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A Note on Type 2 Degenerate Changhee Numbers and Polynomials

Waseem Ahmad Khan

ABSTRACT: In this paper, we introduce type 2 degenerate Changhee numbers and polynomials and investigate some properties of these numbers and polynomials. We introduce higher-order type 2 degenerate Changhee polynomials and numbers and derive their explicit expressions and some identities involving them. In addition, we give some new relations between the type 2 degenerate Changhee polynomials and degenerate Euler polynomials.

Key Words: degenerate Euler polynomials, type 2 degenerate Changhee polynomials, type 2 degenerate Euler polynomials, higher-order type 2 degenerate Changhee polynomials and numbers.

Contents

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the filed of p-adic rational numbers and the completion of an algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\ .\ |_p$ is normalized by $|\ p\ |_p = \frac{1}{p}$. Let $C(\mathbb{Z}_p)$ be the space of continuous function on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows

$$I(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}(x + p^N\mathbb{Z}_p)$$

$$= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{(see [3, 21, 22])} \tag{1.1}$$

From (1.1), we note that

$$I(f_n) + (-1)^{n-1} I(f) = 2\sum_{a=0}^{n-1} (-1)^{n-1-a} f(a), \text{(see [21, 22])}, \tag{1.2}$$

where $f_n(x) = f(x + n), (n \in \mathbb{N})$.

Let the Changhee polynomials are defined by the generating function as follows (see [5, 6])

$$\frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}. \tag{1.3}$$

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Letting $x = 0$, $Ch_n = Ch_n(0), (n \geq 0)$ are called the Changhee numbers. From (1.3), we note that

$$\int_{\mathbb{Z}_p} (1+t)^{x+y} d\mu_{-1}(y) = \frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}. \tag{1.4}$$

Thus, by (1.4), we have

$$\int_{\mathbb{Z}_p} (x+y)_n d\mu_{-1}(y) = Ch_n(x), (n \geq 0), (\text{see [6]}), \tag{1.5}$$

where $(x)_0 = 1, (x)_n = x(x-1)\cdots(x-n+1), (n \geq 1)$,

As well known, the type 2 Euler polynomials are defined by

$$\frac{2}{e^t + e^{-t}} e^{xt} = \sum_{n=0}^{\infty} E_n^*(x)\frac{t^n}{n!}. \tag{1.6}$$

In the case $x = 0$, $E_n^* = E_n^*(0)$ are called the type 2 Euler numbers.

By using (1.1) and (1.6), we note that

$$\int_{\mathbb{Z}_p} e^{(2y+x+1)t} d\mu_{-1}(y) = \frac{2}{e^t + e^{-t}} e^{xt} = \sum_{n=0}^{\infty} E_n^*(x)\frac{t^n}{n!}. \tag{1.7}$$

By (1.7), we get

$$\int_{\mathbb{Z}_p} (2y+x+1)^n d\mu_{-1}(y) = E_n^*(x), (n \geq 0), (\text{see [8, 9, 10]}), \tag{1.8}$$

For $n \geq 0$, the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, (\text{see [1-10]}), \tag{1.9}$$

where $(x)_0 = 1$, and $(x)_n = x(x-1)\cdots(x-n+1), (n \geq 1)$. From (1.9), it is easily to see that

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \quad (k \geq 0), (\text{see [11-20]}). \tag{1.10}$$

In the inverse expression to (1.9), the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, (\text{see [21-26]}). \tag{1.11}$$

From (1.11), it is easily to see that

$$\frac{1}{k!}(e^t - 1)^k = \sum_{n=l}^{\infty} S_2(n,l)\frac{t^n}{n!}, (\text{see [1-15]}). \tag{1.12}$$

From (1.3), (1.6), (1.9) and (1.10), we get

$$Ch_n(x) = \sum_{l=0}^{n} E_l^*(x)S_1(n,l), \tag{1.13}$$

and

$$E_n^*(x) = \sum_{l=0}^{n} Ch_n(x)S_2(n,l), (\text{see [6]}) \tag{1.14}$$

Recently, Kim-Kim [12] introduced the type 2 Changhee polynomials are defined by

$$\int_{\mathbb{Z}_p}(1+t)^{2y+x+1}d\mu_{-1}(y)=\frac{2}{(1+t)+(1+t)^{-1}}(1+t)^{x}$$
$$=\sum_{n=0}^{\infty}c_n(x)\frac{t^n}{n!},\qquad(1.15)$$

When $x=0$, $c_n=c_n(0)$ are called the type 2 Changhee numbers.

For any $\lambda\in\mathbb{R}$, degenerate version of the exponential function $e_\lambda^x(t)$ is defined as follows (see [16-26])

$$e_\lambda^x(t):=(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!},\qquad(1.16)$$

Kim introduced the degenerate Stirling numbers of the second kind (see [24]) are given by

$$\frac{1}{k!}(e_\lambda(t)-1)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!},\quad(k\geq 0).\qquad(1.17)$$

It is clear that $\lim_{\lambda\to 0}S_{2,\lambda}(n,k)=S_2(n,k)$, where $S_2(n,k)$ are called the Stirling numbers of the second.

Recently, Kim [23] introduced the degenerate Changhee polynomials of the second kind are defined by

$$\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{x+y}{\lambda}}d\mu_{-1}(y)=\frac{2}{1+(1+\lambda\log(1+t))}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}$$
$$=\sum_{n=0}^{\infty}Ch_{n,\lambda}(x)\frac{t^n}{n!},\qquad(1.18)$$

where $\lambda\in\mathbb{C}_p$ with $\mid\lambda\mid_p\leq 1$.

When $x=0$, $Ch_{n,\lambda}=Ch_{n,\lambda}(0)$ are called the degenerate Changhee numbers of the second kind.

This paper is organized as follows. In Sect 2, we consider type 2 degenerate Changhee numbers and polynomials and investigate some properties of these numbers and polynomials. In Sect 3, we introduce higher-order type 2 degenerate Changhee polynomials and numbers which can be represented in terms of p-adic integrals on $\mathbb{Z}_p$. We derive their explicit expressions and some other polynomials. Moreover, we obtain identities involving those polynomials and some other special numbers and polynomials.

2. Type 2 degenerate Changhee numbers and polynomials

In this section, let us assume that $\lambda\in\mathbb{C}_p$ and $t\in\mathbb{C}_p$ with the condition $\mid\lambda t\mid_p< p^{-\frac{1}{p-1}}$. As is well-known, the type 2 degenerate Euler polynomials are defined by the generating function

$$\int_{\mathbb{Z}_p}e_\lambda^{2y+x+1}(t)d\mu_{-1}(y)=\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^{x}(t)$$
$$=\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!},\text{(see [8])}.\qquad(2.1)$$

From (2.1), we have

$$\int_{\mathbb{Z}_p}(2y+x+1)_{n,\lambda}d\mu_{-1}(y)=E_{n,\lambda}(x),(n\geq 0).\qquad(2.2)$$

When $x=0$, $E_{n,\lambda}=E_{n,\lambda}(0)$ are called the type 2 degenerate Euler numbers.

Inspired by (1.15) and (2.1), we introduce type 2 degenerate Changhee polynomials are defined by

$$\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{2y+x+1}{\lambda}}d\mu_{-1}(y)=\frac{2}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}+(1+\lambda\log(1+t))^{-\frac{1}{\lambda}}}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}$$
$$=\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.3}$$

Note that, $\lim_{\lambda\to 0}\widehat{C}_{n,\lambda}(x)=c_n(x)$, $(n\geq 0)$, (see [12]). We note that $x=0,\widehat{C}_{n,\lambda}=\widehat{C}_{n,\lambda}(0)$ are called the type 2 degenerate Changhee.

Theorem 2.1. For $n\geq 0$, we have

$$\tilde{C}_{n,\lambda}(x)=\sum_{l=0}^{n}S_1(n,l)\int_{\mathbb{Z}_p}\binom{\frac{2y+x+1}{\lambda}}{l}l!d\mu_{-1}(y)\lambda^l$$
$$=\sum_{l=0}^{n}\int_{\mathbb{Z}_p}(2y+x+1)_{l,\lambda}d\mu_{-1}(y)\lambda^lS_1(n,l), \tag{2.4}$$

where $(x)_{0,\lambda}=1$ and $(x)_{n,\lambda}=x(x-\lambda)\cdots(x-(n-1)\lambda)$ for $n\geq 1$.
Proof. Using (2.3), we note that

$$\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{2y+x+1}{\lambda}}d\mu_{-1}(y)=\sum_{l=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\frac{2y+x+1}{\lambda}}{l}d\mu_{-1}(y)\lambda^l(\log(1+t))^l$$
$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}S_1(n,l)\int_{\mathbb{Z}_p}\binom{\frac{2y+x+1}{\lambda}}{l}l!d\mu_{-1}(y)\lambda^l\right)\frac{t^n}{n!}. \tag{2.5}$$

Comparing the coefficients of on both sides of (2.3) and (2.5), we obtain the result (2.4).
Theorem 2.2. For $n\geq 0$, we have

$$E_{n,\lambda}(x)=\sum_{m=0}^{n}\widehat{C}_{m,\lambda}(x)S_2(n,m). \tag{2.6}$$

Proof By replacing t by e^t-1 in (2.3) and using (1.12), we get

$$\sum_{m=0}^{\infty}\widehat{C}_{m,\lambda}(x)\frac{(e^t-1)^m}{m!}=\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)$$
$$=\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.7}$$

On the other hand,

$$\sum_{m=0}^{\infty}\widehat{C}_{m,\lambda}(x)\frac{(e^t-1)^m}{m!}=\sum_{m=0}^{\infty}\widehat{C}_{m,\lambda}(x)\sum_{n=m}^{\infty}S_2(n,m)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widehat{C}_{m,\lambda}(x)S_2(n,m)\right)\frac{t^n}{n!}. \tag{2.8}$$

By (2.7) and (2.8), we get the result.

Theorem 2.3. For $n \geq 0$, we have

$$\widehat{C}_{n,\lambda}(x)=\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}(x)_{m,\lambda}S_{1,\lambda}(k,m)\widehat{C}_{n-k,\lambda}. \tag{2.9}$$

Proof. From (2.3), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}(x)\frac{t^n}{n!}&=\frac{2}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}+(1+\lambda\log(1+t))^{-\frac{1}{\lambda}}}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\\&=\left(\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\binom{\frac{x}{\lambda}}{m}(\log(1+t))^m\right)\\&=\left(\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}(x)_{m,\lambda}\sum_{k=m}^{\infty}S_1(k,m)\frac{t^k}{k!}\right)\\&=\left(\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}(x)_{m,\lambda}S_1(k,m)\right)\frac{t^k}{k!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}(x)_{m,\lambda}S_{1,\lambda}(k,m)\widehat{C}_{n-k,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

Therefore, by (2.4) and (2.10), we obtain at the required result.

Theorem 2.4. For $n \geq 0$, we have

$$\widehat{C}_{n,\lambda}(1)+\widehat{C}_{n,\lambda}=\begin{cases}2, & \text{if } n=0,\\ 0, & \text{if } n\geq 1,\end{cases} \tag{2.11}$$

Proof. By (1.2), we easily get

$$\int_{\mathbb{Z}_p}f(x+1)d\mu_{-1}(x)+\int_{\mathbb{Z}_p}f(x)d\mu_{-1}(x)=2f(0). \tag{2.12}$$

Now, equation (2.12) can be written as

$$\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{2y+2}{\lambda}}d\mu_{-1}(y)+\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{2y+1}{\lambda}}d\mu_{-1}(y)=2. \tag{2.13}$$

From (2.3) and (2.13), we have

$$\begin{aligned}&\frac{2}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}+(1+\lambda\log(1+t))^{-\frac{1}{\lambda}}}(1+\lambda\log(1+t))^{\frac{1}{\lambda}}\\&+\frac{2}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}+(1+\lambda\log(1+t))^{-\frac{1}{\lambda}}}=2.\end{aligned} \tag{2.14}$$

From (2.11) and (2.14), we have

$$\sum_{n=0}^{\infty}\left(\widehat{C}_{n,\lambda}(1)+\widehat{C}_{n,\lambda}\right)\frac{t^n}{n!}=2. \tag{2.15}$$

In view of (2.15), we complete the proof.

Theorem 2.6. For $n \geq 0$, we have

$$\widehat{C}_{n,\lambda}(x+1)+\widehat{C}_{n,\lambda}(x)=\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k}\widehat{C}_{n-k,\lambda}(x)(1)_{m,\lambda}S_1(k,m). \tag{2.16}$$

Proof. Suppose that

$$\frac{2}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}+(1+\lambda\log(1+t))^{-\frac{1}{\lambda}}}(1+\lambda\log(1+t))^{\frac{x+1}{\lambda}}+\frac{2(1+\lambda\log(1+t))^{\frac{x}{\lambda}}}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}+(1+\lambda\log(1+t))^{-\frac{1}{\lambda}}}$$

$$=\frac{2}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}+(1+\lambda\log(1+t))^{-\frac{1}{\lambda}}}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}(1+\lambda\log(1+t))^{\frac{1}{\lambda}}. \tag{2.17}$$

Thus, by (2.3) and (2.17), we get

$$\sum_{n=0}^{\infty}\left(\widehat{C}_{n,\lambda}(x+1)+\widehat{C}_{n,\lambda}(x)\right)\frac{t^n}{n!}$$

$$=\left(\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}(x)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{1}{m!}(\log(1+t))^m\right)$$

$$=\left(\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}(x)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\sum_{m=0}^{k}(1)_{m,\lambda}S_1(k,m)\frac{t^k}{k!}\right)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k}\widehat{C}_{n-k,\lambda}(x)(1)_{m,\lambda}S_1(k,m)\right)\frac{t^n}{n!}. \tag{2.18}$$

By comparing the coefficients of t, we get (2.16).

3. Type 2 higher-order degenerate Changhee polynomials

In this section, we introduce type 2 degenerate Changhee polynomials of order r which are derived from the multivariate fermionic p-adic integral on $\mathbb{Z}_p$.

For $r\in\mathbb{N}$, we define the type 2 degenerate Changhee polynomials of order r which are given multivariate fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{2(x_1+\cdots+x_r)+x+1}{\lambda}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$

$$=\left(\frac{2}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}+(1+\lambda\log(1+t))^{-\frac{1}{\lambda}}}\right)^r(1+\lambda\log(1+t))^{\frac{x}{\lambda}}$$

$$=\sum_{n=0}^{\infty}\widehat{C}^{(r)}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{3.1}$$

When $x=0,\widehat{C}^{(r)}_{n,\lambda}=\widehat{C}^{(r)}_{n,\lambda}(0)$ are called the type 2 degenerate Changhee numbers of order α.

Theorem 3.1. For $n\geq 0$ and $r\in\mathbb{N}$, we have

$$\widehat{C}^{(r)}_{n,\lambda}(x)=\sum_{m=0}^{n}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(2(x_1+\cdots+x_r)+x+1)_{\lambda,m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)S_1(n,m)$$

Proof. From (3.1), we note that

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda\log_\lambda(1+t))^{\frac{2(x_1+\cdots+x_r)+x+1}{\lambda}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$

$$=\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{\frac{2(x_1+\cdots+x_r)+x+1}{\lambda}}{m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\lambda^m(\log(1+t)))^m$$

$$=\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(2(x_1+\cdots+x_r)+x+1)_{\lambda,m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\frac{1}{m!}(\log(1+t)))^m$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(2(x_1+\cdots+x_r)+x+1)_{\lambda,m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)S_1(n,m)\right)\frac{t^n}{n!}. \tag{3.2}$$

For $r\in\mathbb{N}$, we have

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{2(x_1+\cdots+x_r)+x+1}{\lambda}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$

$$=\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}. \tag{3.3}$$

Now, we define the type 2 degenerate Euler polynomials of order r which are given by

$$\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}E_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{3.4}$$

Thus, by (3.3) and (3.4) we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(2(x_1+\cdots+x_r)+x+1)_{\lambda,m}d\mu_{-1}(x_1)\cdots d\mu_{-1}=E_{n,\lambda}^{(r)}(x),(m\geq 0). \tag{3.5}$$

Theorem 3.2. For $n\geq 0$, we have

$$\widehat{C}_{n,\lambda}^{(r)}(x)=\sum_{m=0}^{n}E_{m,\lambda}^{(r)}S_1(n,m).$$

Proof. By using (3.2), (3.4) and (3.5), we obtain the result.

Theorem 3.3. For $n\geq 0$, we have

$$E_{n,\lambda}^{(r)}(x)=\sum_{m=0}^{n}\widehat{C}_{m,\lambda}^{(r)}(x)S_2(n,m).$$

Proof. By changing t by e^t-1 in (3.1), we have

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{2(x_1+\cdots+x_r)+x+1}{\lambda}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$

$$=\sum_{m=0}^{\infty}\widehat{C}_{m,\lambda}^{(r)}(x)\frac{(e^t-1)^m}{m!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widehat{C}_{m,\lambda}^{(r)}(x)S_2(n,m)\right)\frac{t^n}{n!}. \tag{3.6}$$

Therefore, by (3.3) and (3.6), we get the result.

Theorem 3.4. For $n\geq 0$, we have

$$\widehat{C}_{n,\lambda}^{(r)}(x)=\sum_{l=0}^{n}\binom{n}{l}\widehat{C}_{n-l,\lambda}^{(r)}\widehat{C}_{l,\lambda}^{(r-k)}(x).$$

Proof. From (3.1), we have

$$\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}=\left(\frac{2}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}+(1+\lambda\log(1+t))^{-\frac{1}{\lambda}}}\right)^r(1+\lambda\log(1+t))^{\frac{x}{\lambda}}$$

$$= \left(\frac{2}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}+(1+\lambda\log(1+t))^{-\frac{1}{\lambda}}}\right)^{r}$$

$$\times \left(\frac{2}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}+(1+\lambda\log(1+t))^{-\frac{1}{\lambda}}}\right)^{r-k}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}$$

$$= \left(\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}^{(r)}\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\widehat{C}_{l,\lambda}^{(r-k)}(x)\frac{t^l}{l!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\widehat{C}_{n-l,\lambda}^{(r)}\widehat{C}_{l,\lambda}^{(r-k)}(x)\right)\frac{t^n}{n!}. \tag{3.7}$$

In view of (3.7), we complete the proof.

Theorem 3.5. For $n \geq 0$, we have

$$\widehat{C}_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}(x)_{m,\lambda}S_{1,\lambda}(k,m)\widehat{C}_{n-k,\lambda}^{(r)}. \tag{3.8}$$

Proof. From (3.1), we note that

$$\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} = \left(\frac{2}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}+(1+\lambda\log(1+t))^{-\frac{1}{\lambda}}}\right)^{r}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}$$

$$= \left(\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}^{(r)}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\binom{\frac{x}{\lambda}}{m}(\log(1+t))^m\right)$$

$$= \left(\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}^{(r)}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}(x)_{m,\lambda}\sum_{k=m}^{\infty}S_1(k,m)\frac{t^k}{k!}\right)$$

$$= \left(\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}^{(r)}\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}(x)_{m,\lambda}S_1(k,m)\right)\frac{t^k}{k!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}(x)_{m,\lambda}S_{1,\lambda}(k,m)\widehat{C}_{n-k,\lambda}^{(r)}\right)\frac{t^n}{n!}. \tag{3.9}$$

Therefore, by (3.1) and (3.9), we obtain at the required result.

Theorem 3.6. For $n \geq 0$, we have

$$\sum_{m=0}^{n}\widehat{C}_{m,\lambda}^{(r)}(x)S_2(n,m) = \sum_{m=0}^{n}S_{2,\lambda}(n,m)C_m^{(r)}(x).$$

Proof. Now, we observe that

$$(1+\lambda t)^{\frac{2(x_1+\cdots+x_r)+x+1}{\lambda}} = \left((1+\lambda t)^{\frac{1}{\lambda}}-1+1\right)^{2(x_1+\cdots+x_r)+x+1}$$

$$= \sum_{m=0}^{\infty}\binom{2(x_1+\cdots+x_r)+x+1}{m}((1+\lambda t)^{\frac{1}{\lambda}}-1)^m$$

$$= \sum_{m=0}^{\infty}(2(x_1+\cdots+x_r)+x+1)_m\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_{2,\lambda}(n,m)(2(x_1+\cdots+x_r)+x+1)_m\right)\frac{t^n}{n!}. \tag{3.10}$$

Thus, by (3.5) and (3.10), we get

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{2(x_1+\cdots+x_r++x+1}{\lambda}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_{2,\lambda}(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(2(x_1+\cdots+x_r)+x+1)_m d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\right)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_{2,\lambda}(n,m)C_m^{(r)}(x)\right)\frac{t^n}{n!}. \qquad (3.11)\end{aligned}$$

Therefore, by (3.5) and (3.11), we obtain the result.

Theorem 3.7. For $n \geq 0$, we have

$$\widehat{C}_{n,\lambda}^{(r)}(x)=\sum_{m=0}^{n}E_{m,\lambda}^{(r)}(x)S_1(n,m).$$

Proof. By replacing t by $\log(1+t)$ in (3.4), we have

$$\begin{aligned}\left(\frac{2}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}+(1+\lambda\log(1+t))^{-\frac{1}{\lambda}}}\right)^r(1+\lambda\log(1+t))^{\frac{x}{\lambda}}&=\sum_{m=0}^{\infty}E_{m,\lambda}^{(r)}(x)\frac{(\log(1+t))^m}{m!}\\ &=\sum_{m=0}^{\infty}E_{m,\lambda}^{(r)}(x)\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}E_{m,\lambda}^{(r)}(x)S_1(n,m)\right)\frac{t^n}{n!}. \qquad (3.12)\end{aligned}$$

On the other hand, we have

$$\left(\frac{2}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}+(1+\lambda\log(1+t))^{-\frac{1}{\lambda}}}\right)^r(1+\lambda\log(1+t))^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \qquad (3.13)$$

In view of (3.12) and (3.13), we obtain the result.

Theorem 3.8. For $n \geq 0$, we have

$$\widehat{C}_{n,\lambda}^{(r)}(x+y)=\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}(y)_{m,\lambda}S_1(k,m)\widehat{C}_{n-k,\lambda}^{(r)}(x).$$

Proof. From (3.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}^{(r)}(x+y)\frac{t^n}{n!}&=\left(\frac{2}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}+(1+\lambda\log(1+t))^{-\frac{1}{\lambda}}}\right)^r(1+\lambda\log(1+t))^{\frac{x+y}{\lambda}}\\ &=\left(\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\binom{\frac{y}{\lambda}}{m}(\log(1+t))^m\right)\\ &=\left(\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}^{(r)}\frac{t^n}{n!}(x)\right)\left(\sum_{m=0}^{\infty}(y)_{m,\lambda}\sum_{k=m}^{\infty}S_1(k,m)\frac{t^k}{k!}\right)\\ &=\left(\sum_{n=0}^{\infty}\widehat{C}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}(y)_{m,\lambda}S_1(k,m)\right)\frac{t^k}{k!}\right)\\ &=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}(y)_{m,\lambda}S_1(k,m)\widehat{C}_{n-k,\lambda}^{(r)}(x)\right)\frac{t^n}{n!}. \qquad (3.14)\end{aligned}$$

Thus, by (3.14), we obtain at the required result.

4. Conclusion

In this article, we introduced type 2 degenerate Changhee numbers and polynomials and investigated some properties of these numbers and polynomials. We introduced higher-order type 2 degenerate Changhee polynomials and numbers and derived their explicit expressions and some identities involving them. In addition, we given some new relations between the type 2 degenerate Changhee polynomials and degenerate Euler polynomials.

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Waseem Ahmad Khan,
Department of Electrical Engineering,
Prince Mohammad Bin Fahd University,
P.O Box 1664, Al Khobar 31952
Saudi Arabia.
E-mail address: wkhan1@pmu.edu.sa