



## Derivations in 3-Jordan Algebras

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**ABSTRACT:** In this paper, we study Lie algebras of derivations of a commutative algebra verifying the identity  $(x^3y)x - x^3(xy) = 0$ , known as the 3-Jordan algebra. We characterize the derivations via a quintuplet description, prove that the ideal  $J$  defined in the paper “A Variety containing Jordan and pseudo-composition algebras” is characteristic. We also furnish a necessary and sufficient condition for the ideal  $M$  defined in the aforementioned paper to be  $d$ -invariant for a derivation  $d$  and classify all 3-dimensional dimensionally nilpotent 3-Jordan algebras.

**Key Words:** Derivation, characteristic ideal, dimensionally nilpotent algebra, 3-Jordan algebra.

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### 1. Introduction

Throughout the paper,  $F$  denotes an infinite commutative field of characteristic  $\neq 2, 3$ . A commutative algebra  $\mathcal{A}$  over the field  $F$  is said to be a 3-Jordan algebra, if it satisfies the identity

$$(x^3, y, x) = (x^3y)x - x^3(yx) = 0, \text{ for all } x, y \in \mathcal{A}. \tag{1.1}$$

This identity holds in a pseudo-composition algebra [5]. Moreover, in the same paper, Giuliani and Peresi show that the identities of degree 5 in the pseudo-composition algebra are consequence of the commutativity and 3-Jordan identity. For more about pseudo-composition algebras see [3,4,5,6,8]. The 3-Jordan algebras generalize a Jordan algebra. In [7], the authors give a Peirce decomposition of 3-Jordan algebra and some examples of that algebras. They also prove that a simple 3-Jordan algebra with an idempotent is either a Jordan algebra or a Pseudo-composition algebra and they furnish a necessary and sufficient condition for 3-Jordan algebra with an idempotent to be a Jordan algebra or a Pseudo-composition algebra. In [6], Hentzel and Labra show that the commutative algebra  $\mathcal{A}$  satisfying the identity  $(x^2y)x + tx^3y = 0$  for all  $x, y \in \mathcal{A}$  and  $t \in F \setminus \{-1, 1\}$  is a 3-Jordan algebra. For the cases  $t = 1$  and  $t = -1$ , they find an example of algebras that are not 3-Jordan. Our aim is to study a derivation in the 3-Jordan algebra. A *derivation*  $d$ , in an algebra  $\mathcal{A}$  is an endomorphism of  $\mathcal{A}$  satisfying the equality

$$d(xy) = d(x)y + xd(y) \text{ for all } x, y \in \mathcal{A}.$$

The set  $Der_F(\mathcal{A})$  of all derivation in  $\mathcal{A}$  is a subspace of the associative algebra  $End_F(\mathcal{A})$  of all endomorphisms of  $\mathcal{A}$ . Since the commutator  $[d, d'] = dod' - d'od$  of two derivation in  $\mathcal{A}$  is still a derivation,  $Der_F(\mathcal{A})$  is a subalgebra of  $(End_F(\mathcal{A}), [ , ])$  called the *derivation algebra of  $\mathcal{A}$*  [11]. In [10], Osborn shows that over a perfect field of characteristic not 2 or 3, a dimensionally nilpotent Jordan algebras is either nilpotent or one-dimensional modulo its maximal nilpotent ideal. Dembega et al. study in [1, Theorem 4.9], Lie triple non nil-algebras that are dimensionally nilpotent. Since Jordan algebras are Lie

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triple algebras, they also describe in [1, Corollary 4.11], Jordan non nil-algebras that are dimensionally nilpotent. The paper is organized as follows: in section 2, we recall Peirce decomposition of 3-Jordan algebra with an idempotent. We also give some useful identities. Section 3 is devoted to our main result. We characterize the derivation in the 3-Jordan algebra and give some example. In the last section, we show that ideal  $J$  defined in [7] is a characteristic ideal and we furnish a necessary and sufficient condition for the ideal  $M$  defined in the aforementioned paper to be  $d$ -invariant for a derivation  $d$ . We also give a classification in dimension 3, up to isomorphism, of a 3-Jordan dimensionally nilpotent algebra. This classification shows that a 3-Jordan dimensionally nilpotent algebra is not necessary a Jordan algebra.

## 2. Preliminaries

We recall Peirce decomposition and some identities that hold in 3-Jordan algebra.

**Proposition 2.1** [7, Proposition 1] *Let  $\mathcal{A}$  be a 3-Jordan algebra with a nonzero idempotent  $e$ . Then,  $\mathcal{A}$  admits the following Peirce decomposition:*

$$\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_{-1} \oplus \mathcal{A}_{\frac{1}{2}} \oplus \mathcal{A}_0 \text{ with } \mathcal{A}_i = \{x \in \mathcal{A} \mid ex = ix\} \text{ where } i \in \left\{1, -1, 0, \frac{1}{2}\right\}.$$

Furthermore,

$$\begin{aligned} \mathcal{A}_1^2 &\subset \mathcal{A}_1, \quad \mathcal{A}_1\mathcal{A}_{-1} \subset \mathcal{A}_{-1}, \quad \mathcal{A}_1\mathcal{A}_{\frac{1}{2}} \subset \mathcal{A}_{\frac{1}{2}}, \quad \mathcal{A}_1\mathcal{A}_0 = 0, \\ \mathcal{A}_{\frac{1}{2}}^2 &\subset \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1, \quad \mathcal{A}_{\frac{1}{2}}\mathcal{A}_0 \subset \mathcal{A}_{\frac{1}{2}}, \quad \mathcal{A}_{\frac{1}{2}}\mathcal{A}_{-1} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_{\frac{1}{2}}, \\ &\mathcal{A}_{-1}^2 \subset \mathcal{A}_1 \oplus \mathcal{A}_0, \quad \mathcal{A}_{-1}\mathcal{A}_0 = 0, \\ &\mathcal{A}_0^2 \subset \mathcal{A}_0. \end{aligned}$$

Throughout the paper,  $\mathcal{A}$  denotes a 3-Jordan algebra with an idempotent  $e$ . Furthermore, it admits the Peirce decomposition, given by:  $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_{-1} \oplus \mathcal{A}_{\frac{1}{2}} \oplus \mathcal{A}_0$ .

**Lemma 2.1** [7, Lemma 1] *The following identities hold in the algebra  $\mathcal{A}$ :*

$$(a_1b_1)c_{\frac{1}{2}} = a_1(b_1c_{\frac{1}{2}}) + b_1(a_1c_{\frac{1}{2}}) \quad (2.1)$$

$$a_1[b_{\frac{1}{2}}c_{\frac{1}{2}}]_1 = [(a_1b_{\frac{1}{2}})c_{\frac{1}{2}}]_1 + [(a_1c_{\frac{1}{2}})b_{\frac{1}{2}}]_1 \quad (2.2)$$

$$a_1[b_{\frac{1}{2}}c_{\frac{1}{2}}]_{-1} = -2[(a_1b_{\frac{1}{2}})c_{\frac{1}{2}}]_{-1} = -2[(a_1c_{\frac{1}{2}})b_{\frac{1}{2}}]_{-1} \quad (2.3)$$

$$[(a_1b_{\frac{1}{2}})c_{\frac{1}{2}}]_0 = [(a_1c_{\frac{1}{2}})b_{\frac{1}{2}}]_0 \quad (2.4)$$

$$a_1[b_{\frac{1}{2}}c_{-1}]_1 = 2[(a_1b_{\frac{1}{2}})c_{-1}]_1 = -[(a_1c_{-1})b_{\frac{1}{2}}]_1 \quad (2.5)$$

$$2a_1[b_{\frac{1}{2}}c_{-1}]_{\frac{1}{2}} = 2[(a_1b_{\frac{1}{2}})c_{-1}]_{\frac{1}{2}} = -[(a_1c_{-1})b_{\frac{1}{2}}]_{\frac{1}{2}} \quad (2.6)$$

$$2[(a_1b_{\frac{1}{2}})c_{-1}]_0 = -[(a_1c_{-1})b_{\frac{1}{2}}]_0 \quad (2.7)$$

$$[(a_{\frac{1}{2}}, c_0, b_{\frac{1}{2}})]_1 = 0 \quad (2.8)$$

$$[(a_{\frac{1}{2}}c_0)b_{\frac{1}{2}}]_{-1} = 0 \quad (2.9)$$

$$c_0[a_{\frac{1}{2}}b_{\frac{1}{2}}]_0 = [(c_0a_{\frac{1}{2}})b_{\frac{1}{2}}]_0 + [a_{\frac{1}{2}}(c_0b_{\frac{1}{2}})]_0 \quad (2.10)$$

$$(a_{\frac{1}{2}}c_0)b_{-1} = 0 \quad (2.11)$$

$$(a_{\frac{1}{2}}b_{-1})c_0 = 0 \quad (2.12)$$

$$(a_0b_0)c_{\frac{1}{2}} = a_0(b_0c_{\frac{1}{2}}) + b_0(a_0c_{\frac{1}{2}}) \quad (2.13)$$

$$-3a_{\frac{1}{2}}[b_{-1}c_{-1}]_1 + 3a_{\frac{1}{2}}[b_{-1}c_{-1}]_0 = [[a_{\frac{1}{2}}b_{-1}]_{\frac{1}{2}}c_{-1}]_{\frac{1}{2}} + [[a_{\frac{1}{2}}c_{-1}]_{\frac{1}{2}}b_{-1}]_{\frac{1}{2}} \quad (2.14)$$

$$[(a_{\frac{1}{2}}b_{-1})]_1c_{-1} = -[(a_{\frac{1}{2}}c_{-1})]_1b_{-1} \quad (2.15)$$

$$3[a_{\frac{1}{2}}^2]_1a_{\frac{1}{2}} = 3[a_{\frac{1}{2}}^2]_0a_{\frac{1}{2}} + [[a_{\frac{1}{2}}^2]_{-1}a_{\frac{1}{2}}]_{\frac{1}{2}} \quad (2.16)$$

$$3[a_{\frac{1}{2}}b_{\frac{1}{2}}]_1c_{-1} = -[[a_{\frac{1}{2}}c_{-1}]_{\frac{1}{2}}b_{\frac{1}{2}}]_{-1} - [[b_{\frac{1}{2}}c_{-1}]_{\frac{1}{2}}a_{\frac{1}{2}}]_{-1} \quad (2.17)$$

A partial linearisation of identity (2.16) gives us

**Lemma 2.2** *The following identities hold in the algebra  $\mathcal{A}$ :*

$$6[a_{\frac{1}{2}}b_{\frac{1}{2}}]_1a_{\frac{1}{2}} + 3[a_{\frac{1}{2}}^2]_1b_{\frac{1}{2}} = 6[a_{\frac{1}{2}}b_{\frac{1}{2}}]_0a_{\frac{1}{2}} + 3[a_{\frac{1}{2}}^2]_0b_{\frac{1}{2}} + 2[[a_{\frac{1}{2}}b_{\frac{1}{2}}]_{-1}a_{\frac{1}{2}}]_{\frac{1}{2}} + [[a_{\frac{1}{2}}^2]_{-1}b_{\frac{1}{2}}]_{\frac{1}{2}} \quad (2.18)$$

$$3[a_{\frac{1}{2}}b_{\frac{1}{2}}]_1c_{\frac{1}{2}} + 3[c_{\frac{1}{2}}a_{\frac{1}{2}}]_1b_{\frac{1}{2}} + 3[b_{\frac{1}{2}}c_{\frac{1}{2}}]_1a_{\frac{1}{2}} = 3[a_{\frac{1}{2}}b_{\frac{1}{2}}]_0c_{\frac{1}{2}} + 3[c_{\frac{1}{2}}a_{\frac{1}{2}}]_0b_{\frac{1}{2}} + 3[c_{\frac{1}{2}}b_{\frac{1}{2}}]_0a_{\frac{1}{2}} + [[c_{\frac{1}{2}}b_{\frac{1}{2}}]_{-1}a_{\frac{1}{2}}]_{\frac{1}{2}} + [[a_{\frac{1}{2}}b_{\frac{1}{2}}]_{-1}c_{\frac{1}{2}}]_{\frac{1}{2}} + [[c_{\frac{1}{2}}a_{\frac{1}{2}}]_{-1}b_{\frac{1}{2}}]_{\frac{1}{2}} \quad (2.19)$$

### 3. Characterization of the derivations

In this section, we characterize the derivations in the algebra  $\mathcal{A}$ .

**Theorem 3.1** *Let  $d$  be a derivation of a 3-Jordan  $\mathcal{A}$  with idempotent  $e$ . Then,  $d$  is defined and only determined by the unique quintuplet  $(d(e), \varphi_0^d, \varphi_1^d, \varphi_{-1}^d, \varphi_{\frac{1}{2}}^d)$  satisfying the following assertions :*

- (i)  $d(e) \in \mathcal{A}_{\frac{1}{2}}$  ;
- (ii)  $d(x_i) = \varphi_i^d(x_i) + \frac{2}{2i-1}x_id(e)$  with  $\varphi_i^d \in \text{Der}_F(\mathcal{A}_i)$  for  $i \in \{0, 1\}$  ;
- (iii)  $d(x_i) = \varphi_i^d(x_i) + \sum_{k \in \{0, 1, j\}} \frac{1}{i-k}[x_id(e)]_k$  with  $\varphi_i^d \in \text{End}_F(\mathcal{A}_i)$  ;  $i, j \in \{-1, \frac{1}{2}\}$  pairwise distinct.
- (iv)  $\varphi_j^d(x_jx_i) = \varphi_j^d(x_j)x_i + x_j\varphi_i^d(x_i)$  for  $i \in \{0, 1\}$  and  $j \in \{-1, \frac{1}{2}\}$  ;
- (v)  $\varphi_i^d([x_{-1}y_{-1}]_i) = [\varphi_{-1}^d(x_{-1})y_{-1} + x_{-1}\varphi_{-1}^d(y_{-1}) - \frac{2}{3}y_{-1}[x_{-1}d(e)]_{\frac{1}{2}} - \frac{2}{3}x_{-1}[y_{-1}d(e)]_{\frac{1}{2}}]_i$  for  $i \in \{0, 1\}$  ;
- (vi)  $\varphi_i^d([x_{\frac{1}{2}}y_j]_i) = [\varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}})y_j + x_{\frac{1}{2}}\varphi_j^d(y_j) + \frac{2}{3}y_j[x_{\frac{1}{2}}d(e)]_{-1} + \frac{1}{i-k}[x_{\frac{1}{2}}y_j]_kd(e) + \frac{1}{j-k}x_{\frac{1}{2}}[y_jd(e)]_k]_i$  with  $i \in \{0, 1\}$  and  $j, k \in \{-1, \frac{1}{2}\}$  pairwise distinct ;
- (vii)  $\varphi_{\frac{1}{2}}^d([x_{\frac{1}{2}}y_{-1}]_{\frac{1}{2}}) = [\varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}})y_{-1} + x_{\frac{1}{2}}\varphi_{-1}^d(y_{-1})]_{\frac{1}{2}} + 2[x_{\frac{1}{2}}y_{-1}]_0d(e) - 2[x_{\frac{1}{2}}y_{-1}]_1d(e) - [y_{-1}d(e)]_0x_{\frac{1}{2}} - \frac{1}{2}[y_{-1}d(e)]_1x_{\frac{1}{2}}$  ;
- (viii)  $\varphi_{-1}^d([x_{\frac{1}{2}}y_{\frac{1}{2}}]_{-1}) = [\varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}})y_{\frac{1}{2}} + x_{\frac{1}{2}}\varphi_{\frac{1}{2}}^d(y_{\frac{1}{2}})]_{-1}$ .

**Proof:** Let  $d \in \text{Der}_F(\mathcal{A})$  and  $x \in \mathcal{A}$ . Set  $d(x) = \sum_{k \in \{-1, 0, 1, \frac{1}{2}\}} [d(x)]_k$  with  $[d(x)]_k \in \mathcal{A}_k$ .

- We have  $d(e) = d(e^2) = 2ed(e)$ . So  $d(e) \in \mathcal{A}_{\frac{1}{2}}$  and we get the assertion (i).
- Let  $i, j \in \{0, 1\}$  pairwise distinct and  $x_i, y_i \in \mathcal{A}_i$ . We have  $id(x_i) = d(ex_i) = d(e)x_i + ed(x_i) = d(e)x_i - [d(x_i)]_{-1} + \frac{1}{2}[d(x_i)]_{\frac{1}{2}} + \sum_{k \in \{0, 1\}} k[d(x_i)]_k$  and by identification, we get  $[d(x_i)]_{\frac{1}{2}} = \frac{2}{2i-1}d(e)x_i$ ,  $[d(x_i)]_{-1} = 0$  and  $[d(x_i)]_j = 0$ . It follows that

$$d(x_i) = \varphi_i^d(x_i) + \frac{2}{2i-1}x_id(e) \quad (3.1)$$

where  $\varphi_i^d \in \text{End}(\mathcal{A}_i)$  is defined by  $\varphi_i^d(x_i) = [d(x_i)]_i$ . Moreover, we have  $\varphi_i^d(x_iy_i) + \frac{2}{2i-1}(x_iy_i)d(e) = d(x_iy_i) = (\varphi_i^d(x_i)y_i + x_i\varphi_i^d(y_i)) + \frac{2}{2i-1}((x_id(e))y_i + x_i(y_id(e)))$ . The identities (2.1) and (2.13) tell us that  $(x_id(e))y_i + x_i(y_id(e)) = (x_iy_i)d(e)$ . We deduce that

$$\varphi_i^d(x_iy_i) = \varphi_i^d(x_i)y_i + x_i\varphi_i^d(y_i). \quad (3.2)$$

The relations (3.1) and (3.2) give us the assertion (ii).

- Let  $i, j \in \{-1, \frac{1}{2}\}$  pairwise distinct and  $x_i \in \mathcal{A}_i$ . We have  $id(x_i) = d(ex_i) = d(e)x_i + ed(x_i) = i[d(x_i)]_i + \sum_{k \in \{0, 1, j\}} [(d(e)x_i) + k(d(x_i))]_k$ . By identification, we get  $[d(x_i)]_k = \frac{1}{i-k}[x_id(e)]_k$  for  $k \in \{0, 1, j\}$  and it follows that

$$d(x_i) = \varphi_i^d(x_i) + \sum_{k \in \{0, 1, j\}} \frac{1}{i-k}[x_id(e)]_k \text{ where } \varphi_i^d \in \text{End}(\mathcal{A}_i) \text{ and } \varphi_i^d(x_i) = [d(x_i)]_i.$$

Hence, we have the assertion (iii).

- Let  $i \in \{0, 1\}$ ,  $j, k \in \{-1, \frac{1}{2}\}$  pairwise distinct,  $x_i \in \mathcal{A}_i$  and  $x_j \in \mathcal{A}_j$ . We have

$$\begin{aligned} d(x_i x_j) &= d(x_i) x_j + x_i d(x_j) \\ &= \varphi_i^d(x_i) x_j + x_i \varphi_j^d(x_j) + \frac{2}{2i-1} x_j (x_i d(e)) + \sum_{t \in \{0, 1, k\}} \frac{1}{j-t} x_i [x_j d(e)]_t \end{aligned}$$

Set  $H_{(i,j)} = \frac{2}{2i-1} x_j (x_i d(e)) + \sum_{t \in \{0, 1, k\}} \frac{1}{j-t} x_i [x_j d(e)]_t$ . Using the relation  $\mathcal{A}_0 \mathcal{A}_1 = \mathcal{A}_0 \mathcal{A}_{-1} = 0$  and the identities (2.2) to (2.12), we get

$$\begin{aligned} H_{(1, \frac{1}{2})} &= 2[(x_1 x_{\frac{1}{2}}) d(e)]_0 - 2[(x_1 x_{\frac{1}{2}}) d(e)]_1 + \frac{2}{3} [(x_1 x_{\frac{1}{2}}) d(e)]_{-1} \\ H_{(1, -1)} &= -[(x_1 x_{-1}) d(e)]_0 - \frac{1}{2} [(x_1 x_{-1}) d(e)]_1 - \frac{2}{3} [(x_1 x_{-1}) d(e)]_{\frac{1}{2}} \\ H_{(0, \frac{1}{2})} &= 2[(x_0 x_{\frac{1}{2}}) d(e)]_0 - 2[(x_0 x_{\frac{1}{2}}) d(e)]_1 ; \\ H_{(0, -1)} &= 0. \end{aligned}$$

It follows that

$$H_{(i,j)} = \sum_{t \in \{0, 1, k\}} \frac{1}{j-t} [(x_i x_j) d(e)]_t$$

and

$$d(x_i x_j) = \varphi_i^d(x_i) x_j + x_i \varphi_j^d(x_j) + \sum_{t \in \{0, 1, k\}} \frac{1}{j-t} [(x_i x_j) d(e)]_t.$$

Since  $\mathcal{A}_i \mathcal{A}_j \subset \mathcal{A}_j$ , we also have

$$d(x_i x_j) = \varphi_j^d(x_i x_j) + \sum_{t \in \{0, 1, k\}} \frac{1}{j-t} [(x_i x_j) d(e)]_t.$$

Thus  $\varphi_j^d(x_i x_j) = \varphi_i^d(x_i) x_j + x_i \varphi_j^d(x_j)$  and we obtain the assertion (iv).

- Let  $i \in \{0, 1\}$ . Taking account to the relation  $\mathcal{A}_0 \mathcal{A}_{-1} = 0$  and the identities (2.14), (2.15), we have

$$\begin{aligned} d(x_{-1} y_{-1}) &= d(x_{-1}) y_{-1} + x_{-1} d(y_{-1}) \\ &= \varphi_{-1}^d(x_{-1}) y_{-1} + x_{-1} \varphi_{-1}^d(y_{-1}) - \sum_{k \in \{0, 1, \frac{1}{2}\}} \frac{1}{1+k} \left( [x_{-1} d(e)]_k y_{-1} + x_{-1} [y_{-1} d(e)]_k \right) \\ &= \varphi_{-1}^d(x_{-1}) y_{-1} + x_{-1} \varphi_{-1}^d(y_{-1}) - \frac{2}{3} \left( [x_{-1} d(e)]_{\frac{1}{2}} y_{-1} + x_{-1} [y_{-1} d(e)]_{\frac{1}{2}} \right) \\ &= \varphi_{-1}^d(x_{-1}) y_{-1} + x_{-1} \varphi_{-1}^d(y_{-1}) - \frac{2}{3} \sum_{k \in \{0, 1, \frac{1}{2}\}} \left[ [x_{-1} d(e)]_{\frac{1}{2}} y_{-1} + x_{-1} [y_{-1} d(e)]_{\frac{1}{2}} \right]_k \\ &= \varphi_{-1}^d(x_{-1}) y_{-1} + x_{-1} \varphi_{-1}^d(y_{-1}) + 2[x_{-1} y_{-1}]_1 d(e) - 2[x_{-1} y_{-1}]_0 d(e) \\ &\quad - \frac{2}{3} \sum_{k=0}^1 \left[ [x_{-1} d(e)]_{\frac{1}{2}} y_{-1} + x_{-1} [y_{-1} d(e)]_{\frac{1}{2}} \right]_k. \end{aligned}$$

We also have

$$\begin{aligned} d(x_{-1} y_{-1}) &= \sum_{i=0}^1 d([x_{-1} y_{-1}]_i) \\ &= \sum_{i=0}^1 \left( \varphi_i^d([x_{-1} y_{-1}]_i) + \frac{2}{2i-1} [x_{-1} y_{-1}]_i d(e) \right) \\ &= \sum_{i=0}^1 \left( \varphi_i^d([x_{-1} y_{-1}]_i) \right) - 2[x_{-1} y_{-1}]_0 d(e) + 2[x_{-1} y_{-1}]_1 d(e) \end{aligned}$$

We deduce that

$$\sum_{i=0}^1 \left( \varphi_i^d([x_{-1}y_{-1}]_i) \right) = \varphi_{-1}^d(x_{-1})y_{-1} + x_{-1}\varphi_{-1}^d(y_{-1}) - \frac{2}{3} \sum_{k=0}^1 \left[ [x_{-1}d(e)]_{\frac{1}{2}}y_{-1} + x_{-1}[y_{-1}d(e)]_{\frac{1}{2}} \right]_k.$$

By identification, we obtain the assertion (v).

- Let  $i, j \in \{-1, \frac{1}{2}\}$  pairwise distinct. We have

$$\begin{aligned} d(x_{\frac{1}{2}}y_j) &= d(x_{\frac{1}{2}})y_j + x_{\frac{1}{2}}d(y_j) \\ &= \varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}})y_j + x_{\frac{1}{2}}\varphi_j^d(y_j) + \sum_{k \in \{0,1,-1\}} \frac{2}{1-2k}y_j[x_{\frac{1}{2}}d(e)]_k + \sum_{k \in \{0,1,i\}} \frac{1}{j-k}x_{\frac{1}{2}}[y_jd(e)]_k \\ &= \varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}})y_j + x_{\frac{1}{2}}\varphi_j^d(y_j) + \frac{2}{3}y_j[x_{\frac{1}{2}}d(e)]_{-1} + \frac{1}{j-i}x_{\frac{1}{2}}[y_jd(e)]_i + \\ &\quad \sum_{k=0}^1 \left( \frac{2}{1-2k}y_j[x_{\frac{1}{2}}d(e)]_k + \frac{1}{j-k}x_{\frac{1}{2}}[y_jd(e)]_k \right). \end{aligned}$$

We also have

$$\begin{aligned} d(x_{\frac{1}{2}}y_j) &= d([x_{\frac{1}{2}}y_j]_i) + \sum_{k=0}^1 d([x_{\frac{1}{2}}y_j]_k) \\ &= \varphi_i^d([x_{\frac{1}{2}}y_j]_i) + \frac{1}{i-j}[x_{\frac{1}{2}}y_j]_i d(e)]_j + \sum_{k=0}^1 \left( \varphi_k^d([x_{\frac{1}{2}}y_j]_k) + \frac{1}{i-k}[[x_{\frac{1}{2}}y_j]_i d(e)]_k + \right. \\ &\quad \left. \frac{2}{2k-1}[x_{\frac{1}{2}}y_j]_k d(e) \right). \end{aligned}$$

Taking account to the relation  $\mathcal{A}_0\mathcal{A}_{-1} = 0$ , we obtain the assertions (vi), (vii) and (viii), by identifications. Conversely, suppose that the endomorphism  $d$  satisfies the assertion (i) to the assertion (viii). Let  $x = x_0 + x_1 + x_{-1} + x_{\frac{1}{2}}$ ,  $y = y_0 + y_1 + y_{-1} + y_{\frac{1}{2}} \in \mathcal{A}$ . By a direct calculation and according to the Proposition 2.1, the Lemma 2.1 and the Lemma 2.2, we show that  $d(x_i y_j) = d(x_i)y_j + x_i d(y_j)$  for all  $i, j \in \{0, 1, -1, \frac{1}{2}\}$ . It follows that  $d(xy) = d(x)y + xd(y)$  and  $d$  is a derivation in the 3-Jordan algebra  $\mathcal{A}$ .  $\square$

**Note 1** By "defined and only determined" in the Theorem 3.1, we mean that the action of  $d$  on the whole algebra  $\mathcal{A}$  is fully determined by the data  $(d(e), \varphi_0^d, \varphi_1^d, \varphi_{-1}^d, \varphi_{\frac{1}{2}}^d)$  together with the listed conditions. Also, this conditions are intrinsic parts of the characterization itself, i.e. necessary to reconstruct  $d$ .

**Note 2** By "unique quintuplet" in the Theorem 3.1, we mean that if two derivations  $d$  and  $d'$  of  $\mathcal{A}$  have the same quintuple, then  $d = d'$ .

**Example 3.1** Let  $\mathcal{A}$  be the commutative algebra with the base  $\{e, e_1, e_2, e_3, e_4\}$  and the nonzero products given by:  $e^2 = e$ ,  $ee_1 = \frac{1}{2}e_1$ ,  $ee_2 = -e_2$ ,  $ee_3 = -e_3$ ,  $ee_4 = \frac{1}{2}e_4$ ,  $e_1^2 = e_3$ ,  $e_1e_2 = ke_4$  where  $k \in F^*$ . Thus, we have  $\mathcal{A}_0 = \mathcal{A}_1 = 0$ ;  $\mathcal{A}_{\frac{1}{2}} = \langle e_1, e_4 \rangle$  and  $\mathcal{A}_{-1} = \langle e_2, e_3 \rangle$ . This algebra is 3-Jordan [7, Example 3]. Let  $d$  be a derivation in  $\mathcal{A}$  and  $(d(e), \varphi_{\frac{1}{2}}^d, \varphi_{-1}^d, \varphi_1^d, \varphi_0^d)$  the unique quintuplet associated with it. Since,  $\varphi_i^d(x_i) \in \mathcal{A}_i$  for  $x_i \in \mathcal{A}_i$  and  $i \in \{-1, 0, \frac{1}{2}, 1\}$ , it follows that  $\varphi_0^d = \varphi_1^d = 0$  and we set  $d(e) = \alpha_1e_1 + \alpha_4e_4$ ,  $\varphi_{\frac{1}{2}}^d(e_1) = \beta_{11}e_1 + \beta_{14}e_4$ ,  $\varphi_{\frac{1}{2}}^d(e_4) = \beta_{41}e_1 + \beta_{44}e_4$ ,  $\varphi_{-1}^d(e_2) = \gamma_{22}e_2 + \gamma_{23}e_3$ ,  $\varphi_{-1}^d(e_3) = \gamma_{32}e_2 + \gamma_{33}e_3$ . We have

- $k\varphi_{\frac{1}{2}}^d(e_4) = \varphi_{\frac{1}{2}}^d([e_1e_2]_{\frac{1}{2}}) = [\varphi_{\frac{1}{2}}^d(e_1)e_2 + e_1\varphi_{-1}^d(e_2)]_{\frac{1}{2}} = k(\beta_{11} + \gamma_{22})e_4$ , i.e.  $\varphi_{\frac{1}{2}}^d(e_4) = (\beta_{11} + \gamma_{22})e_4$ .
- $\varphi_{-1}^d(e_3) = \varphi_{-1}^d([e_1^2]_{-1}) = 2[e_1\varphi_{\frac{1}{2}}^d(e_1)]_{-1} = 2\beta_{11}e_3$ .

We deduce that  $d(e) = \alpha_1 e_1 + \alpha_4 e_4$ ,  $d(e_1) = \beta_{11} e_1 + \beta_{14} e_4 + \frac{2}{3} \alpha_1 e_3$ ,  $d(e_2) = \gamma_{22} e_2 + \gamma_{23} e_3 - \frac{2}{3} \alpha_1 k e_4$ ,  $d(e_3) = 2\beta_{11} e_3$ ,  $d(e_4) = (\beta_{11} + \gamma_{22}) e_4$ . The matrix of  $d$  in the base  $(e, e_1, e_2, e_3, e_4)$  is given by:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & \beta_{11} & 0 & 0 & 0 \\ 0 & 0 & \gamma_{22} & 0 & 0 \\ 0 & \frac{2}{3} \alpha_1 & \gamma_{23} & 2\beta_{11} & 0 \\ \alpha_2 & \beta_{14} & -\frac{2}{3} \alpha_1 k & 0 & \beta_{11} + \gamma_{22} \end{pmatrix}$$

Set  $d_1 = E_{21} + \frac{2}{3} E_{42} - \frac{2}{3} k E_{53}$ ,  $d_2 = E_{51}$ ,  $d_3 = E_{22} + 2E_{44} + E_{55}$ ,  $d_4 = E_{52}$ ,  $d_5 = E_{33} + E_{55}$ ,  $d_6 = E_{43}$  where  $E_{ij}$  is an elementary matrix. The family  $\{d_1, \dots, d_6\}$  is a base of  $Der_F(\mathcal{A})$  and its nonzero products are given by  $[d_1, d_3] = -d_1$ ,  $[d_1, d_4] = -d_2$ ,  $[d_2, d_3] = [d_2, d_5] = -d_2$ ,  $[d_3, d_6] = 2d_6$ ,  $[d_4, d_5] = -d_4$ ,  $[d_5, d_6] = -d_6$ .

In the following we give the characterization of  $d \in Der_F(\mathcal{A})$  for  $d(e) = 0$ .

**Corollary 3.1** *Let  $d$  be an endomorphism of the 3-Jordan algebra  $\mathcal{A}$  such that  $d(e) = 0$ . Then,  $d$  is a derivation of  $\mathcal{A}$  if, and only if, the following assertions are satisfied:*

- (i) the Peirce subspaces  $\mathcal{A}_i$  for  $i \in \{-1, 0, 1, \frac{1}{2}\}$  are invariants with respect to  $d$  ;
- (ii) Let set  $d_i = d_{\mathcal{A}_i}$  the endomorphism induced by  $d$  on the Peirce Subspace  $\mathcal{A}_i$ . We have
  - (a) the endomorphisms  $d_0$  and  $d_1$  are derivations of  $\mathcal{A}_0$  and  $\mathcal{A}_1$  respectively ;
  - (b)  $d_j(x_j x_i) = d_j(x_j) x_i + x_j d_i(x_i)$  for  $i \in \{0, 1\}$  and  $j \in \{-1, \frac{1}{2}\}$  ;
  - (c)  $d_i([x_{-1} y_{-1}]_i) = [d_{-1}(x_{-1}) y_{-1} + x_{-1} d_{-1}(y_{-1})]_i$  for  $i \in \{0, 1\}$  ;
  - (d)  $d_i([x_{\frac{1}{2}} y_j]_i) = [d_{\frac{1}{2}}(x_{\frac{1}{2}}) y_j + x_{\frac{1}{2}} d_j(x_j)]_i$  for  $i \in \{-1, 0, 1, \frac{1}{2}\}$  and  $j \in \{-1, \frac{1}{2}\}$ .

**Proof:** Just take  $d(e) = 0$ , in the Theorem 3.1. □

Let  $d, d' \in Der_F(\mathcal{A})$  be two derivations in the 3-Jordan algebra  $\mathcal{A}$ . We give below, the quintuplet associated to the derivation  $[d, d'] = d \circ d' - d' \circ d$ .

**Proposition 3.1** *Let  $([d, d'](e), \varphi_0^{[d, d']}, \varphi_1^{[d, d']}, \varphi_{-1}^{[d, d']}, \varphi_{\frac{1}{2}}^{[d, d']})$  be the quintuplet associated with the derivation  $[d, d']$  of the 3-Jordan algebra  $\mathcal{A}$ . We have:*

- (i)  $[d, d'](e) = (\varphi_{\frac{1}{2}}^d \circ d' - \varphi_{\frac{1}{2}}^{d'} \circ d)(e)$  ;
- (ii)  $\varphi_i^{[d, d']}(x_i) = [\varphi_i^d, \varphi_i^{d'}](x_i) - \frac{4}{(2i-1)^2} [(x_i d'(e)) d(e) - (x_i d(e)) d'(e)]_i$  for  $i \in \{0, 1\}$  ;
- (iii)  $\varphi_i^{[d, d']}(x_i) = [\varphi_i^d, \varphi_i^{d'}](x_i) - \frac{1}{(i-j)^2} [[x_i d'(e)]_j d(e) - [x_i d(e)]_j d'(e)]_i + \epsilon_i \sum_{k=0}^1 \frac{2}{(i-k)(2k-1)} ([x_i d'(e)]_k d(e) - [x_i d(e)]_k d'(e))$  where  $i \in \{-1, \frac{1}{2}\}$  and  $\epsilon_i = \begin{cases} 1 & \text{for } i = \frac{1}{2} ; \\ 0 & \text{for } i = -1 \end{cases}$

**Proof:** Denote  $p_i : \mathcal{A} \rightarrow \mathcal{A}_i$  the projection of  $\mathcal{A}$  on  $\mathcal{A}_i$  for  $i \in \{0, 1, -1, \frac{1}{2}\}$ .

- The assertion (i) is obtain by direct calculation of  $[d, d'](e)$ .
- For  $i \in \{0, 1\}$ , we have  $d \circ d'(x_i) = d\left[\varphi_i^{d'}(x_i) + \frac{2}{2i-1} x_i d'(e)\right] = \varphi_i^d \circ \varphi_i^{d'}(x_i) + \frac{2}{2i-1} \left[\varphi_i^{d'}(x_i) d(e) + \varphi_{\frac{1}{2}}^d(x_i d'(e)) + \sum_{k \in \{0, 1, -1\}} \frac{2}{1-2k} [(x_i d'(e)) d(e)]_k\right]$  and  $p_i(d \circ d'(x_i)) = \varphi_i^d \circ \varphi_i^{d'}(x_i) - \frac{4}{(2i-1)^2} [(x_i d'(e)) d(e)]_i$ . We deduce that  $\varphi_i^{[d, d']}(x_i) = p_i([d, d'](x_i)) = [\varphi_i^d, \varphi_i^{d'}](x_i) - \frac{4}{(2i-1)^2} [(x_i d'(e)) d(e) - (x_i d(e)) d'(e)]_i$  and we get assertion (ii).
- For  $i, j \in \{-1, \frac{1}{2}\}$  pairwise distinct, we have  $d \circ d'(x_i) = d\left(\varphi_i^{d'}(x_i) + \sum_{k \in \{0, 1, j\}} \frac{1}{i-k} [x_i d'(e)]_k\right) = \varphi_i^d \circ \varphi_i^{d'}(x_i) + \sum_{t \in \{0, 1, j\}} \frac{1}{i-t} [\varphi_i^{d'}(x_i) d(e)]_t + \frac{1}{i-j} \left(\varphi_j^d([x_i d'(e)]_j) + \sum_{t \in \{0, 1, i\}} \frac{1}{j-t} [[x_i d'(e)]_j d(e)]_t\right) + \sum_{k=0}^1 \frac{1}{i-k} [\varphi_k([x_i d'(e)]_k) + \frac{2}{2k-1} [x_i d'(e)]_k d(e)]$  and  $p_i(d \circ d')(x_i) = \varphi_i^d \circ \varphi_i^{d'}(x_i) - \frac{1}{(i-j)^2} [[x_i d'(e)]_j d(e)]_i +$

$\epsilon_i \sum_{k \in \{0,1\}} \frac{2}{(i-k)(2k-1)} [x_i d'(e)]_k d(e)$ . Hence  $\varphi_i^{[d,d']}(x_i) = p_i([d, d'])(x_i) = [\varphi_i^d, \varphi_i^{d'}](x_i) - \frac{1}{(i-j)^2} [[x_i d'(e)]_j d(e) - [x_i d(e)]_j d'(e)]_i + \epsilon_i \sum_{k \in \{0,1\}} \frac{2}{(i-k)(2k-1)} ([x_i d'(e)]_k d(e) - [x_i d(e)]_k d'(e))$ . We obtain the assertion (iii).  $\square$

#### 4. Characteristic ideals and dimensionally nilpotent algebras

In this section, we give an example of characteristic ideal in 3-Jordan algebra  $\mathcal{A}$  and classify dimensionally nilpotent 3-Jordan algebra in dimensional 3, up to isomorphism.

##### 4.1. Characteristic ideals

**Definition 4.1** *Let  $d$  be a derivation of an algebra  $\mathcal{U}$ . An ideal  $I$  of the algebra  $\mathcal{U}$  is called  $d$ -invariant if  $d(I) \subset I$ .*

**Definition 4.2** [11, Page 145] *An ideal  $I$  of an algebra  $\mathcal{U}$  is called characteristic ideal if  $d(I) \subset I$  for every derivation  $d$  of  $\mathcal{U}$ .*

Set  $I = \{x \in \mathcal{A}_1 \mid x\mathcal{A}_{-1} = 0\}$ ,  $J = I + \mathcal{A}_{\frac{1}{2}}I + [\mathcal{A}_{\frac{1}{2}}(\mathcal{A}_{\frac{1}{2}}I)]_0$ ,  $M = \{y \in \mathcal{A}_{\frac{1}{2}} \mid y\mathcal{A}_{\frac{1}{2}} \subset \mathcal{A}_1, y\mathcal{A}_{-1} = 0\}$  in the 3-Jordan algebra  $\mathcal{A}$ .

**Lemma 4.1** [7, Lemma 3, Lemma 4]

- The set  $J$  is an ideal of  $\mathcal{A}$ .
- If  $\mathcal{A}_0 = 0$  and  $I = 0$ , then the set  $M$  is an ideal of  $\mathcal{A}$ .

**Lemma 4.2** *Let  $d$  be a derivation of the 3-Jordan algebra  $\mathcal{A}$  and  $(d(e), \varphi_0^d, \varphi_1^d, \varphi_{-1}^d, \varphi_{\frac{1}{2}}^d)$  the unique quintuplet associated with it. The subset  $I$  is invariant with respect to the derivation  $\varphi_1^d$  of the algebra  $\mathcal{A}_1$ .*

**Proof:** Let  $x_1 \in I$  and  $x_{-1} \in \mathcal{A}_{-1}$ . According to the equalities  $x_1 x_{-1} = 0$ ;  $\mathcal{A}_1 \mathcal{A}_0 = 0$  and the identities (2.5), (2.6), (2.7), we have

$$\begin{aligned}
0 &= d(x_{-1}x_1) \\
&= d(x_{-1})x_1 + x_{-1}d(x_1) \\
&= \left[ \varphi_{-1}^d(x_{-1}) - \sum_{k \in \{0,1,\frac{1}{2}\}} \frac{1}{1+k} [x_{-1}d(e)]_k \right] x_1 + x_{-1} \left[ \varphi_1^d(x_1) + 2x_1d(e) \right] \\
&= -\frac{1}{2} ([x_{-1}d(e)]_1) x_1 - \frac{2}{3} ([x_{-1}d(e)]_{\frac{1}{2}}) x_1 + x_{-1} \varphi_1^d(x_1) + 2x_{-1}(x_1d(e)) \\
&= \frac{1}{2} [d(e)(x_1x_{-1})]_1 + \frac{1}{3} [d(e)(x_1x_{-1})]_{\frac{1}{2}} + x_{-1} \varphi_1^d(x_1) - d(e)(x_1x_{-1}) \\
&= x_{-1} \varphi_1^d(x_1)
\end{aligned}$$

So  $\varphi_1^d(I) \subset I$  and we get the lemma.  $\square$

**Theorem 4.1** *The ideal  $J$  is a characteristic ideal.*

**Proof:** Let  $d$  be a derivation of the 3-Jordan algebra  $\mathcal{A}$ .

- Let  $x_1 \in I$  and  $x_{-1} \in \mathcal{A}_{-1}$ . According to the Lemma 4.2 and the identities (2.5), (2.6), (2.7), we have  $x_{-1}d(x_1) = x_{-1}\varphi_1^d(x_1) + 2x_{-1}(x_1d(e)) = -[d(e)(x_{-1}x_1)] = 0$ . So  $d(x_1) \in I \subset J$ .

- Let  $x_1 \in I$  and  $x_{\frac{1}{2}} \in \mathcal{A}_{\frac{1}{2}}$ . According to the identities (2.2), (2.3), (2.4), we have

$$\begin{aligned}
d(x_1 x_{\frac{1}{2}}) &= d(x_1) x_{\frac{1}{2}} + x_1 d(x_{\frac{1}{2}}) \\
&= \varphi_1^d(x_1) x_{\frac{1}{2}} + x_1 \varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}}) - 2 \left[ x_1 [x_{\frac{1}{2}} d(e)]_1 - (x_1 d(e)) x_{\frac{1}{2}} \right] + \frac{2}{3} x_1 [x_{\frac{1}{2}} d(e)]_{-1} \\
&= \varphi_1^d(x_1) x_{\frac{1}{2}} + x_1 \varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}}) - 2 \left[ x_1 [x_{\frac{1}{2}} d(e)]_1 - [(x_1 d(e)) x_{\frac{1}{2}}]_1 \right] + 2[(x_1 d(e)) x_{\frac{1}{2}}]_0 \\
&\quad + 2[(x_1 d(e)) x_{\frac{1}{2}}]_{-1} \\
&= \varphi_1^d(x_1) x_{\frac{1}{2}} + x_1 \varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}}) - 2[(x_1 x_{\frac{1}{2}}) d(e)]_1 + 2[(x_1 d(e)) x_{\frac{1}{2}}]_0 - x_1 [x_{\frac{1}{2}} d(e)]_{-1} \\
&= \varphi_1^d(x_1) x_{\frac{1}{2}} + x_1 \varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}}) - 2[(x_1 x_{\frac{1}{2}}) d(e)]_1 + 2[(x_1 d(e)) x_{\frac{1}{2}}]_0.
\end{aligned}$$

We have  $\varphi_1^d(x_1) x_{\frac{1}{2}} + x_1 \varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}}) \in \mathcal{A}_{\frac{1}{2}} I$  and  $[(x_1 d(e)) x_{\frac{1}{2}}]_0 \in [\mathcal{A}_{\frac{1}{2}} (\mathcal{A}_{\frac{1}{2}} I)]_0$ . Let show that  $[(x_1 x_{\frac{1}{2}}) d(e)]_1 \in I$ . For  $x_{-1} \in \mathcal{A}_{-1}$  and according to the identities (2.17), (2.3), (2.6), we have

$$\begin{aligned}
x_{-1} [(x_1 x_{\frac{1}{2}}) d(e)]_1 &= -\frac{1}{3} \left[ [x_{-1} d(e)]_{\frac{1}{2}} (x_1 x_{\frac{1}{2}}) \right]_{-1} - \frac{1}{3} \left[ d(e) [x_{-1} (x_1 x_{\frac{1}{2}})]_{\frac{1}{2}} \right]_{-1} \\
&= \frac{1}{6} x_1 \left[ x_{\frac{1}{2}} [x_{-1} d(e)]_{\frac{1}{2}} \right]_{-1} + \frac{1}{6} \left[ d(e) [x_{\frac{1}{2}} (x_{-1} x_1)]_{\frac{1}{2}} \right]_{-1} \\
&= 0.
\end{aligned}$$

We deduce that  $[(x_1 x_{\frac{1}{2}}) d(e)]_1 \in I \subset J$  and it follows that  $d(x_1 x_{\frac{1}{2}}) \in J$ .

- Let  $x_1 \in I, x_{\frac{1}{2}}, y_{\frac{1}{2}} \in \mathcal{A}_{\frac{1}{2}}$ . According the identities (2.7), (2.3) we have

$$\begin{aligned}
d([y_{\frac{1}{2}} (x_{\frac{1}{2}} x_1)]_0) &= \varphi_0^d([y_{\frac{1}{2}} (x_{\frac{1}{2}} x_1)]_0) - 2[y_{\frac{1}{2}} (x_{\frac{1}{2}} x_1)]_0 d(e) \\
&= \left[ \varphi_{\frac{1}{2}}^d(y_{\frac{1}{2}}) (x_{\frac{1}{2}} x_1) + y_{\frac{1}{2}} \varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}} x_1) + \frac{2}{3} (x_{\frac{1}{2}} x_1) [y_{\frac{1}{2}} d(e)]_{-1} + [y_{\frac{1}{2}} (x_{\frac{1}{2}} x_1)]_{-1} d(e) \right. \\
&\quad \left. + \frac{2}{3} y_{\frac{1}{2}} [(x_{\frac{1}{2}} x_1) d(e)]_{-1} \right]_0 - 2[y_{\frac{1}{2}} (x_{\frac{1}{2}} x_1)]_0 d(e) \\
&= \left[ \varphi_{\frac{1}{2}}^d(y_{\frac{1}{2}}) (x_{\frac{1}{2}} x_1) + y_{\frac{1}{2}} (\varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}}) x_1) + y_{\frac{1}{2}} (x_{\frac{1}{2}} \varphi_1^d(x_1)) + \frac{2}{3} (x_{\frac{1}{2}} x_1) [y_{\frac{1}{2}} d(e)]_{-1} + \right. \\
&\quad \left. [y_{\frac{1}{2}} (x_{\frac{1}{2}} x_1)]_{-1} d(e) + \frac{2}{3} y_{\frac{1}{2}} [(x_{\frac{1}{2}} x_1) d(e)]_{-1} \right]_0 - 2[y_{\frac{1}{2}} (x_{\frac{1}{2}} x_1)]_0 d(e) \\
&= \left[ \varphi_{\frac{1}{2}}^d(y_{\frac{1}{2}}) (x_{\frac{1}{2}} x_1) + y_{\frac{1}{2}} (\varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}}) x_1) + y_{\frac{1}{2}} (x_{\frac{1}{2}} \varphi_1^d(x_1)) - \frac{1}{3} x_{\frac{1}{2}} (x_1 [y_{\frac{1}{2}} d(e)]_{-1}) \right. \\
&\quad \left. - \frac{1}{2} (x_1 [y_{\frac{1}{2}} x_{\frac{1}{2}}]_{-1}) d(e) - \frac{1}{3} y_{\frac{1}{2}} (x_1 [x_{\frac{1}{2}} d(e)]_{-1}) \right]_0 - 2[y_{\frac{1}{2}} (x_{\frac{1}{2}} x_1)]_0 d(e) \\
&= \left[ \varphi_{\frac{1}{2}}^d(y_{\frac{1}{2}}) (x_{\frac{1}{2}} x_1) + y_{\frac{1}{2}} (\varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}}) x_1) + y_{\frac{1}{2}} (x_{\frac{1}{2}} \varphi_1^d(x_1)) \right]_0 - 2[y_{\frac{1}{2}} (x_{\frac{1}{2}} x_1)]_0 d(e).
\end{aligned}$$

We have  $\left[ \varphi_{\frac{1}{2}}^d(y_{\frac{1}{2}}) (x_{\frac{1}{2}} x_1) + y_{\frac{1}{2}} (\varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}}) x_1) + y_{\frac{1}{2}} (x_{\frac{1}{2}} \varphi_1^d(x_1)) \right]_0 \in [\mathcal{A}_{\frac{1}{2}} (\mathcal{A}_{\frac{1}{2}} I)]_0$  and [7, p77] tells us that  $[y_{\frac{1}{2}} (x_{\frac{1}{2}} x_1)]_0 d(e) \in \mathcal{A}_{\frac{1}{2}} I$ . So  $d([y_{\frac{1}{2}} (x_{\frac{1}{2}} x_1)]_0) \in J$ .  
From above, it follows that the ideal  $J$  is a characteristic ideal.  $\square$

Now, we assume that  $\mathcal{A}_0 = 0$  and  $I = 0$ .

**Lemma 4.3** *Let  $d$  be a derivation of the 3-Jordan algebra  $\mathcal{A}$  and  $(d(e), \varphi_0^d, \varphi_1^d, \varphi_{-1}^d, \varphi_{\frac{1}{2}}^d)$  the unique quintuplet associated with it. The ideal  $M$  is invariant with respect to the endomorphism  $\varphi_{\frac{1}{2}}^d$  if, and only if  $[x_{-1} d(e)]_1 y = 0$  for  $y \in M$ .*

**Proof:** Let  $y \in M, x_{\frac{1}{2}} \in \mathcal{A}_{\frac{1}{2}}$ . We have

$$\begin{aligned}
 d(x_{\frac{1}{2}}y) &= d(x_{\frac{1}{2}})y + x_{\frac{1}{2}}d(y) \\
 &= \varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}})y + x_{\frac{1}{2}}\varphi_{\frac{1}{2}}^d(y) - 2\left[[x_{\frac{1}{2}}d(e)]_1y + x_{\frac{1}{2}}[yd(e)]_1\right] + \frac{2}{3}\left[[x_{\frac{1}{2}}d(e)]_{-1}y + x_{\frac{1}{2}}[yd(e)]_{-1}\right] \\
 &= \varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}})y + x_{\frac{1}{2}}\varphi_{\frac{1}{2}}^d(y) - 2\left[[x_{\frac{1}{2}}d(e)]_1y + x_{\frac{1}{2}}[yd(e)]_1\right]
 \end{aligned}$$

Since  $x_{\frac{1}{2}}y \in \mathcal{A}_1$ , we also have

$$\begin{aligned}
 d(x_{\frac{1}{2}}y) &= \varphi_1^d(x_{\frac{1}{2}}y) - 2(x_{\frac{1}{2}}y)d(e) \\
 &= \left[\varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}})y + x_{\frac{1}{2}}\varphi_{\frac{1}{2}}^d(y) + \frac{2}{3}y[x_{\frac{1}{2}}d(e)]_{-1} + \frac{1}{2}[yx_{\frac{1}{2}}]_{-1}d(e) + \frac{2}{3}x_{\frac{1}{2}}[yd(e)]_{-1}\right]_1 \\
 &\quad - 2(x_{\frac{1}{2}}y)d(e) \\
 &= \varphi_{\frac{1}{2}}^d(x_{\frac{1}{2}})y + \left[x_{\frac{1}{2}}\varphi_{\frac{1}{2}}^d(y)\right]_1 - 2(x_{\frac{1}{2}}y)d(e).
 \end{aligned}$$

and by identification, we obtain

$$[x_{\frac{1}{2}}d(e)]_1y = 0, [x_{\frac{1}{2}}, y, d(e)] = 0 \text{ and } [x_{\frac{1}{2}}\varphi_{\frac{1}{2}}^d(y)]_{-1} = 0 \text{ for all } x_{\frac{1}{2}} \in \mathcal{A}_{\frac{1}{2}}. \quad (4.1)$$

Thus  $x_{\frac{1}{2}}\varphi_{\frac{1}{2}}^d(y) \in \mathcal{A}_1$ . For  $x_{-1} \in \mathcal{A}_{-1}$ , we also have

$$\begin{aligned}
 0 &= d(x_{-1}y) \\
 &= d(x_{-1})y + x_{-1}d(y) \\
 &= \varphi_{-1}^d(x_{-1})y + x_{-1}\varphi_{\frac{1}{2}}^d(y) - \frac{1}{2}[x_{-1}d(e)]_1y - 2x_{-1}[yd(e)]_1 - \frac{2}{3}[x_{-1}d(e)]_{\frac{1}{2}}y + \frac{2}{3}x_{-1}[yd(e)]_{-1} \\
 &= x_{-1}\varphi_{\frac{1}{2}}^d(y) - \frac{1}{2}[x_{-1}d(e)]_1y - 2x_{-1}[yd(e)]_1 - \frac{2}{3}[x_{-1}d(e)]_{\frac{1}{2}}y.
 \end{aligned}$$

According to [7, Proof of Lemma 4], we have  $\mathcal{A}_{\frac{1}{2}}M \subset I = 0$ . It follows that  $[x_{-1}d(e)]_{\frac{1}{2}}y = 0$ . Identity (2.17) give us

$$x_{-1}[yd(e)]_1 = -\frac{1}{3}\left[[x_{-1}y]_{\frac{1}{2}}d(e)\right]_{-1} - \frac{1}{3}\left[[x_{-1}d(e)]_{\frac{1}{2}}y\right]_{-1} = 0$$

It follows that  $x_{-1}\varphi_{\frac{1}{2}}^d(y) - \frac{1}{2}[x_{-1}d(e)]_1y = 0$ . From above, we get the lemma.  $\square$

**Proposition 4.1** *Let  $d$  be a derivation of the 3-Jordan algebra  $\mathcal{A}$  and  $(d(e), \varphi_0^d, \varphi_1^d, \varphi_{-1}^d, \varphi_{\frac{1}{2}}^d)$  the unique quintuplet associated with it. The ideal  $M$  is  $d$ -invariant if, and only if,  $M$  is invariant with respect to the endomorphism  $\varphi_{\frac{1}{2}}^d$ .*

**Proof:** Let  $y \in M$ , we have  $d(y) = \varphi_{\frac{1}{2}}^d(y) - 2[yd(e)]_1 + \frac{2}{3}[yd(e)]_{-1} = \varphi_{\frac{1}{2}}^d(y) - 2[yd(e)]_1$ . The proof of the Lemma 4.3 tells us  $[yd(e)]_1 \in I = \{0\}$ . It follows that  $d(y) = \varphi_{\frac{1}{2}}^d(y)$  and we obtain the desired result.  $\square$

## 4.2. Dimensionally nilpotent algebras

**Definition 4.3** *Let  $\mathcal{U}$  be a finite  $n$ -dimensional algebra over the field  $F$ . We said that The algebra  $\mathcal{U}$  is dimensionally nilpotent if it admits a nilpotent derivation  $d$  such that  $d^n = 0$  and  $d^{n-1} \neq 0$ . In this case, there is a base  $(e_0, \dots, e_{n-1})$  such that  $e_i = d^i(e_0) = d(e_{i-1})$  for  $0 \leq i \leq n-1$  and  $d(e_{n-1}) = 0$ . The base  $(e_0, \dots, e_{n-1})$  is called adapted base of the derivation  $d$ .*

Dimensionally nilpotent algebra has also been studied for Lie triple and genetics algebras [2,9].

**Theorem 4.2** *Let  $\mathcal{A}$  be a dimensionally nilpotent 3-Jordan algebra of dimension 3 with a nonzero idempotent and with an adapted base  $(e_0, e_1, e_2)$ . Then  $\mathcal{A}$  is isomorphic to one and only one of the following algebras (the products not mentioned are void):*

- $\mathcal{A}_1(\alpha)$  :  $e_0^2 = e_0$ ,  $e_0e_1 = \frac{1}{2}e_1$ ,  $e_0e_2 = \alpha e_2$ ,  $e_1^2 = e_2$  with  $\alpha \in \{0, 1, -1\}$  ;
- $\mathcal{A}_2$  :  $e_0^2 = e_0$ ,  $e_0e_1 = \frac{1}{2}e_1$ ,  $e_0e_2 = \frac{1}{2}e_2$ .

**Proof:** Assume that  $e_0$  is an idempotent of the algebra  $\mathcal{A}$ . We have  $e_1 = d(e_0) \in \mathcal{A}_{\frac{1}{2}}$  ;  $e_2 = d(e_1) = \varphi_{\frac{1}{2}}^d(e_1) + 2[e_1^2]_0 - 2[e_1^2]_1 + \frac{2}{3}[e_1^2]_{-1}$ . Using the Theorem 3.1 and the identity (2.16), we get

$$\begin{aligned}
e_3 &= d(e_2) \\
&= (\varphi_{\frac{1}{2}}^d)^2(e_1) + 2[e_1\varphi_{\frac{1}{2}}^d(e_1)]_0 - 2[e_1\varphi_{\frac{1}{2}}^d(e_1)]_1 + \frac{2}{3}[e_1\varphi_{\frac{1}{2}}^d(e_1)]_{-1} + \frac{2}{3}[\varphi_{-1}^d([e_1^2]_{-1}) - [e_1[e_1^2]_{-1}]_0 \\
&\quad - \frac{1}{2}[e_1[e_1^2]_{-1}]_1 - \frac{2}{3}[e_1[e_1^2]_{-1}]_{\frac{1}{2}}] + 2[\varphi_0^d([e_1^2]_0) - 2e_1[e_1^2]_0] - 2[\varphi_1^d([e_1^2]_1) + 2e_1[e_1^2]_1] \\
&= (\varphi_{\frac{1}{2}}^d)^2(e_1) + 2[e_1\varphi_{\frac{1}{2}}^d(e_1)]_0 - 2[e_1\varphi_{\frac{1}{2}}^d(e_1)]_1 + \frac{2}{3}[e_1\varphi_{\frac{1}{2}}^d(e_1)]_{-1} + \frac{2}{3}[2[e_1\varphi_{\frac{1}{2}}^d(e_1)]_{-1} - \\
&\quad [e_1[e_1^2]_{-1}]_0 - \frac{1}{2}[e_1[e_1^2]_{-1}]_1 - \frac{2}{3}[e_1[e_1^2]_{-1}]_{\frac{1}{2}}] - 4e_1[e_1^2]_0 - 4e_1[e_1^2]_1 + \\
&\quad 2[2e_1\varphi_{\frac{1}{2}}^d(e_1) + \frac{7}{3}e_1[e_1^2]_{-1}]_0 - 2[2e_1\varphi_{\frac{1}{2}}^d(e_1) + \frac{11}{6}e_1[e_1^2]_{-1}]_1 \\
&= (\varphi_{\frac{1}{2}}^d)^2(e_1) + 6[e_1\varphi_{\frac{1}{2}}^d(e_1)]_0 - 6[e_1\varphi_{\frac{1}{2}}^d(e_1)]_1 + 2[e_1\varphi_{\frac{1}{2}}^d(e_1)]_{-1} + 4[e_1[e_1^2]_{-1}]_0 - 4[e_1[e_1^2]_{-1}]_1 \\
&\quad - \frac{4}{9}[e_1[e_1^2]_{-1}]_{\frac{1}{2}} - 4e_1[e_1^2]_0 - 4e_1[e_1^2]_1 \\
&= (\varphi_{\frac{1}{2}}^d)^2(e_1) - \frac{8}{3}e_1[e_1^2]_0 - \frac{16}{3}e_1[e_1^2]_1 + 2[e_1\varphi_{\frac{1}{2}}^d(e_1)]_{-1} + 6[e_1\varphi_{\frac{1}{2}}^d(e_1) + \frac{2}{3}e_1[e_1^2]_{-1}]_0 \\
&\quad - 6[e_1\varphi_{\frac{1}{2}}^d(e_1) + \frac{2}{3}e_1[e_1^2]_{-1}]_1
\end{aligned}$$

The algebra  $\mathcal{A}$  being 3-dimensional dimensionally nilpotent implies that  $0 = d(e_2) = e_3$ . Hence  $e_1\varphi_{\frac{1}{2}}^d(e_1) = -\frac{2}{3}[e_1[e_1^2]_{-1}]_0 - \frac{2}{3}[e_1[e_1^2]_{-1}]_1$  and  $(\varphi_{\frac{1}{2}}^d)^2(e_1) = \frac{8}{3}e_1[e_1^2]_0 + \frac{16}{3}e_1[e_1^2]_1$ . It follows that  $e_1e_2 = e_1\varphi_{\frac{1}{2}}^d(e_1) + 2e_1[e_1^2]_0 - 2e_1[e_1^2]_1 + \frac{2}{3}e_1[e_1^2]_{-1} = e_1\varphi_{\frac{1}{2}}^d(e_1) + \frac{2}{3}[e_1[e_1^2]_{-1}]_0 + \frac{2}{3}[e_1[e_1^2]_{-1}]_1 = 0$  and  $e_2^2 = d(e_1e_2) = 0$ . Since  $d(e_0e_2) = d(e_0)e_2 = e_1e_2 = 0$ , it follows that  $e_0e_2 = \alpha e_2$  with  $\alpha \in \{0, 1, -1, \frac{1}{2}\}$ .

- For  $\alpha = 0$ , we have  $e_2 = 2[e_1^2]_0$  and  $0 = \varphi_{\frac{1}{2}}^d(e_1) = [e_1^2]_1 = [e_1^2]_{-1}$ . Thus  $e_1^2 = [e_1^2]_0 = \frac{1}{2}e_2$  and  $e_0e_2 = 0$ .  
By setting  $u = \frac{1}{2}e_2$ , the family  $(e_0, e_1, u)$  is an adapted base of  $\mathcal{A}$  and its multiplication table is given by  $e_0^2 = e_0$ ,  $e_0e_1 = \frac{1}{2}e_1$ ,  $e_1^2 = u$ ,  $e_0u = e_1u = u^2 = 0$ . So the algebra  $\mathcal{A}$  is isomorphic to the algebra  $\mathcal{A}_1(0)$ .
- For  $\alpha = 1$ , we have  $e_2 = -2[e_1^2]_1$  and  $0 = \varphi_{\frac{1}{2}}^d(e_1) = [e_1^2]_0 = [e_1^2]_{-1}$ . It follows that  $e_1^2 = [e_1^2]_1 = -\frac{1}{2}e_2$  and  $e_0e_2 = e_2$ . Set  $u = -\frac{1}{2}e_2$ , the family  $(e_0, e_1, u)$  is an adapted base of  $\mathcal{A}$  and its multiplication table is given by  $e_0^2 = e_0$ ,  $e_0e_1 = \frac{1}{2}e_1$ ,  $e_0u = u$ ,  $e_1^2 = u$ ,  $e_1u = u^2 = 0$ . The algebra  $\mathcal{A}$  is isomorphic to the algebra  $\mathcal{A}_1(1)$ .
- For  $\alpha = -1$ , we have  $e_2 = \frac{2}{3}[e_1^2]_{-1}$  and  $0 = \varphi_{\frac{1}{2}}^d(e_1) = [e_1^2]_0 = [e_1^2]_1$ . We deduce that  $e_1^2 = [e_1^2]_{-1} = \frac{3}{2}e_2$  and  $e_0e_2 = -e_2$ . Set  $u = \frac{3}{2}e_2$ , the family  $(e_0, e_1, u)$  is an adapted base of  $\mathcal{A}$  and its multiplication table is given by  $e_0^2 = e_0$ ,  $e_0e_1 = \frac{1}{2}e_1$ ,  $e_0u = -u$ ,  $e_1^2 = u$ ,  $e_1u = u^2 = 0$ . The algebra  $\mathcal{A}$  is isomorphic to the algebra  $\mathcal{A}_1(-1)$ .
- For  $\alpha = \frac{1}{2}$ , we have  $e_2 = \varphi_{\frac{1}{2}}^d(e_1)$  and  $0 = [e_1^2]_0 = [e_1^2]_1 = [e_1^2]_{-1}$ . It follows that  $e_1^2 = 0$  and  $e_0e_2 = \frac{1}{2}e_2$ . It follows that the algebra  $\mathcal{A}$  is isomorphic to the algebra  $\mathcal{A}_2$ . □

**Remark 4.1** *The 3-Jordan algebras  $\mathcal{A}_2$  ;  $\mathcal{A}_1(0)$  and  $\mathcal{A}_1(1)$  are also Jordan algebras isomorphic to algebras given in [1, Assertion 2) of Corollary 4.11] for  $p = 1$ . The algebra  $\mathcal{A}_1(-1)$  is the only 3-dimensional dimensionally nilpotent 3-Jordan algebra that is not a Jordan algebra.*

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