrak{M}_i(\zeta),\ \forall\ i \in I_{\mathfrak{M}}^+ \cup I_{\mathfrak{M}\mathfrak{N}}^{+0}$ and $\mathfrak{N}_i(\xi) \leq \mathfrak{N}_i(\zeta),\ \forall\ i \in I_{\mathfrak{N}}^+ \cup I_{\mathfrak{M}\mathfrak{N}}^{0+}$. Since all of these functions are quasiconvex, we get

$$\left\langle \zeta^*_{\mathfrak{M}_i}, \xi - \zeta \right\rangle \leq 0,\ \forall\ \zeta^*_{\mathfrak{M}_i} \in \partial_T \mathfrak{M}_i(\zeta),\ \forall\ i \in I_{\mathfrak{M}}^+ \cup I_{\mathfrak{M}\mathfrak{N}}^{+0}, \tag{4.5}$$

$$\left\langle \zeta^*_{\mathfrak{N}_i}, \xi - \zeta \right\rangle \leq 0,\ \forall\ \zeta^*_{\mathfrak{N}_i} \in \partial_T \mathfrak{N}_i(\zeta),\ \forall\ i \in I_{\mathfrak{N}}^+ \cup I_{\mathfrak{M}\mathfrak{N}}^{0+}. \tag{4.6}$$

Since $I_{\mathfrak{M}}^{-} \cup I_{\mathfrak{N}}^{-} \cup I_{\mathfrak{M}\mathfrak{N}}^{0-} \cup I_{\mathfrak{M}\mathfrak{N}}^{-0} = \emptyset$, we get

$$\left\langle \sum_{I_{\mathfrak{M}} \cup I_{\mathfrak{M}\mathfrak{N}}} \lambda_i^{\mathfrak{M}} \zeta_{\mathfrak{M}_i}^*, \xi - \zeta \right\rangle \leq 0, \quad \forall\, \zeta_{\mathfrak{M}_i}^* \in \partial_T \mathfrak{M}_i(\zeta),$$

$$\left\langle \sum_{I_{\mathfrak{M}\mathfrak{N}} \cup I_{\mathfrak{N}}} \lambda_i^{\mathfrak{N}} \zeta_{\mathfrak{N}_i}^*, \xi - \zeta \right\rangle \leq 0, \quad \forall\, \zeta_{\mathfrak{N}_i}^* \in \partial_T \mathfrak{N}_i(\zeta),$$

$$\left\langle \sum_{s \in S(\zeta)} \lambda_s^{\phi} \zeta_{\phi_s}^*, \xi - \zeta \right\rangle \leq 0, \quad \forall\, \zeta_{\phi_s}^* \in \partial_T \phi_s(\zeta),$$

$$\left\langle \sum_{k=1}^{q} \lambda_k^{\psi} \zeta_{\psi_k}^*, \xi - \zeta \right\rangle \leq 0, \quad \forall\, \zeta_{\psi_k}^* \in \partial_T \psi_k(\zeta).$$

Adding above inequalities with (4.1) we get

$$\left\langle \sum_{j \in J} \lambda_j^L \zeta_{f_j^L}^* + \sum_{j \in J} \lambda_j^U \zeta_{f_j^U}^* + \sum_{I_{\mathfrak{M}} \cup I_{\mathfrak{M}\mathfrak{N}}} \lambda_i^{\mathfrak{M}} \zeta_{\mathfrak{M}_i}^* + \sum_{I_{\mathfrak{M}\mathfrak{N}} \cup I_{\mathfrak{N}}} \lambda_i^{\mathfrak{N}} \zeta_{\mathfrak{N}_i}^* + \sum_{s \in S(\zeta)} \lambda_s^{\phi} \zeta_{\phi_s}^* + \sum_{k=1}^{q} \lambda_k^{\psi} \zeta_{\psi_k}^*, \xi - \zeta \right\rangle < 0$$

for any $\zeta_{f_j^L}^* \in \partial_T f_j^L(\zeta), \zeta_{f_j^U}^* \in \partial_T f_j^U(\zeta), \zeta_{\mathfrak{M}_i}^* \in \partial_T \mathfrak{M}_i, \zeta_{\mathfrak{N}_i}^* \in \partial_T \mathfrak{N}_i, \zeta_{\phi_s}^* \in \partial_T \phi_s(\zeta)$ and $\zeta_{\psi_k}^* \in \partial_T \psi_k(\zeta)$, which gives a contradiction that M-stationary conditions holds at ζ for IVNMPSC. Hence ζ is a weakly LU-efficient solution of IVNMPSC. This completes the proof. □

5. Conclusion

This paper investigates the optimality conditions for IVNMPSC. We establish necessary M-stationary conditions by applying an appropriate constraint qualification and utilizing tangential subdifferentials. Additionally, we develop sufficient optimality conditions through generalized convexity assumptions. Our theoretical findings are clarified through the illustrative examples. An interesting direction for future research is to expand the present results to the setting of approximate solutions of IVNMPSC. The approaches and findings presented in [49,50] could be explored within this context. Furthermore, the findings presented in this paper could be further expanded by utilizing the framework of directional convexificators [51,52] and by employing quasidifferential tools [53] to study approximate solutions.

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