



Molodtsov’s Soft Topological Structures

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ABSTRACT: This study revisits several foundational concepts in soft topological structures introduced by Molodtsov in 2015, which are based on proximity mappings rather than classical open-set axioms. In particular, the paper explores proximity mappings, $\tilde{\tau}$ -neighborhoods of elements and sets, $\tilde{\tau}$ -closure, $\tilde{\tau}$ -interior, $\tilde{\tau}$ -boundary, $\tilde{\tau}$ -border set, $\tilde{\tau}$ -hidden set, $\tilde{\tau}$ -accumulation points, $\tilde{\tau}$ -cluster points, $\tilde{\tau}$ -isolated points, and $\tilde{\tau}$ -separated points. The study’s originality lies in the rigorous development, extension, and unification of these concepts, many of which have remained underexplored owing to their limited presence in the relevant literature and the limited accessibility of the few existing sources. Beyond these, another originality of the paper is introducing novel concepts that have not been previously considered in the literature: $\tilde{\tau}$ -disjointness, $\tilde{\tau}$ -exterior, and $\tilde{\tau}$ -condensation points of a set, together with $\tilde{\tau}$ - C -closed, $\tilde{\tau}$ - B -open, $\tilde{\tau}$ - T -closed, and $\tilde{\tau}$ - I -open sets and the corresponding $\tilde{\tau}$ - C -, $\tilde{\tau}$ - B -, $\tilde{\tau}$ - T -, and $\tilde{\tau}$ - I -soft topologies. This study thoroughly investigates the fundamental properties of the newly proposed concepts and supports all theoretical developments with illustrative examples. The paper concludes by outlining potential directions for future research within Molodtsov’s soft topological framework.

Keywords: Soft sets, proximity mappings, $\tilde{\tau}$ -neighborhood, $\tilde{\tau}$ -closure, $\tilde{\tau}$ - C - and $\tilde{\tau}$ - T -closed sets, $\tilde{\tau}$ - C - and $\tilde{\tau}$ - T -soft topologies.

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1. Introduction

As an alternative mathematical framework to fuzzy sets [3], the concept of soft sets was introduced by Molodtsov [2] in 1999. This concept was applied to various topics and fields in [1,2], including soft transformations, soft limits, soft continuity, soft derivatives, soft integrals, soft probability, soft game theory, soft equilibrium, and soft mathematics in portfolio management. However, these applications were addressed at a foundational level in the aforementioned studies.

Around the same period, soft topology has been studied from two distinct perspectives. In one approach, Çağman et al. [5] propounded the concept of soft topology on arbitrary soft sets and investigated classical topological concepts in this context. In another approach, Shabir and Naz [4] introduced soft topology on a universal set via soft sets, exploring topological concepts in terms of the elements of the universal set. While these studies hold a significant place in the relevant literature, both approaches essentially construct soft counterparts of classical topology based on open-set axioms.

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In contrast, Molodtsov's approach [6] diverges from these perspectives. He has defined two operators, denoted Close and Touch (see Definitions 2.3 and 2.4), which are analogous to the classical closure operator, using proximity mappings and the Kuratowski axioms [8]. These operators underlie his soft topological structures. This operator-based formulation reveals a novel framework that is conceptually and methodologically distinct from both classical topology and earlier soft topological approaches. However, since [6] was published in Russian, their reach and impact in international literature have remained limited, and many of the properties of the introduced notions have not yet been systematically investigated. Therefore, a thorough investigation of these properties is essential to establish a solid theoretical foundation for soft topological structures. Furthermore, elucidating these properties is crucial for understanding the connection between soft analysis and soft topology.

The first aim of this study is to provide a detailed and systematic analysis of several foundational notions introduced in [6,7], including proximity mappings, $\tilde{\tau}$ -neighborhoods, $\tilde{\tau}$ -closure, $\tilde{\tau}$ -interior, $\tilde{\tau}$ -boundary, $\tilde{\tau}$ -border set, $\tilde{\tau}$ -hidden set, and various types of points defined within this framework. Many structural properties of these notions are established here for the first time, thereby strengthening the theoretical basis of Molodtsov's soft topological structures. The main originality of the paper lies in going beyond this analysis by introducing new concepts that have not previously appeared in the relevant literature: $\tilde{\tau}$ -disjointness, $\tilde{\tau}$ -exterior, and $\tilde{\tau}$ -condensation points of a set. Furthermore, new families of sets— $\tilde{\tau}$ - C -closed, $\tilde{\tau}$ - B -open, $\tilde{\tau}$ - T -closed, and $\tilde{\tau}$ - I -open—are defined, and the corresponding $\tilde{\tau}$ - C -, $\tilde{\tau}$ - B -, $\tilde{\tau}$ - T -, and $\tilde{\tau}$ - I -soft topologies are constructed and analyzed. By thoroughly analyzing their properties and interrelations, this study establishes a solid theoretical foundation for Molodtsov's framework. Hence, the present study contributes to the broader dissemination and deeper understanding of this framework.

The remainder of the paper is organized as follows. Section 2 provides an overview of fundamental concepts of soft topological structures in [6]. Section 3 provides a detailed theoretical analysis of these concepts. Section 4 defines $\tilde{\tau}$ -disjointness, $\tilde{\tau}$ -exterior, and $\tilde{\tau}$ -condensation points of a set and explores some of their basic properties. Section 5 introduces $\tilde{\tau}$ - C -closed, $\tilde{\tau}$ - B -open, $\tilde{\tau}$ - T -closed, and $\tilde{\tau}$ - I -open sets and proposes the corresponding $\tilde{\tau}$ - C -, $\tilde{\tau}$ - B -, $\tilde{\tau}$ - T -, and $\tilde{\tau}$ - I -soft topologies. Moreover, it analyzes fundamental properties of these concepts. Section 6 provides illustrative examples that clarify the theoretical aspects of each of the aforementioned concepts. The last section addresses the need for further exploration in this framework.

2. Preliminaries

This section presents some fundamental definitions and properties from [6] that are required in the subsequent sections. Across this study, let $X \neq \emptyset$ and $P(X)$ denote the set of all classical subsets of X .

Definition 2.1 [6] *Let $\tilde{\tau} : X \rightarrow P(X)$ be a function. If $x \in \tilde{\tau}(x)$, for all $x \in X$, then the function $\tilde{\tau}$ is called a proximity mapping on X . Here, each point of $\tilde{\tau}(x)$, for all $x \in X$, is interpreted as a $\tilde{\tau}$ -close point to the point x .*

Throughout this study, unless otherwise specified, let $\tilde{\tau}$ and $\tilde{\lambda}$ be two proximity mappings on X .

Definition 2.2 [6] *Let $x \in X$. Then, the set $\tilde{\tau}(x)$ is called the $\tilde{\tau}$ -neighborhood of the point x .*

Definition 2.3 [6] *Let $A \subseteq X$. Then, the $\tilde{\tau}$ -neighborhood of the set A is defined by*

$$\text{Close}(A, X, \tilde{\tau}) := \bigcup_{x \in A} \tilde{\tau}(x) = \tilde{\tau}(A)$$

Definition 2.4 [6] *Let $x \in X$ and $A \subseteq X$. If $\tilde{\tau}(x) \cap A \neq \emptyset$, then the point x is called a $\tilde{\tau}$ -closure point or a $\tilde{\tau}$ -tangency point of the set A . The set of all $\tilde{\tau}$ -closure points of the set A is denoted by $\text{Touch}(A, X, \tilde{\tau})$. In other words,*

$$\text{Touch}(A, X, \tilde{\tau}) := \{x \in X : \tilde{\tau}(x) \cap A \neq \emptyset\}$$

Definition 2.5 [6] *Let $x \in X$ and $A \subseteq X$. If $\tilde{\tau}(x) \subseteq A$, then the point x is called a $\tilde{\tau}$ -interior point of the set A . The set of all $\tilde{\tau}$ -interior points of the set A is denoted by $\text{Int}(A, X, \tilde{\tau})$. In other words,*

$$\text{Int}(A, X, \tilde{\tau}) := \{x \in X : \tilde{\tau}(x) \subseteq A\}$$

Definition 2.6 [6] Let $x \in X$ and $A \subseteq X$. If $\tilde{\tau}(x) \cap A \neq \emptyset$ and $\tilde{\tau}(x) \cap (X \setminus A) \neq \emptyset$, then the point x is called a $\tilde{\tau}$ -boundary point or a $\tilde{\tau}$ -frontier point of the set A . The set of all $\tilde{\tau}$ -boundary points of the set A is denoted by $\text{Fr}(A, X, \tilde{\tau})$. In other words,

$$\text{Fr}(A, X, \tilde{\tau}) := \{x \in X : \tilde{\tau}(x) \cap A \neq \emptyset \wedge \tilde{\tau}(x) \cap (X \setminus A) \neq \emptyset\}$$

Definition 2.7 [6] Let $A \subseteq X$. Then, the $\tilde{\tau}$ -border set of the set A is defined by

$$\text{Bord}(A, X, \tilde{\tau}) := \text{Close}(A, X, \tilde{\tau}) \cap \text{Close}(X \setminus A, X, \tilde{\tau})$$

Here, the elements of the set $\text{Bord}(A, X, \tilde{\tau})$ are called $\tilde{\tau}$ -border points of the set A .

Definition 2.8 [6] Let $A \subseteq X$. Then, the $\tilde{\tau}$ -hidden set of the set A is defined by

$$\text{Bow}(A, X, \tilde{\tau}) := A \setminus \text{Bord}(A, X, \tilde{\tau})$$

Here, the elements of the set $\text{Bow}(A, X, \tilde{\tau})$ are called $\tilde{\tau}$ -hidden points of the set A .

Theorem 2.1 establishes the monotonicity properties of the aforementioned notions.

Theorem 2.1 [6] Let $A, B \subseteq X$. Then, the following hold:

- i. If $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, and $A \subseteq B$, then $\text{Close}(A, X, \tilde{\tau}) \subseteq \text{Close}(B, X, \tilde{\lambda})$.
- ii. If $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, and $A \subseteq B$, then $\text{Touch}(A, X, \tilde{\tau}) \subseteq \text{Touch}(B, X, \tilde{\lambda})$.
- iii. If $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, and $A \subseteq B$, then $\text{Int}(A, X, \tilde{\tau}) \subseteq \text{Int}(B, X, \tilde{\lambda})$.
- iv. If $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, then $\text{Fr}(A, X, \tilde{\tau}) \subseteq \text{Fr}(A, X, \tilde{\lambda})$.
- v. If $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, then $\text{Bord}(A, X, \tilde{\tau}) \subseteq \text{Bord}(A, X, \tilde{\lambda})$.
- vi. If $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, and $A \subseteq B$, then $\text{Bow}(A, X, \tilde{\tau}) \subseteq \text{Bow}(B, X, \tilde{\lambda})$.

Proposition 2.1 presents the properties of the aforementioned notions with respect to union and intersection operations.

Proposition 2.1 [6] Let I be an index set and $A_i \subseteq X$, for all $i \in I$. Then, the following hold:

- i. $\text{Close}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) = \bigcup_{i \in I} \text{Close}(A_i, X, \tilde{\tau})$
- ii. $\text{Close}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \subseteq \bigcap_{i \in I} \text{Close}(A_i, X, \tilde{\tau})$
- iii. $\text{Touch}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) = \bigcup_{i \in I} \text{Touch}(A_i, X, \tilde{\tau})$
- iv. $\text{Touch}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \subseteq \bigcap_{i \in I} \text{Touch}(A_i, X, \tilde{\tau})$
- v. $\bigcup_{i \in I} \text{Int}(A_i, X, \tilde{\tau}) \subseteq \text{Int}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right)$
- vi. $\text{Int}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) = \bigcap_{i \in I} \text{Int}(A_i, X, \tilde{\tau})$
- vii. $\text{Fr}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) \subseteq \bigcup_{i \in I} \text{Fr}(A_i, X, \tilde{\tau})$
- viii. $\text{Fr}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \subseteq \bigcup_{i \in I} \text{Fr}(A_i, X, \tilde{\tau})$
- ix. $\text{Bord}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) \subseteq \bigcup_{i \in I} \text{Bord}(A_i, X, \tilde{\tau})$

$$x. \text{Bord} \left(\bigcap_{i \in I} A_i, X, \tilde{\tau} \right) \subseteq \bigcup_{i \in I} \text{Bord} (A_i, X, \tilde{\tau})$$

$$xi. \bigcup_{i \in I} \text{Bow} (A_i, X, \tilde{\tau}) \subseteq \text{Bow} \left(\bigcup_{i \in I} A_i, X, \tilde{\tau} \right)$$

$$xii. \text{Bow} \left(\bigcap_{i \in I} A_i, X, \tilde{\tau} \right) = \bigcap_{i \in I} \text{Bow} (A_i, X, \tilde{\tau})$$

Proposition 2.2 presents several properties of the aforementioned notions.

Proposition 2.2 [6] *Let $A, B \subseteq X$. Then, the following hold:*

$$i. \text{Close}(\emptyset, X, \tilde{\tau}) = \emptyset$$

$$ii. A \subseteq \text{Close}(A, X, \tilde{\tau})$$

$$iii. \text{Close}(\text{Close}(A, X, \tilde{\tau}), X, \tilde{\lambda}) = \text{Close}(A, X, \tilde{\lambda} \circ \tilde{\tau})$$

$$iv. A \subseteq \text{Touch}(A, X, \tilde{\tau})$$

$$v. \text{Touch}(\emptyset, X, \tilde{\tau}) = \emptyset$$

$$vi. \text{Touch}(\text{Touch}(A, X, \tilde{\tau}), X, \tilde{\lambda}) = \text{Touch}(A, X, \tilde{\tau} \circ \tilde{\lambda})$$

$$vii. \text{Int}(X, X, \tilde{\tau}) = X$$

$$viii. \text{Int}(A, X, \tilde{\tau}) \subseteq A$$

$$ix. \text{Int}(A, X, \tilde{\tau}) \cap \text{Int}(X \setminus A, X, \tilde{\tau}) = \emptyset$$

$$x. \text{Close}(\text{Int}(A, X, \tilde{\tau}), X, \tilde{\tau}) \subseteq A \subseteq \text{Int}(\text{Close}(A, X, \tilde{\tau}), X, \tilde{\tau})$$

$$xi. \text{Int}(A, X, \tilde{\tau}) = A \setminus \text{Touch}(X \setminus A, X, \tilde{\tau})$$

$$xii. \text{Touch}(A, X, \tilde{\tau}) = X \setminus \text{Int}(X \setminus A, X, \tilde{\tau})$$

$$xiii. \text{Fr}(A, X, \tilde{\tau}) = \text{Fr}(X \setminus A, X, \tilde{\tau})$$

$$xiv. \text{Fr}(A, X, \tilde{\tau}) \cup \text{Fr}(B, X, \tilde{\tau}) = \text{Fr}(A \cup B, X, \tilde{\tau}) \cup \text{Fr}(A \cap B, X, \tilde{\tau}) \cup [\text{Fr}(A, X, \tilde{\tau}) \cap \text{Fr}(B, X, \tilde{\tau})]$$

$$xv. \text{Fr}(A, X, \tilde{\tau}) = \text{Touch}(A, X, \tilde{\tau}) \cap \text{Touch}(X \setminus A, X, \tilde{\tau})$$

$$xvi. \text{Int}(A, X, \tilde{\tau}) \cap \text{Fr}(A, X, \tilde{\tau}) = \emptyset$$

$$xvii. X = \text{Int}(A, X, \tilde{\tau}) \cup \text{Fr}(A, X, \tilde{\tau}) \cup \text{Int}(X \setminus A, X, \tilde{\tau})$$

$$xviii. \text{Touch}(A, X, \tilde{\tau}) = \text{Int}(A, X, \tilde{\tau}) \cup \text{Fr}(A, X, \tilde{\tau})$$

$$xix. \text{Bord}(A, X, \tilde{\tau}) = \text{Bord}(X \setminus A, X, \tilde{\tau})$$

$$xx. \text{Bord}(A, X, \tilde{\tau}) \cup \text{Bord}(B, X, \tilde{\tau}) = \text{Bord}(A \cup B, X, \tilde{\tau}) \cup \text{Bord}(A \cap B, X, \tilde{\tau}) \cup [\text{Bord}(A, X, \tilde{\tau}) \cap \text{Bord}(B, X, \tilde{\tau})]$$

$$xxi. \text{Bow}(A, X, \tilde{\tau}) \subseteq A$$

$$xxii. \text{Bow}(X, X, \tilde{\tau}) = X$$

$$xxiii. \text{Bow}(A, X, \tilde{\tau}) = A \setminus \text{Close}(X \setminus A, X, \tilde{\tau})$$

$$xxiv. \text{Bow}(A, X, \tilde{\tau}) \cap \text{Bow}(X \setminus A, X, \tilde{\tau}) = \emptyset$$

$$xxv. \text{Touch}(\text{Bow}(A, X, \tilde{\tau}), X, \tilde{\tau}) \subseteq A \subseteq \text{Bow}(\text{Touch}(A, X, \tilde{\tau}), X, \tilde{\tau})$$

$$xxvi. \text{Close}(A, X, \tilde{\tau}) = \text{Bow}(A, X, \tilde{\tau}) \cup \text{Bord}(A, X, \tilde{\tau})$$

$$xxvii. X = \text{Bow}(A, X, \tilde{\tau}) \cup \text{Bord}(A, X, \tilde{\tau}) \cup \text{Bow}(X \setminus A, X, \tilde{\tau})$$

Here, for all $x \in X$, $(\tilde{\tau} \circ \tilde{\lambda})(x) = \tilde{\tau}(\tilde{\lambda}(x))$ (see Example 6.3).

Definition 2.9 [6] *Let $A \subseteq X$ and $a \in A$.*

i. If $a \in \text{Touch}(A \setminus \{a\}, X, \tilde{\tau})$, then the point a is called a $\tilde{\tau}$ -accumulation point of the set A .

ii. If $a \in \text{Close}(A \setminus \{a\}, X, \tilde{\tau})$, then the point a is called a $\tilde{\tau}$ -cluster point of the set A .

iii. If the point a is not a $\tilde{\tau}$ -accumulation point of the set A , then the point a is called a $\tilde{\tau}$ -isolated point of the set A .

iv. If the point a is not a $\tilde{\tau}$ -cluster point of the set A , then the point a is called a $\tilde{\tau}$ -separated point of the set A .

Note 1 [6] The operators Touch and Close can be considered analogs of the closure operator in classical topology. Similarly, the operators Int and Bow can be regarded as analogs of the interior operator in classical topology.

The operators Fr and Bord can be regarded as analogs of the boundary operator in classical topology. Although $\tilde{\tau}$ -accumulation, $\tilde{\tau}$ -cluster, $\tilde{\tau}$ -isolated, and $\tilde{\tau}$ -separated points are similar to their classical counterparts, they belong only to the set under consideration, as defined in Definition 2.9, and not to its complement. Therefore, this study adopts Definition 2.10 rather than Definition 2.9 to ensure consistency among the concepts presented herein and their analogies to their classical counterparts.

Definition 2.10 Let $A \subseteq X$ and $a \in X$.

i. If $a \in \text{Touch}(A \setminus \{a\}, X, \tilde{\tau})$, then the point a is called a $\tilde{\tau}$ -accumulation point of the set A . The set of all $\tilde{\tau}$ -accumulation points of the set A is denoted by $\text{Accu}(A, X, \tilde{\tau})$. In other words,

$$\text{Accu}(A, X, \tilde{\tau}) := \{a \in X : a \in \text{Touch}(A \setminus \{a\}, X, \tilde{\tau})\}$$

ii. If $a \in \text{Close}(A \setminus \{a\}, X, \tilde{\tau})$, then the point a is called a $\tilde{\tau}$ -cluster point of the set A . The set of all $\tilde{\tau}$ -cluster points of the set A is denoted by $\text{Crowd}(A, X, \tilde{\tau})$. In other words,

$$\text{Crowd}(A, X, \tilde{\tau}) := \{a \in X : a \in \text{Close}(A \setminus \{a\}, X, \tilde{\tau})\}$$

iii. If $a \in A \setminus \text{Accu}(A, X, \tilde{\tau})$, then the point a is called a $\tilde{\tau}$ -isolated point of the set A . The set of all $\tilde{\tau}$ -isolated points of the set A is denoted by $\text{Isol}(A, X, \tilde{\tau})$. In other words,

$$\text{Isol}(A, X, \tilde{\tau}) := A \setminus \text{Accu}(A, X, \tilde{\tau})$$

iv. If $a \in A \setminus \text{Crowd}(A, X, \tilde{\tau})$, then the point a is called a $\tilde{\tau}$ -separated point of the set A . The set of all $\tilde{\tau}$ -separated points of the set A is denoted by $\text{Sep}(A, X, \tilde{\tau})$. In other words,

$$\text{Sep}(A, X, \tilde{\tau}) := A \setminus \text{Crowd}(A, X, \tilde{\tau})$$

Theorem 2.2 [6] Let $A \subseteq X$ and $a \in X$. Then, $a \in \text{Accu}(A, X, \tilde{\tau})$ if and only if $(\tilde{\tau}(a) \setminus \{a\}) \cap A \neq \emptyset$.

Theorem 2.3 [6] Let $A \subseteq X$ and $a \in X$. Then, the following are equivalent:

- i. $a \in \text{Crowd}(A, X, \tilde{\tau})$
- ii. There exists a point $b \in A \setminus \{a\}$ such that $a \in \tilde{\tau}(b)$.
- iii. There exists a point $b \in A$ such that $a \in \tilde{\tau}(b) \setminus \{b\}$.

Corollary 2.1 [6] Let $A \subseteq X$. Then, $\text{Crowd}(A, X, \tilde{\tau}) = \bigcup_{x \in A} (\tilde{\tau}(x) \setminus \{x\})$.

Proposition 2.3 [6] Let $A, B \subseteq X$, I be an index set, and $A_i \subseteq X$, for all $i \in I$. Then, the following hold:

- i. If $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, and $A \subseteq B$, then $\text{Crowd}(A, X, \tilde{\tau}) \subseteq \text{Crowd}(B, X, \tilde{\lambda})$.
- ii. $\bigcup_{i \in I} \text{Crowd}(A_i, X, \tilde{\tau}) = \text{Crowd}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right)$
- iii. $\text{Crowd}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \subseteq \bigcap_{i \in I} \text{Crowd}(A_i, X, \tilde{\tau})$

Theorem 2.4 [6] Let $A \subseteq X$ and $a \in A$. Then, $a \in \text{Isol}(A, X, \tilde{\tau})$ if and only if $(\tilde{\tau}(a) \setminus \{a\}) \cap A = \emptyset$.

Proposition 2.4 [6] Let $A \subseteq X$. Then, $\text{Sep}(A, X, \tilde{\tau}) = A \setminus \left(\bigcup_{x \in A} (\tilde{\tau}(x) \setminus \{x\})\right)$.

3. Main Results

This section studies the properties of the concepts presented in the previous section and develops the relationships among them.

Proposition 3.1 *Let $A \subseteq X$. Then, the following hold:*

- i.* $\text{Int}(\emptyset, X, \tilde{\tau}) = \text{Fr}(\emptyset, X, \tilde{\tau}) = \text{Fr}(X, X, \tilde{\tau}) = \text{Bord}(\emptyset, X, \tilde{\tau}) = \text{Bord}(X, X, \tilde{\tau}) = \text{Bow}(\emptyset, X, \tilde{\tau}) = \emptyset$
- ii.* $\text{Close}(X, X, \tilde{\tau}) = \text{Touch}(X, X, \tilde{\tau}) = X$
- iii.* $X \setminus \text{Close}(A, X, \tilde{\tau}) \subseteq \text{Close}(X \setminus A, X, \tilde{\tau})$
- iv.* $\text{Close}(A, X, \tilde{\tau}) \cup \text{Close}(X \setminus A, X, \tilde{\tau}) = X$
- v.* $X \setminus \text{Touch}(A, X, \tilde{\tau}) \subseteq \text{Touch}(X \setminus A, X, \tilde{\tau})$
- vi.* $\text{Touch}(A, X, \tilde{\tau}) \cup \text{Touch}(X \setminus A, X, \tilde{\tau}) = X$
- vii.* $X \setminus \text{Int}(A, X, \tilde{\tau}) = \text{Touch}(X \setminus A, X, \tilde{\tau})$
- viii.* $\text{Fr}(A, X, \tilde{\tau}) \subseteq \text{Touch}(A, X, \tilde{\tau})$
- ix.* $\text{Fr}(A, X, \tilde{\tau}) = \text{Touch}(A, X, \tilde{\tau}) \setminus \text{Int}(A, X, \tilde{\tau})$
- x.* $\text{Fr}(A, X, \tilde{\tau}) = (X \setminus \text{Int}(A, X, \tilde{\tau})) \cap (X \setminus \text{Int}(X \setminus A, X, \tilde{\tau}))$
- xi.* $\text{Bord}(A, X, \tilde{\tau}) \subseteq \text{Close}(A, X, \tilde{\tau})$
- xii.* $\text{Bow}(X \setminus A, X, \tilde{\tau}) = X \setminus \text{Close}(A, X, \tilde{\tau}) = \bigcap_{x \in A} (X \setminus \tilde{\tau}(x))$
- xiii.* $X \setminus \text{Bow}(A, X, \tilde{\tau}) = \text{Close}(X \setminus A, X, \tilde{\tau})$
- xiv.* $\text{Bow}(A, X, \tilde{\tau}) \cap \text{Bord}(A, X, \tilde{\tau}) = \emptyset$
- xv.* $\text{Bord}(A, X, \tilde{\tau}) = (X \setminus \text{Bow}(A, X, \tilde{\tau})) \cap (X \setminus \text{Bow}(X \setminus A, X, \tilde{\tau}))$
- xvi.* $\text{Bord}(A, X, \tilde{\tau}) = \text{Close}(A, X, \tilde{\tau}) \setminus \text{Bow}(A, X, \tilde{\tau})$
- xvii.* $\text{Int}(\text{Int}(A, X, \tilde{\tau}), X, \tilde{\lambda}) = \text{Int}(A, X, \tilde{\tau} \circ \tilde{\lambda})$
- xviii.* $\text{Bow}(\text{Bow}(A, X, \tilde{\tau}), X, \tilde{\lambda}) = \text{Bow}(A, X, \tilde{\lambda} \circ \tilde{\tau})$

Proof: Let $A \subseteq X$. The proofs of *i* and *ii* follow from the corresponding definitions.

iii. From Proposition 2.2 *ii*,

$$\begin{aligned} a \in X \setminus \text{Close}(A, X, \tilde{\tau}) &\Rightarrow a \in X \setminus A \\ &\Rightarrow a \in \text{Close}(X \setminus A, X, \tilde{\tau}) \end{aligned}$$

Hence, $X \setminus \text{Close}(A, X, \tilde{\tau}) \subseteq \text{Close}(X \setminus A, X, \tilde{\tau})$.

iv. By the definition of Close , it follows that $\text{Close}(A, X, \tilde{\tau}) \cup \text{Close}(X \setminus A, X, \tilde{\tau}) \subseteq X$. From Proposition 2.2 *ii*,

$$\begin{aligned} a \in X &\Rightarrow a \in A \vee a \in X \setminus A \\ &\Rightarrow a \in \text{Close}(A, X, \tilde{\tau}) \vee a \in \text{Close}(X \setminus A, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Close}(A, X, \tilde{\tau}) \cup \text{Close}(X \setminus A, X, \tilde{\tau}) \end{aligned}$$

Thus, $X \subseteq \text{Close}(A, X, \tilde{\tau}) \cup \text{Close}(X \setminus A, X, \tilde{\tau})$. Hence, $\text{Close}(A, X, \tilde{\tau}) \cup \text{Close}(X \setminus A, X, \tilde{\tau}) = X$.

v. From Proposition 2.2 *iv*,

$$\begin{aligned} a \in X \setminus \text{Touch}(A, X, \tilde{\tau}) &\Rightarrow a \in X \setminus A \\ &\Rightarrow a \in \text{Touch}(X \setminus A, X, \tilde{\tau}) \end{aligned}$$

Hence, $X \setminus \text{Touch}(A, X, \tilde{\tau}) \subseteq \text{Touch}(X \setminus A, X, \tilde{\tau})$.

vi. By the definition of Touch, it follows that $\text{Touch}(A, X, \tilde{\tau}) \cup \text{Touch}(X \setminus A, X, \tilde{\tau}) \subseteq X$. From Proposition 2.2 iv,

$$\begin{aligned} a \in X &\Rightarrow a \in A \vee a \in X \setminus A \\ &\Rightarrow a \in \text{Touch}(A, X, \tilde{\tau}) \vee a \in \text{Touch}(X \setminus A, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Touch}(A, X, \tilde{\tau}) \cup \text{Touch}(X \setminus A, X, \tilde{\tau}) \end{aligned}$$

Thus, $X \subseteq \text{Touch}(A, X, \tilde{\tau}) \cup \text{Touch}(X \setminus A, X, \tilde{\tau})$. Hence, $\text{Touch}(A, X, \tilde{\tau}) \cup \text{Touch}(X \setminus A, X, \tilde{\tau}) = X$.

vii. From Proposition 2.2 iv and xi,

$$X \setminus \text{Int}(A, X, \tilde{\tau}) = X \setminus (A \setminus \text{Touch}(X \setminus A, X, \tilde{\tau})) = (X \setminus A) \cup \text{Touch}(X \setminus A, X, \tilde{\tau}) = \text{Touch}(X \setminus A, X, \tilde{\tau})$$

viii. From Proposition 2.2 xv,

$$\begin{aligned} a \in \text{Fr}(A, X, \tilde{\tau}) &\Rightarrow a \in \text{Touch}(A, X, \tilde{\tau}) \cap \text{Touch}(X \setminus A, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Touch}(A, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Fr}(A, X, \tilde{\tau}) \subseteq \text{Touch}(A, X, \tilde{\tau})$.

ix. From Proposition 2.2 xii and xv,

$$\begin{aligned} \text{Fr}(A, X, \tilde{\tau}) &= \text{Touch}(A, X, \tilde{\tau}) \cap \text{Touch}(X \setminus A, X, \tilde{\tau}) \\ &= \text{Touch}(A, X, \tilde{\tau}) \cap (X \setminus \text{Int}(A, X, \tilde{\tau})) \\ &= \text{Touch}(A, X, \tilde{\tau}) \setminus \text{Int}(A, X, \tilde{\tau}) \end{aligned}$$

x. From Proposition 2.2 xiii and xv,

$$\text{Fr}(A, X, \tilde{\tau}) = \text{Touch}(A, X, \tilde{\tau}) \cap \text{Touch}(X \setminus A, X, \tilde{\tau}) = (X \setminus \text{Int}(A, X, \tilde{\tau})) \cap (X \setminus \text{Int}(X \setminus A, X, \tilde{\tau}))$$

xi. From the definition of Bord,

$$\begin{aligned} a \in \text{Bord}(A, X, \tilde{\tau}) &\Rightarrow a \in \text{Close}(A, X, \tilde{\tau}) \cap \text{Close}(X \setminus A, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Close}(A, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Bord}(A, X, \tilde{\tau}) \subseteq \text{Close}(A, X, \tilde{\tau})$.

xii. From Proposition 2.2 ii,

$$\begin{aligned} \text{Bow}(X \setminus A, X, \tilde{\tau}) &= (X \setminus A) \setminus \text{Bord}(X \setminus A, X, \tilde{\tau}) \\ &= (X \setminus A) \setminus (\text{Close}(A, X, \tilde{\tau}) \cap \text{Close}(X \setminus A, X, \tilde{\tau})) \\ &= [(X \setminus A) \cap (X \setminus \text{Close}(A, X, \tilde{\tau}))] \cup [(X \setminus A) \cap (X \setminus \text{Close}(X \setminus A, X, \tilde{\tau}))] \\ &= (X \setminus \text{Close}(A, X, \tilde{\tau})) \cup \emptyset \\ &= X \setminus \text{Close}(A, X, \tilde{\tau}) \\ &= X \setminus \bigcup_{x \in A} \tilde{\tau}(x) \\ &= \bigcap_{x \in A} (X \setminus \tilde{\tau}(x)) \end{aligned}$$

xiii. From Proposition 2.2 ii,

$$\begin{aligned} X \setminus \text{Bow}(A, X, \tilde{\tau}) &= X \setminus (A \setminus \text{Bord}(A, X, \tilde{\tau})) \\ &= (X \setminus A) \cup \text{Bord}(A, X, \tilde{\tau}) \\ &= (X \setminus A) \cup (\text{Close}(A, X, \tilde{\tau}) \cap \text{Close}(X \setminus A, X, \tilde{\tau})) \\ &= [(X \setminus A) \cup \text{Close}(A, X, \tilde{\tau})] \cap [(X \setminus A) \cup \text{Close}(X \setminus A, X, \tilde{\tau})] \\ &= X \cap \text{Close}(X \setminus A, X, \tilde{\tau}) \\ &= \text{Close}(X \setminus A, X, \tilde{\tau}) \end{aligned}$$

xiv. From the definition of Bow,

$$\text{Bow}(A, X, \tilde{\tau}) \cap \text{Bord}(A, X, \tilde{\tau}) = (A \setminus \text{Bord}(A, X, \tilde{\tau})) \cap \text{Bord}(A, X, \tilde{\tau}) = \emptyset$$

xv. From Proposition 2.2 *ii* and *xxiii*,

$$\begin{aligned} a \in \text{Bord}(A, X, \tilde{\tau}) &\Leftrightarrow a \in \text{Close}(A, X, \tilde{\tau}) \cap \text{Close}(X \setminus A, X, \tilde{\tau}) \\ &\Leftrightarrow a \in \text{Close}(A, X, \tilde{\tau}) \wedge a \in \text{Close}(X \setminus A, X, \tilde{\tau}) \\ &\Leftrightarrow a \in A \cup \text{Close}(A, X, \tilde{\tau}) \wedge a \in (X \setminus A) \cup \text{Close}(X \setminus A, X, \tilde{\tau}) \\ &\Leftrightarrow a \in X \setminus [(X \setminus A) \cap (X \setminus \text{Close}(A, X, \tilde{\tau}))] \wedge a \in X \setminus [A \cap (X \setminus \text{Close}(X \setminus A, X, \tilde{\tau}))] \\ &\Leftrightarrow a \in X \setminus ((X \setminus A) \setminus \text{Close}(A, X, \tilde{\tau})) \wedge a \in X \setminus (A \setminus \text{Close}(X \setminus A, X, \tilde{\tau})) \\ &\Leftrightarrow a \in X \setminus \text{Bow}(X \setminus A, X, \tilde{\tau}) \wedge a \in X \setminus \text{Bow}(A, X, \tilde{\tau}) \\ &\Leftrightarrow a \in (X \setminus \text{Bow}(A, X, \tilde{\tau})) \cap (X \setminus \text{Bow}(X \setminus A, X, \tilde{\tau})) \end{aligned}$$

Hence, $\text{Bord}(A, X, \tilde{\tau}) = (X \setminus \text{Bow}(A, X, \tilde{\tau})) \cap (X \setminus \text{Bow}(X \setminus A, X, \tilde{\tau}))$.

xvi. From Proposition 3.1 *xiii*,

$$\begin{aligned} \text{Bord}(A, X, \tilde{\tau}) &= \text{Close}(A, X, \tilde{\tau}) \cap \text{Close}(X \setminus A, X, \tilde{\tau}) \\ &= \text{Close}(A, X, \tilde{\tau}) \cap (X \setminus \text{Bow}(A, X, \tilde{\tau})) \\ &= \text{Close}(A, X, \tilde{\tau}) \setminus \text{Bow}(A, X, \tilde{\tau}) \end{aligned}$$

xvii. From the definition of Int,

$$\begin{aligned} a \in \text{Int}(\text{Int}(A, X, \tilde{\tau}), X, \tilde{\lambda}) &\Leftrightarrow \tilde{\lambda}(a) \subseteq \text{Int}(A, X, \tilde{\tau}) \\ &\Leftrightarrow \tilde{\tau}(\tilde{\lambda}(a)) \subseteq A \\ &\Leftrightarrow (\tilde{\tau} \circ \tilde{\lambda})(a) \subseteq A \\ &\Leftrightarrow a \in \text{Int}(A, X, \tilde{\tau} \circ \tilde{\lambda}) \end{aligned}$$

Hence, $\text{Int}(\text{Int}(A, X, \tilde{\tau}), X, \tilde{\lambda}) = \text{Int}(A, X, \tilde{\tau} \circ \tilde{\lambda})$.

xviii. From Propositions 2.2 *iii* and 3.1 *xiii*,

$$\begin{aligned} \text{Bow}(\text{Bow}(A, X, \tilde{\tau}), X, \tilde{\lambda}) &= X \setminus \text{Close}(X \setminus \text{Bow}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \\ &= X \setminus \text{Close}(\text{Close}(X \setminus A, X, \tilde{\tau}), X, \tilde{\lambda}) \\ &= X \setminus \text{Close}(X \setminus A, X, \tilde{\lambda} \circ \tilde{\tau}) \\ &= \text{Bow}(A, X, \tilde{\lambda} \circ \tilde{\tau}) \end{aligned}$$

□

Corollary 3.1 *Let $A, B \subseteq X$. Then, the following hold:*

- i.* $A \subseteq B \Rightarrow \text{Int}(\text{Close}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq \text{Int}(\text{Close}(B, X, \tilde{\tau}), X, \tilde{\lambda})$
- ii.* $A \subseteq B \Rightarrow \text{Close}(\text{Int}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq \text{Close}(\text{Int}(B, X, \tilde{\tau}), X, \tilde{\lambda})$
- iii.* $A \subseteq B \Rightarrow \text{Bow}(\text{Touch}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq \text{Bow}(\text{Touch}(B, X, \tilde{\tau}), X, \tilde{\lambda})$
- iv.* $A \subseteq B \Rightarrow \text{Touch}(\text{Bow}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq \text{Touch}(\text{Bow}(B, X, \tilde{\tau}), X, \tilde{\lambda})$
- v.* $\text{Int}(A, X, \tilde{\tau}) \subseteq \text{Int}(\text{Close}(A, X, \tilde{\lambda}), X, \tilde{\tau})$

- vi. $\text{Int}(\text{Int}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq \text{Int}(A, X, \tilde{\lambda})$
- vii. $\text{Bow}(A, X, \tilde{\tau}) \subseteq \text{Bow}(\text{Touch}(A, X, \tilde{\lambda}), X, \tilde{\tau})$
- viii. $\text{Bow}(\text{Bow}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq \text{Bow}(A, X, \tilde{\lambda})$

Proposition 3.2 *Let $A, B \subseteq X$. Then, the following hold:*

- i. $\text{Close}(A, X, \tilde{\tau}) \setminus \text{Close}(B, X, \tilde{\tau}) \subseteq \text{Close}(A \setminus B, X, \tilde{\tau})$
- ii. $\text{Touch}(A, X, \tilde{\tau}) \setminus \text{Touch}(B, X, \tilde{\tau}) \subseteq \text{Touch}(A \setminus B, X, \tilde{\tau})$
- iii. $\text{Int}(A \setminus B, X, \tilde{\tau}) \subseteq \text{Int}(A, X, \tilde{\tau}) \setminus \text{Int}(B, X, \tilde{\tau})$
- iv. $\text{Bow}(A \setminus B, X, \tilde{\tau}) \subseteq \text{Bow}(A, X, \tilde{\tau}) \setminus \text{Bow}(B, X, \tilde{\tau})$

Proof: Let $A, B \subseteq X$.

- i. Since $A \cup B = (A \setminus B) \cup B$, then $\text{Close}(A \cup B, X, \tilde{\tau}) = \text{Close}((A \setminus B) \cup B, X, \tilde{\tau})$. From Proposition 2.1 i,

$$\text{Close}(A, X, \tilde{\tau}) \cup \text{Close}(B, X, \tilde{\tau}) = \text{Close}(A \setminus B, X, \tilde{\tau}) \cup \text{Close}(B, X, \tilde{\tau})$$

Hence, $(\text{Close}(A, X, \tilde{\tau}) \cup \text{Close}(B, X, \tilde{\tau})) \cap (X \setminus \text{Close}(B, X, \tilde{\tau})) = (\text{Close}(A \setminus B, X, \tilde{\tau}) \cup \text{Close}(B, X, \tilde{\tau})) \cap (X \setminus \text{Close}(B, X, \tilde{\tau}))$. Thereby,

$$\text{Close}(A, X, \tilde{\tau}) \cap (X \setminus \text{Close}(B, X, \tilde{\tau})) = \text{Close}(A \setminus B, X, \tilde{\tau}) \cap (X \setminus \text{Close}(B, X, \tilde{\tau})) \subseteq \text{Close}(A \setminus B, X, \tilde{\tau})$$

Therefore,

$$\text{Close}(A, X, \tilde{\tau}) \setminus \text{Close}(B, X, \tilde{\tau}) \subseteq \text{Close}(A \setminus B, X, \tilde{\tau})$$

- ii. The proof is similar to that of i.

iii. From the definition of Int,

$$\begin{aligned} a \in \text{Int}(A \setminus B, X, \tilde{\tau}) &\Rightarrow \tilde{\tau}(a) \subseteq A \setminus B \\ &\Rightarrow \tilde{\tau}(a) \subseteq A \wedge \tilde{\tau}(a) \not\subseteq B \\ &\Rightarrow a \in \text{Int}(A, X, \tilde{\tau}) \wedge a \notin \text{Int}(B, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Int}(A, X, \tilde{\tau}) \setminus \text{Int}(B, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Int}(A \setminus B, X, \tilde{\tau}) \subseteq \text{Int}(A, X, \tilde{\tau}) \setminus \text{Int}(B, X, \tilde{\tau})$.

- iv. From Propositions 2.1 i, 2.2 ii and xxi, and 3.1 xii,

$$\begin{aligned} \text{Bow}(A \setminus B, X, \tilde{\tau}) &= X \setminus \text{Close}(X \setminus (A \setminus B), X, \tilde{\tau}) \\ &= X \setminus \text{Close}((X \setminus A) \cup B, X, \tilde{\tau}) \\ &= X \setminus (\text{Close}(X \setminus A, X, \tilde{\tau}) \cup \text{Close}(B, X, \tilde{\tau})) \\ &= (X \setminus \text{Close}(X \setminus A, X, \tilde{\tau})) \cap (X \setminus \text{Close}(B, X, \tilde{\tau})) \\ &= \text{Bow}(A, X, \tilde{\tau}) \cap (X \setminus \text{Close}(B, X, \tilde{\tau})) \\ &= \text{Bow}(A, X, \tilde{\tau}) \setminus \text{Close}(B, X, \tilde{\tau}) \\ &\subseteq \text{Bow}(A, X, \tilde{\tau}) \setminus \text{Bow}(B, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Bow}(A \setminus B, X, \tilde{\tau}) \subseteq \text{Bow}(A, X, \tilde{\tau}) \setminus \text{Bow}(B, X, \tilde{\tau})$. □

Theorem 3.1 *Let $A, B \subseteq X$. Then, the following hold:*

- i. If $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, and $A \subseteq B$, then $\text{Accu}(A, X, \tilde{\tau}) \subseteq \text{Accu}(B, X, \tilde{\lambda})$.

ii. If $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, then $\text{Isol}(A, X, \tilde{\lambda}) \subseteq \text{Isol}(A, X, \tilde{\tau})$.

iii. If $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, then $\text{Sep}(A, X, \tilde{\lambda}) \subseteq \text{Sep}(A, X, \tilde{\tau})$.

Proof: Let $A, B \subseteq X$.

i. Assume that $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, and $A \subseteq B$. From Theorem 2.1 ii,

$$\begin{aligned} a \in \text{Accu}(A, X, \tilde{\tau}) &\Rightarrow a \in \text{Touch}(A \setminus \{a\}, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Touch}(B \setminus \{a\}, X, \tilde{\lambda}) \\ &\Rightarrow a \in \text{Accu}(B, X, \tilde{\lambda}) \end{aligned}$$

Hence, $\text{Accu}(A, X, \tilde{\tau}) \subseteq \text{Accu}(B, X, \tilde{\lambda})$.

ii. Assume that $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$. From Theorem 3.1 i,

$$\begin{aligned} a \in \text{Isol}(A, X, \tilde{\lambda}) &\Rightarrow a \in A \setminus \text{Accu}(A, X, \tilde{\lambda}) \\ &\Rightarrow a \in A \setminus \text{Accu}(A, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Isol}(A, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Isol}(A, X, \tilde{\lambda}) \subseteq \text{Isol}(A, X, \tilde{\tau})$.

iii. Assume that $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$. From Proposition 2.3 i,

$$\begin{aligned} a \in \text{Sep}(A, X, \tilde{\lambda}) &\Rightarrow a \in A \setminus \text{Crowd}(A, X, \tilde{\lambda}) \\ &\Rightarrow a \in A \setminus \text{Crowd}(A, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Sep}(A, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Sep}(A, X, \tilde{\lambda}) \subseteq \text{Sep}(A, X, \tilde{\tau})$.

□

Proposition 3.3 Let I be an index set and $A_i \subseteq X$, for all $i \in I$. Then, the following hold:

i. $\text{Accu}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) = \bigcup_{i \in I} \text{Accu}(A_i, X, \tilde{\tau})$

ii. $\text{Accu}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \subseteq \bigcap_{i \in I} \text{Accu}(A_i, X, \tilde{\tau})$

iii. $\text{Isol}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) \subseteq \bigcup_{i \in I} \text{Isol}(A_i, X, \tilde{\tau})$

iv. $\bigcap_{i \in I} \text{Isol}(A_i, X, \tilde{\tau}) \subseteq \text{Isol}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right)$

v. $\text{Sep}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) \subseteq \bigcup_{i \in I} \text{Sep}(A_i, X, \tilde{\tau})$

vi. $\bigcap_{i \in I} \text{Sep}(A_i, X, \tilde{\tau}) \subseteq \text{Sep}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right)$

Proof: Let I be an index set and $A_i \subseteq X$, for all $i \in I$.

i. From Proposition 2.1 *iii*,

$$\begin{aligned}
a \in \text{Accu} \left(\bigcup_{i \in I} A_i, X, \tilde{\tau} \right) &\Leftrightarrow a \in \text{Touch} \left(\left(\bigcup_{i \in I} A_i \right) \setminus \{a\}, X, \tilde{\tau} \right) \\
&\Leftrightarrow a \in \text{Touch} \left(\bigcup_{i \in I} (A_i \setminus \{a\}), X, \tilde{\tau} \right) \\
&\Leftrightarrow a \in \bigcup_{i \in I} \text{Touch} (A_i \setminus \{a\}, X, \tilde{\tau}) \\
&\Leftrightarrow \exists i \in I \ni a \in \text{Touch} (A_i \setminus \{a\}, X, \tilde{\tau}) \\
&\Leftrightarrow \exists i \in I \ni a \in \text{Accu} (A_i, X, \tilde{\tau}) \\
&\Leftrightarrow a \in \bigcup_{i \in I} \text{Accu} (A_i, X, \tilde{\tau})
\end{aligned}$$

Hence, $\text{Accu} \left(\bigcup_{i \in I} A_i, X, \tilde{\tau} \right) = \bigcup_{i \in I} \text{Accu} (A_i, X, \tilde{\tau})$.

ii. From Proposition 2.1 *iv*,

$$\begin{aligned}
a \in \text{Accu} \left(\bigcap_{i \in I} A_i, X, \tilde{\tau} \right) &\Rightarrow a \in \text{Touch} \left(\left(\bigcap_{i \in I} A_i \right) \setminus \{a\}, X, \tilde{\tau} \right) \\
&\Rightarrow a \in \text{Touch} \left(\bigcap_{i \in I} (A_i \setminus \{a\}), X, \tilde{\tau} \right) \\
&\Rightarrow a \in \bigcap_{i \in I} \text{Touch} (A_i \setminus \{a\}, X, \tilde{\tau}) \\
&\Rightarrow \forall i \in I, a \in \text{Touch} (A_i \setminus \{a\}, X, \tilde{\tau}) \\
&\Rightarrow \forall i \in I, a \in \text{Accu} (A_i, X, \tilde{\tau}) \\
&\Rightarrow a \in \bigcap_{i \in I} \text{Accu} (A_i, X, \tilde{\tau})
\end{aligned}$$

Hence, $\text{Accu} \left(\bigcap_{i \in I} A_i, X, \tilde{\tau} \right) \subseteq \bigcap_{i \in I} \text{Accu} (A_i, X, \tilde{\tau})$.

iii. From Theorem 2.2,

$$\begin{aligned}
a \in \text{Isol} \left(\bigcup_{i \in I} A_i, X, \tilde{\tau} \right) &\Rightarrow a \in \left(\bigcup_{i \in I} A_i \right) \setminus \text{Accu} \left(\bigcup_{i \in I} A_i, X, \tilde{\tau} \right) \\
&\Rightarrow a \in \bigcup_{i \in I} A_i \wedge a \notin \text{Accu} \left(\bigcup_{i \in I} A_i, X, \tilde{\tau} \right) \\
&\Rightarrow (\exists i \in I \ni a \in A_i) \wedge (\tilde{\tau}(a) \setminus \{a\}) \cap \left(\bigcup_{i \in I} A_i \right) = \emptyset \\
&\Rightarrow (\exists i \in I \ni a \in A_i) \wedge \bigcup_{i \in I} ((\tilde{\tau}(a) \setminus \{a\}) \cap A_i) = \emptyset \\
&\Rightarrow (\exists i \in I \ni a \in A_i) \wedge (\forall k \in I, (\tilde{\tau}(a) \setminus \{a\}) \cap A_k = \emptyset) \\
&\Rightarrow \exists i \in I \ni a \in A_i \wedge (\tilde{\tau}(a) \setminus \{a\}) \cap A_i = \emptyset \\
&\Rightarrow \exists i \in I \ni a \in A_i \wedge a \notin \text{Accu}(A_i, X, \tilde{\tau}) \\
&\Rightarrow \exists i \in I \ni a \in A_i \setminus \text{Accu}(A_i, X, \tilde{\tau}) \\
&\Rightarrow \exists i \in I \ni a \in \text{Isol}(A_i, X, \tilde{\tau}) \\
&\Rightarrow a \in \bigcup_{i \in I} \text{Isol}(A_i, X, \tilde{\tau})
\end{aligned}$$

Hence, $\text{Isol}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) \subseteq \bigcup_{i \in I} \text{Isol}(A_i, X, \tilde{\tau})$.

iv. From Proposition 3.3 *ii*,

$$\begin{aligned}
a \in \bigcap_{i \in I} \text{Isol}(A_i, X, \tilde{\tau}) &\Rightarrow \forall i \in I, a \in \text{Isol}(A_i, X, \tilde{\tau}) \\
&\Rightarrow \forall i \in I, a \in A_i \setminus \text{Accu}(A_i, X, \tilde{\tau}) \\
&\Rightarrow \forall i \in I, a \in A_i \wedge a \notin \text{Accu}(A_i, X, \tilde{\tau}) \\
&\Rightarrow a \in \bigcap_{i \in I} A_i \wedge a \notin \bigcap_{i \in I} \text{Accu}(A_i, X, \tilde{\tau}) \\
&\Rightarrow a \in \bigcap_{i \in I} A_i \wedge a \notin \text{Accu}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \\
&\Rightarrow a \in \left(\bigcap_{i \in I} A_i\right) \setminus \text{Accu}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \\
&\Rightarrow a \in \text{Isol}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right)
\end{aligned}$$

Hence, $\bigcap_{i \in I} \text{Isol}(A_i, X, \tilde{\tau}) \subseteq \text{Isol}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right)$.

v. From Proposition 2.3 *ii*,

$$\begin{aligned}
a \in \text{Sep}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) &\Rightarrow a \in \left(\bigcup_{i \in I} A_i\right) \setminus \text{Crowd}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) \\
&\Rightarrow a \in \bigcup_{i \in I} A_i \wedge a \notin \text{Crowd}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) \\
&\Rightarrow a \in \bigcup_{i \in I} A_i \wedge a \notin \bigcup_{i \in I} \text{Crowd}(A_i, X, \tilde{\tau}) \\
&\Rightarrow (\exists i \in I \ni a \in A_i) \wedge (\forall k \in I, a \notin \text{Crowd}(A_k, X, \tilde{\tau})) \\
&\Rightarrow \exists i \in I \ni a \in A_i \wedge a \notin \text{Crowd}(A_i, X, \tilde{\tau}) \\
&\Rightarrow \exists i \in I \ni a \in A_i \setminus \text{Crowd}(A_i, X, \tilde{\tau}) \\
&\Rightarrow \exists i \in I \ni a \in \text{Sep}(A_i, X, \tilde{\tau}) \\
&\Rightarrow a \in \bigcup_{i \in I} \text{Sep}(A_i, X, \tilde{\tau})
\end{aligned}$$

Hence, $\text{Sep}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) \subseteq \bigcup_{i \in I} \text{Sep}(A_i, X, \tilde{\tau})$.

vi. From Proposition 2.3 *iii*,

$$\begin{aligned}
a \in \bigcap_{i \in I} \text{Sep}(A_i, X, \tilde{\tau}) &\Rightarrow \forall i \in I, a \in \text{Sep}(A_i, X, \tilde{\tau}) \\
&\Rightarrow \forall i \in I, a \in A_i \setminus \text{Crowd}(A_i, X, \tilde{\tau}) \\
&\Rightarrow \forall i \in I, a \in A_i \wedge a \notin \text{Crowd}(A_i, X, \tilde{\tau}) \\
&\Rightarrow a \in \bigcap_{i \in I} A_i \wedge a \notin \bigcap_{i \in I} \text{Crowd}(A_i, X, \tilde{\tau}) \\
&\Rightarrow a \in \bigcap_{i \in I} A_i \wedge a \notin \text{Crowd}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \\
&\Rightarrow a \in \left(\bigcap_{i \in I} A_i\right) \setminus \text{Crowd}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \\
&\Rightarrow a \in \text{Sep}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right)
\end{aligned}$$

Hence, $\bigcap_{i \in I} \text{Sep}(A_i, X, \tilde{\tau}) \subseteq \text{Sep}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right)$.

□

Proposition 3.4 *Let $A, B \subseteq X$. Then, the following hold:*

- i. $\text{Accu}(A, X, \tilde{\tau}) \setminus \text{Accu}(B, X, \tilde{\tau}) \subseteq \text{Accu}(A \setminus B, X, \tilde{\tau})$
- ii. $\text{Crowd}(A, X, \tilde{\tau}) \setminus \text{Crowd}(B, X, \tilde{\tau}) \subseteq \text{Crowd}(A \setminus B, X, \tilde{\tau})$

Proof: Let $A, B \subseteq X$.

- i. From Theorem 2.2,

$$\begin{aligned}
a \in \text{Accu}(A, X, \tilde{\tau}) \setminus \text{Accu}(B, X, \tilde{\tau}) &\Rightarrow a \in \text{Accu}(A, X, \tilde{\tau}) \wedge a \notin \text{Accu}(B, X, \tilde{\tau}) \\
&\Rightarrow (\tilde{\tau}(a) \setminus \{a\}) \cap A \neq \emptyset \wedge (\tilde{\tau}(a) \setminus \{a\}) \cap B = \emptyset \\
&\Rightarrow (\tilde{\tau}(a) \setminus \{a\}) \cap A \neq \emptyset \wedge (\tilde{\tau}(a) \setminus \{a\}) \setminus B = \tilde{\tau}(a) \setminus \{a\} \\
&\Rightarrow (\tilde{\tau}(a) \setminus \{a\}) \cap (A \setminus B) \neq \emptyset \\
&\Rightarrow a \in \text{Accu}(A \setminus B, X, \tilde{\tau})
\end{aligned}$$

Hence, $\text{Accu}(A, X, \tilde{\tau}) \setminus \text{Accu}(B, X, \tilde{\tau}) \subseteq \text{Accu}(A \setminus B, X, \tilde{\tau})$.

ii. Let $a \in \text{Crowd}(A, X, \tilde{\tau}) \setminus \text{Crowd}(B, X, \tilde{\tau})$. Then, $a \in \text{Crowd}(A, X, \tilde{\tau})$ and $a \notin \text{Crowd}(B, X, \tilde{\tau})$. From Theorem 2.3, there exists a point $b_1 \in A$ such that $a \in \tilde{\tau}(b_1) \setminus \{b_1\}$ and for all $b \in B$, $a \notin \tilde{\tau}(b) \setminus \{b\}$. Hence, there exists a point $b_1 \in A \setminus B$ such that $a \in \tilde{\tau}(b_1) \setminus \{b_1\}$. Therefore, $a \in \text{Crowd}(A \setminus B, X, \tilde{\tau})$ from Theorem 2.3.

□

Proposition 3.5 *Let $A \subseteq X$. Then, the following hold:*

- i. $\text{Accu}(\emptyset, X, \tilde{\tau}) = \text{Crowd}(\emptyset, X, \tilde{\tau}) = \emptyset$
- ii. $x \notin \text{Accu}(\{x\}, X, \tilde{\tau})$, for all $x \in X$
- iii. $x \notin \text{Crowd}(\{x\}, X, \tilde{\tau})$, for all $x \in X$
- iv. $\text{Accu}(A, X, \tilde{\tau}) \subseteq \text{Touch}(A, X, \tilde{\tau})$
- v. $\text{Crowd}(A, X, \tilde{\tau}) \subseteq \text{Close}(A, X, \tilde{\tau})$
- vi. $\text{Accu}(A, X, \tilde{\tau}) \cap \text{Accu}(X \setminus A, X, \tilde{\tau}) \subseteq \text{Fr}(A, X, \tilde{\tau})$
- vii. $\text{Crowd}(A, X, \tilde{\tau}) \cap \text{Crowd}(X \setminus A, X, \tilde{\tau}) \subseteq \text{Bord}(A, X, \tilde{\tau})$

viii. $\text{Touch}(A, X, \tilde{\tau}) = A \cup \text{Accu}(A, X, \tilde{\tau})$

ix. $\text{Close}(A, X, \tilde{\tau}) = A \cup \text{Crowd}(A, X, \tilde{\tau})$

Proof: Let $A \subseteq X$. The proof of *i* follows from the corresponding definitions.

ii. Assume that there exists an $x \in X$ such that $x \in \text{Accu}(\{x\}, X, \tilde{\tau})$. Then, $x \in \text{Touch}(\{x\} \setminus \{x\}, X, \tilde{\tau}) = \text{Touch}(\emptyset, X, \tilde{\tau}) = \emptyset$ from Proposition 2.2 v. This is a contradiction. Thus, $x \notin \text{Accu}(\{x\}, X, \tilde{\tau})$, for all $x \in X$.

iii. By Proposition 2.2 i, the proof is similar to that of *ii*.

iv. From Theorem 2.2,

$$\begin{aligned} a \in \text{Accu}(A, X, \tilde{\tau}) &\Rightarrow (\tilde{\tau}(a) \setminus \{a\}) \cap A \neq \emptyset \\ &\Rightarrow \tilde{\tau}(a) \cap A \neq \emptyset \\ &\Rightarrow a \in \text{Touch}(A, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Accu}(A, X, \tilde{\tau}) \subseteq \text{Touch}(A, X, \tilde{\tau})$.

v. Since $\tilde{\tau}(x) \setminus \{x\} \subseteq \tilde{\tau}(x)$, for all $x \in A$, then $\bigcup_{x \in A} (\tilde{\tau}(x) \setminus \{x\}) \subseteq \bigcup_{x \in A} \tilde{\tau}(x)$. Thus, $\text{Crowd}(A, X, \tilde{\tau}) \subseteq \text{Close}(A, X, \tilde{\tau})$.

vi. From Propositions 2.2 xv and 3.5 iv,

$$\begin{aligned} a \in \text{Accu}(A, X, \tilde{\tau}) \cap \text{Accu}(X \setminus A, X, \tilde{\tau}) &\Rightarrow a \in \text{Touch}(A, X, \tilde{\tau}) \cap \text{Touch}(X \setminus A, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Fr}(A, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Accu}(A, X, \tilde{\tau}) \cap \text{Accu}(X \setminus A, X, \tilde{\tau}) \subseteq \text{Fr}(A, X, \tilde{\tau})$.

vii. From Proposition 3.5 v,

$$\begin{aligned} a \in \text{Crowd}(A, X, \tilde{\tau}) \cap \text{Crowd}(X \setminus A, X, \tilde{\tau}) &\Rightarrow a \in \text{Close}(A, X, \tilde{\tau}) \cap \text{Close}(X \setminus A, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Bord}(A, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Crowd}(A, X, \tilde{\tau}) \cap \text{Crowd}(X \setminus A, X, \tilde{\tau}) \subseteq \text{Bord}(A, X, \tilde{\tau})$.

viii. Assume that $a \in \text{Touch}(A, X, \tilde{\tau})$. Then, there are two cases: $a \in A$ or $a \notin A$.

Case 1: Let $a \in A$. Then, $a \in A \cup \text{Accu}(A, X, \tilde{\tau})$. Therefore, $\text{Touch}(A, X, \tilde{\tau}) \subseteq A \cup \text{Accu}(A, X, \tilde{\tau})$.

Case 2: Let $a \notin A$. Since $a \in \text{Touch}(A, X, \tilde{\tau})$, then $\tilde{\tau}(a) \cap A \neq \emptyset$. Therefore, $\tilde{\tau}(a) \cap (A \setminus \{a\}) \neq \emptyset$ because $a \notin A$. Thus, $a \in \text{Touch}(A \setminus \{a\}, X, \tilde{\tau})$. Thereby, $a \in \text{Accu}(A, X, \tilde{\tau})$. Hence, $a \in A \cup \text{Accu}(A, X, \tilde{\tau})$. Consequently, $\text{Touch}(A, X, \tilde{\tau}) \subseteq A \cup \text{Accu}(A, X, \tilde{\tau})$.

Conversely, assume that $a \in A \cup \text{Accu}(A, X, \tilde{\tau})$. Then, $a \in A$ or $a \in \text{Accu}(A, X, \tilde{\tau})$. If $a \in A$, then $a \in \text{Touch}(A, X, \tilde{\tau})$ from Proposition 2.2 iv. Moreover, if $a \in \text{Accu}(A, X, \tilde{\tau})$, then $a \in \text{Touch}(A, X, \tilde{\tau})$ from Proposition 3.5 iv. Hence, $A \cup \text{Accu}(A, X, \tilde{\tau}) \subseteq \text{Touch}(A, X, \tilde{\tau})$. Consequently, $\text{Touch}(A, X, \tilde{\tau}) = A \cup \text{Accu}(A, X, \tilde{\tau})$.

ix. From Corollary 2.1,

$$A \cup \text{Crowd}(A, X, \tilde{\tau}) = A \cup \bigcup_{x \in A} (\tilde{\tau}(x) \setminus \{x\}) = \left(\bigcup_{x \in A} \{x\} \right) \cup \left(\bigcup_{x \in A} (\tilde{\tau}(x) \setminus \{x\}) \right) = \bigcup_{x \in A} \tilde{\tau}(x) = \text{Close}(A, X, \tilde{\tau})$$

□

Corollary 3.2 *Let $A \subseteq X$ and $a \in X$. Then, the following hold:*

i. $\text{Isol}(\emptyset, X, \tilde{\tau}) = \text{Sep}(\emptyset, X, \tilde{\tau}) = \emptyset$

ii. $\text{Isol}(\{a\}, X, \tilde{\tau}) = \{a\}$

- iii. $\text{Sep}(\{a\}, X, \tilde{\tau}) = \{a\}$
- iv. $\text{Isol}(A, X, \tilde{\tau}) \cap \text{Accu}(A, X, \tilde{\tau}) = \emptyset$
- v. $\text{Sep}(A, X, \tilde{\tau}) \cap \text{Crowd}(A, X, \tilde{\tau}) = \emptyset$

Theorem 3.2 *Let $A \subseteq X$ and $a \in A$. Then, the following hold:*

- i. $a \notin \text{Isol}(A, X, \tilde{\tau})$ if and only if $a \in \text{Accu}(A, X, \tilde{\tau})$.
- ii. $a \notin \text{Sep}(A, X, \tilde{\tau})$ if and only if $a \in \text{Crowd}(A, X, \tilde{\tau})$.

Proof: Let $A \subseteq X$ and $a \in A$.

i. From the definition of Isol ,

$$\begin{aligned} a \notin \text{Isol}(A, X, \tilde{\tau}) &\Leftrightarrow a \notin A \setminus \text{Accu}(A, X, \tilde{\tau}) \\ &\Leftrightarrow a \in X \setminus A \vee a \in \text{Accu}(A, X, \tilde{\tau}) \\ &\Leftrightarrow a \in \text{Accu}(A, X, \tilde{\tau}) \end{aligned}$$

ii. The proof is similar to that of *i*.

□

4. $\tilde{\tau}$ -Disjointness, $\tilde{\tau}$ -Exterior, and $\tilde{\tau}$ -Condensation Points

This section introduces the notions of $\tilde{\tau}$ -disjointness, $\tilde{\tau}$ -exterior, and $\tilde{\tau}$ -condensation points of a set and establishes their fundamental properties.

Definition 4.1 *Let $A \subseteq X$. Then, the $\tilde{\tau}$ -disjointness of the set A is defined by*

$$\text{Disj}(A, X, \tilde{\tau}) := \text{Int}(X \setminus A, X, \tilde{\tau})$$

Here, the elements of the set $\text{Disj}(A, X, \tilde{\tau})$ are called $\tilde{\tau}$ -disjointness points of the set A .

Definition 4.2 *Let $A \subseteq X$. Then, the $\tilde{\tau}$ -exterior of the set A is defined by*

$$\text{Ext}(A, X, \tilde{\tau}) := \text{Bow}(X \setminus A, X, \tilde{\tau})$$

Here, the elements of the set $\text{Ext}(A, X, \tilde{\tau})$ are called $\tilde{\tau}$ -exterior points of the set A .

Proposition 4.1 *Let $A \subseteq X$. Then, the following hold:*

- i. $\text{Disj}(X, X, \tilde{\tau}) = \text{Ext}(X, X, \tilde{\tau}) = \emptyset$
- ii. $\text{Disj}(\emptyset, X, \tilde{\tau}) = \text{Ext}(\emptyset, X, \tilde{\tau}) = X$
- iii. $\text{Disj}(A, X, \tilde{\tau}) \subseteq X \setminus A$
- iv. $\text{Ext}(A, X, \tilde{\tau}) \subseteq X \setminus A$

Proof: Let $A \subseteq X$. The proofs of *i* and *ii* follow from the corresponding definitions.

iii. From the relevant definitions,

$$\begin{aligned} a \in \text{Disj}(A, X, \tilde{\tau}) &\Rightarrow a \in \text{Int}(X \setminus A, X, \tilde{\tau}) \\ &\Rightarrow \tilde{\tau}(a) \subseteq X \setminus A \\ &\Rightarrow a \in X \setminus A \end{aligned}$$

Hence, $\text{Disj}(A, X, \tilde{\tau}) \subseteq X \setminus A$.

iv. From Proposition 2.2 *xxiii*,

$$\begin{aligned} a \in \text{Ext}(A, X, \tilde{\tau}) &\Rightarrow a \in \text{Bow}(X \setminus A, X, \tilde{\tau}) \\ &\Rightarrow a \in (X \setminus A) \setminus \text{Close}(A, X, \tilde{\tau}) \\ &\Rightarrow a \in X \setminus A \end{aligned}$$

Hence, $\text{Ext}(A, X, \tilde{\tau}) \subseteq X \setminus A$.

□

Proposition 4.2 *Let I be an index set and $A_i \subseteq X$, for all $i \in I$. Then, the following hold:*

i. $\text{Disj}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) = \bigcap_{i \in I} \text{Disj}(A_i, X, \tilde{\tau})$

ii. $\bigcup_{i \in I} \text{Disj}(A_i, X, \tilde{\tau}) \subseteq \text{Disj}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right)$

iii. $\text{Ext}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) = \bigcap_{i \in I} \text{Ext}(A_i, X, \tilde{\tau})$

iv. $\bigcup_{i \in I} \text{Ext}(A_i, X, \tilde{\tau}) \subseteq \text{Ext}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right)$

Proof: Let I be an index set and $A_i \subseteq X$, for all $i \in I$.

i. From Proposition 2.1 *vi*,

$$\begin{aligned} a \in \text{Disj}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) &\Leftrightarrow a \in \text{Int}\left(X \setminus \bigcup_{i \in I} A_i, X, \tilde{\tau}\right) \\ &\Leftrightarrow a \in \text{Int}\left(\bigcap_{i \in I} (X \setminus A_i), X, \tilde{\tau}\right) \\ &\Leftrightarrow a \in \bigcap_{i \in I} \text{Int}(X \setminus A_i, X, \tilde{\tau}) \\ &\Leftrightarrow \forall i \in I, a \in \text{Int}(X \setminus A_i, X, \tilde{\tau}) \\ &\Leftrightarrow \forall i \in I, a \in \text{Disj}(A_i, X, \tilde{\tau}) \\ &\Leftrightarrow a \in \bigcap_{i \in I} \text{Disj}(A_i, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Disj}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) = \bigcap_{i \in I} \text{Disj}(A_i, X, \tilde{\tau})$.

ii. From Proposition 2.1 *v*,

$$\begin{aligned} a \in \bigcup_{i \in I} \text{Disj}(A_i, X, \tilde{\tau}) &\Rightarrow \exists i \in I \ni a \in \text{Disj}(A_i, X, \tilde{\tau}) \\ &\Rightarrow \exists i \in I \ni a \in \text{Int}(X \setminus A_i, X, \tilde{\tau}) \\ &\Rightarrow a \in \bigcup_{i \in I} \text{Int}(X \setminus A_i, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Int}\left(\bigcup_{i \in I} (X \setminus A_i), X, \tilde{\tau}\right) \\ &\Rightarrow a \in \text{Int}\left(X \setminus \bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \\ &\Rightarrow a \in \text{Disj}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \end{aligned}$$

Hence, $\bigcup_{i \in I} \text{Disj}(A_i, X, \tilde{\tau}) \subseteq \text{Disj}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right)$.

iii. From Proposition 2.1 *xii*,

$$\begin{aligned} a \in \text{Ext}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) &\Leftrightarrow a \in \text{Bow}\left(X \setminus \bigcup_{i \in I} A_i, X, \tilde{\tau}\right) \\ &\Leftrightarrow a \in \text{Bow}\left(\bigcap_{i \in I} (X \setminus A_i), X, \tilde{\tau}\right) \\ &\Leftrightarrow a \in \bigcap_{i \in I} \text{Bow}(X \setminus A_i, X, \tilde{\tau}) \\ &\Leftrightarrow \forall i \in I, a \in \text{Bow}(X \setminus A_i, X, \tilde{\tau}) \\ &\Leftrightarrow \forall i \in I, a \in \text{Ext}(A_i, X, \tilde{\tau}) \\ &\Leftrightarrow a \in \bigcap_{i \in I} \text{Ext}(A_i, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Ext}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) = \bigcap_{i \in I} \text{Ext}(A_i, X, \tilde{\tau})$.

iv. From Proposition 2.1 *xi*,

$$\begin{aligned} a \in \bigcup_{i \in I} \text{Ext}(A_i, X, \tilde{\tau}) &\Rightarrow \exists i \in I \ni a \in \text{Ext}(A_i, X, \tilde{\tau}) \\ &\Rightarrow \exists i \in I \ni a \in \text{Bow}(X \setminus A_i, X, \tilde{\tau}) \\ &\Rightarrow a \in \bigcup_{i \in I} \text{Bow}(X \setminus A_i, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Bow}\left(\bigcup_{i \in I} (X \setminus A_i), X, \tilde{\tau}\right) \\ &\Rightarrow a \in \text{Bow}\left(X \setminus \bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \\ &\Rightarrow a \in \text{Ext}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \end{aligned}$$

Hence, $\bigcup_{i \in I} \text{Ext}(A_i, X, \tilde{\tau}) \subseteq \text{Ext}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right)$.

□

Theorem 4.1 *Let $A, B \subseteq X$. Then, the following hold:*

i. *If $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, and $A \subseteq B$, then $\text{Disj}(B, X, \tilde{\lambda}) \subseteq \text{Disj}(A, X, \tilde{\tau})$.*

ii. *If $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, and $A \subseteq B$, then $\text{Ext}(B, X, \tilde{\lambda}) \subseteq \text{Ext}(A, X, \tilde{\tau})$.*

Proof: Let $A, B \subseteq X$.

i. Assume that $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, and $A \subseteq B$. From Theorem 2.1 *iii*,

$$\begin{aligned} a \in \text{Disj}(B, X, \tilde{\lambda}) &\Rightarrow a \in \text{Int}(X \setminus B, X, \tilde{\lambda}) \\ &\Rightarrow a \in \text{Int}(X \setminus A, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Disj}(A, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Disj}(B, X, \tilde{\lambda}) \subseteq \text{Disj}(A, X, \tilde{\tau})$.

ii. Assume that $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, and $A \subseteq B$. From Theorem 2.1 vi,

$$\begin{aligned} a \in \text{Ext}(B, X, \tilde{\lambda}) &\Rightarrow a \in \text{Bow}(X \setminus B, X, \tilde{\lambda}) \\ &\Rightarrow a \in \text{Bow}(X \setminus A, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Ext}(A, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Ext}(B, X, \tilde{\lambda}) \subseteq \text{Ext}(A, X, \tilde{\tau})$.

□

Proposition 4.3 *Let $A \subseteq X$. Then, the following hold:*

- i. $\text{Disj}(A, X, \tilde{\tau}) \subseteq \text{Touch}(X \setminus A, X, \tilde{\tau})$
- ii. $\text{Disj}(A, X, \tilde{\tau}) \cap \text{Isol}(A, X, \tilde{\tau}) = \emptyset$
- iii. $\text{Isol}(A, X, \tilde{\tau}) = \text{Disj}(A \setminus \{a\}, X, \tilde{\tau})$, for all $a \in A$
- iv. $X \setminus \text{Disj}(A, X, \tilde{\tau}) = \text{Touch}(A, X, \tilde{\tau})$
- v. $\text{Disj}(\text{Touch}(A, X, \tilde{\tau}), X, \tilde{\lambda}) = \text{Disj}(A, X, \tilde{\tau} \circ \tilde{\lambda})$
- vi. $\text{Disj}(\text{Disj}(A, X, \tilde{\tau}), X, \tilde{\lambda}) = \text{Int}(\text{Touch}(A, X, \tilde{\tau}), X, \tilde{\lambda})$
- vii. $\text{Disj}(\text{Disj}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq \text{Touch}(A, X, \tilde{\tau} \circ \tilde{\lambda})$
- viii. $\text{Fr}(A, X, \tilde{\tau}) = X \setminus (\text{Int}(A, X, \tilde{\tau}) \cup \text{Disj}(A, X, \tilde{\tau}))$

Proof: Let $A \subseteq X$.

i. From the relevant definitions,

$$\begin{aligned} a \in \text{Disj}(A, X, \tilde{\tau}) &\Rightarrow a \in \text{Int}(X \setminus A, X, \tilde{\tau}) \\ &\Rightarrow \tilde{\tau}(a) \subseteq X \setminus A \\ &\Rightarrow \tilde{\tau}(a) \cap (X \setminus A) \neq \emptyset \\ &\Rightarrow a \in \text{Touch}(X \setminus A, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Disj}(A, X, \tilde{\tau}) \subseteq \text{Touch}(X \setminus A, X, \tilde{\tau})$.

ii. Since $\text{Disj}(A, X, \tilde{\tau}) \subseteq X \setminus A$ from Proposition 4.1 iii and $\text{Isol}(A, X, \tilde{\tau}) \subseteq A$ from Definition 2.10 iii, then $\text{Disj}(A, X, \tilde{\tau}) \cap \text{Isol}(A, X, \tilde{\tau}) = \emptyset$.

iii. Assume that $a \in A$. From Theorem 2.4,

$$\begin{aligned} a \in \text{Isol}(A, X, \tilde{\tau}) &\Leftrightarrow (\tilde{\tau}(a) \setminus \{a\}) \cap A = \emptyset \\ &\Leftrightarrow \tilde{\tau}(a) \cap (A \setminus \{a\}) = \emptyset \\ &\Leftrightarrow \tilde{\tau}(a) \subseteq X \setminus (A \setminus \{a\}) \\ &\Leftrightarrow a \in \text{Int}(X \setminus (A \setminus \{a\}), X, \tilde{\tau}) \\ &\Leftrightarrow a \in \text{Disj}(A \setminus \{a\}, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Isol}(A, X, \tilde{\tau}) = \text{Disj}(A \setminus \{a\}, X, \tilde{\tau})$, for all $a \in A$.

iv. From Proposition 2.2 xii,

$$\begin{aligned} a \in X \setminus \text{Disj}(A, X, \tilde{\tau}) &\Leftrightarrow a \in X \setminus \text{Int}(X \setminus A, X, \tilde{\tau}) \\ &\Leftrightarrow a \in \text{Touch}(A, X, \tilde{\tau}) \end{aligned}$$

Hence, $X \setminus \text{Disj}(A, X, \tilde{\tau}) = \text{Touch}(A, X, \tilde{\tau})$.

v. From Proposition 2.2 *xii*,

$$\begin{aligned}
a \in \text{Disj}(\text{Touch}(A, X, \tilde{\tau}), X, \tilde{\lambda}) &\Leftrightarrow a \in \text{Int}(X \setminus \text{Touch}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \\
&\Leftrightarrow \tilde{\lambda}(a) \subseteq X \setminus \text{Touch}(A, X, \tilde{\tau}) \\
&\Leftrightarrow \tilde{\lambda}(a) \subseteq \text{Int}(X \setminus A, X, \tilde{\tau}) \\
&\Leftrightarrow \tilde{\tau}(\tilde{\lambda}(a)) \subseteq X \setminus A \\
&\Leftrightarrow a \in \text{Int}(X \setminus A, X, \tilde{\tau} \circ \tilde{\lambda}) \\
&\Leftrightarrow a \in \text{Disj}(A, X, \tilde{\tau} \circ \tilde{\lambda})
\end{aligned}$$

Hence, $\text{Disj}(\text{Touch}(A, X, \tilde{\tau}), X, \tilde{\lambda}) = \text{Disj}(A, X, \tilde{\tau} \circ \tilde{\lambda})$.

vi. From Proposition 4.3 *iv*,

$$\text{Disj}(\text{Disj}(A, X, \tilde{\tau}), X, \tilde{\lambda}) = \text{Disj}(X \setminus \text{Touch}(A, X, \tilde{\tau}), X, \tilde{\lambda}) = \text{Int}(\text{Touch}(A, X, \tilde{\tau}), X, \tilde{\lambda})$$

vii. From Propositions 2.2 *vi* and 4.3 *vi*,

$$\begin{aligned}
a \in \text{Disj}(\text{Disj}(A, X, \tilde{\tau}), X, \tilde{\lambda}) &\Rightarrow a \in \text{Int}(\text{Touch}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \\
&\Rightarrow \tilde{\lambda}(a) \subseteq \text{Touch}(A, X, \tilde{\tau}) \\
&\Rightarrow \tilde{\lambda}(a) \cap \text{Touch}(A, X, \tilde{\tau}) \neq \emptyset \\
&\Rightarrow a \in \text{Touch}(\text{Touch}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \\
&\Rightarrow a \in \text{Touch}(A, X, \tilde{\tau} \circ \tilde{\lambda})
\end{aligned}$$

Hence, $\text{Disj}(\text{Disj}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq \text{Touch}(A, X, \tilde{\tau} \circ \tilde{\lambda})$.

viii. From Proposition 3.1 *x*,

$$\begin{aligned}
\text{Fr}(A, X, \tilde{\tau}) &= (X \setminus \text{Int}(A, X, \tilde{\tau})) \cap (X \setminus \text{Int}(X \setminus A, X, \tilde{\tau})) \\
&= X \setminus (\text{Int}(A, X, \tilde{\tau}) \cup \text{Int}(X \setminus A, X, \tilde{\tau})) \\
&= X \setminus (\text{Int}(A, X, \tilde{\tau}) \cup \text{Disj}(A, X, \tilde{\tau}))
\end{aligned}$$

□

Proposition 4.4 *Let $A \subseteq X$. Then, the following hold:*

- i. $\text{Ext}(A, X, \tilde{\tau}) \subseteq \text{Close}(X \setminus A, X, \tilde{\tau})$
- ii. $\text{Ext}(A, X, \tilde{\tau}) \cap \text{Sep}(A, X, \tilde{\tau}) = \emptyset$
- iii. $\text{Sep}(A, X, \tilde{\tau}) = \text{Ext}(A \setminus \{a\}, X, \tilde{\tau})$, for all $a \in A$
- iv. $X \setminus \text{Ext}(A, X, \tilde{\tau}) = \text{Close}(A, X, \tilde{\tau})$
- v. $\text{Ext}(\text{Close}(A, X, \tilde{\tau}), X, \tilde{\lambda}) = \text{Ext}(A, X, \tilde{\lambda} \circ \tilde{\tau})$
- vi. $\text{Ext}(\text{Ext}(A, X, \tilde{\tau}), X, \tilde{\lambda}) = \text{Bow}(\text{Close}(A, X, \tilde{\tau}), X, \tilde{\lambda})$
- vii. $\text{Ext}(\text{Ext}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq \text{Close}(A, X, \tilde{\lambda} \circ \tilde{\tau})$
- viii. $\text{Bord}(A, X, \tilde{\tau}) = X \setminus (\text{Bow}(A, X, \tilde{\tau}) \cup \text{Ext}(A, X, \tilde{\tau}))$

Proof: Let $A \subseteq X$.

i. From Proposition 3.1 *iii* and *xii*,

$$\text{Ext}(A, X, \tilde{\tau}) = \text{Bow}(X \setminus A, X, \tilde{\tau}) = X \setminus \text{Close}(A, X, \tilde{\tau}) \subseteq \text{Close}(X \setminus A, X, \tilde{\tau})$$

ii. Since $\text{Ext}(A, X, \tilde{\tau}) \subseteq X \setminus A$ from Proposition 4.1 iv and $\text{Sep}(A, X, \tilde{\tau}) \subseteq A$ from Definition 2.10 iv, then $\text{Ext}(A, X, \tilde{\tau}) \cap \text{Sep}(A, X, \tilde{\tau}) = \emptyset$.

iii. Assume that $a \in A$. From Proposition 2.2 xxiii,

$$\begin{aligned} a \in \text{Sep}(A, X, \tilde{\tau}) &\Leftrightarrow a \in A \setminus \text{Crowd}(A, X, \tilde{\tau}) \\ &\Leftrightarrow a \in A \wedge a \notin \text{Crowd}(A, X, \tilde{\tau}) \\ &\Leftrightarrow a \notin A \setminus \{a\} \wedge a \notin \text{Close}(A \setminus \{a\}, X, \tilde{\tau}) \\ &\Leftrightarrow a \in X \setminus (A \setminus \{a\}) \wedge a \notin \text{Close}(A \setminus \{a\}, X, \tilde{\tau}) \\ &\Leftrightarrow a \in (X \setminus (A \setminus \{a\})) \setminus \text{Close}(A \setminus \{a\}, X, \tilde{\tau}) \\ &\Leftrightarrow a \in \text{Bow}(X \setminus (A \setminus \{a\}), X, \tilde{\tau}) \\ &\Leftrightarrow a \in \text{Ext}(A \setminus \{a\}, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Sep}(A, X, \tilde{\tau}) = \text{Ext}(A \setminus \{a\}, X, \tilde{\tau})$, for all $a \in A$.

iv. From Proposition 3.1 xii,

$$\begin{aligned} a \in X \setminus \text{Ext}(A, X, \tilde{\tau}) &\Leftrightarrow a \in X \setminus \text{Bow}(X \setminus A, X, \tilde{\tau}) \\ &\Leftrightarrow a \in \text{Close}(A, X, \tilde{\tau}) \end{aligned}$$

Hence, $X \setminus \text{Ext}(A, X, \tilde{\tau}) = \text{Close}(A, X, \tilde{\tau})$.

v. From Propositions 2.2 iii and 4.4 iv,

$$\text{Ext}(\text{Close}(A, X, \tilde{\tau}), X, \tilde{\lambda}) = X \setminus \text{Close}(\text{Close}(A, X, \tilde{\tau}), X, \tilde{\lambda}) = X \setminus \text{Close}(A, X, \tilde{\lambda} \circ \tilde{\tau}) = \text{Ext}(A, X, \tilde{\lambda} \circ \tilde{\tau})$$

vi. From Proposition 4.4 iv,

$$\text{Ext}(\text{Ext}(A, X, \tilde{\tau}), X, \tilde{\lambda}) = \text{Ext}(X \setminus \text{Close}(A, X, \tilde{\tau}), X, \tilde{\lambda}) = \text{Bow}(\text{Close}(A, X, \tilde{\tau}), X, \tilde{\lambda})$$

vii. From Propositions 2.2 iii, 3.1 iii and xii, and 4.4 vi,

$$\begin{aligned} a \in \text{Ext}(\text{Ext}(A, X, \tilde{\tau}), X, \tilde{\lambda}) &\Rightarrow a \in \text{Bow}(\text{Close}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \\ &\Rightarrow a \in X \setminus \text{Close}(X \setminus \text{Close}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \\ &\Rightarrow a \in \text{Close}(\text{Close}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \\ &\Rightarrow a \in \text{Close}(A, X, \tilde{\lambda} \circ \tilde{\tau}) \end{aligned}$$

Hence, $\text{Ext}(\text{Ext}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq \text{Close}(A, X, \tilde{\lambda} \circ \tilde{\tau})$.

viii. From Proposition 3.1 xv,

$$\begin{aligned} \text{Bord}(A, X, \tilde{\tau}) &= (X \setminus \text{Bow}(A, X, \tilde{\tau})) \cap (X \setminus \text{Bow}(X \setminus A, X, \tilde{\tau})) \\ &= X \setminus (\text{Bow}(A, X, \tilde{\tau}) \cup \text{Bow}(X \setminus A, X, \tilde{\tau})) \\ &= X \setminus (\text{Bow}(A, X, \tilde{\tau}) \cup \text{Ext}(A, X, \tilde{\tau})) \end{aligned}$$

□

Definition 4.3 Let $x \in X$ and $A \subseteq X$. If $\tilde{\tau}(x) \cap A$ is an uncountable set, then the point x is called a $\tilde{\tau}$ -condensation point of the set A . The set of all $\tilde{\tau}$ -condensation points of the set A is denoted by $\text{Cond}(A, X, \tilde{\tau})$. In other words,

$$\text{Cond}(A, X, \tilde{\tau}) := \{x \in X : \tilde{\tau}(x) \cap A \text{ is an uncountable set}\}$$

Proposition 4.5 *Let $A \subseteq X$. Then, the following hold:*

- i.* $\text{Cond}(A, X, \tilde{\tau}) \subseteq \text{Accu}(A, X, \tilde{\tau})$
- ii.* $\text{Cond}(A, X, \tilde{\tau}) \subseteq \text{Touch}(A, X, \tilde{\tau})$
- iii.* $\text{Cond}(A, X, \tilde{\tau}) \subseteq X \setminus \text{Isol}(A, X, \tilde{\tau})$
- iv.* $\text{Cond}(\text{Cond}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq \text{Cond}(A, X, \tilde{\tau} \circ \tilde{\lambda})$
- v.* *If the set A is a countable set, then $\text{Cond}(A, X, \tilde{\tau}) = \emptyset$*

Proof: Let $A \subseteq X$.

i. From Theorem 2.2,

$$\begin{aligned} a \in \text{Cond}(A, X, \tilde{\tau}) &\Rightarrow \tilde{\tau}(a) \cap A \text{ is an uncountable set} \\ &\Rightarrow \tilde{\tau}(a) \cap A \neq \emptyset \\ &\Rightarrow (\tilde{\tau}(a) \setminus \{a\}) \cap A \neq \emptyset \\ &\Rightarrow a \in \text{Accu}(A, X, \tilde{\tau}) \end{aligned}$$

Hence, $\text{Cond}(A, X, \tilde{\tau}) \subseteq \text{Accu}(A, X, \tilde{\tau})$.

ii. From Propositions 3.5 *iv* and 4.5 *i*, $\text{Cond}(A, X, \tilde{\tau}) \subseteq \text{Touch}(A, X, \tilde{\tau})$.

iii. Since $X \setminus \text{Isol}(A, X, \tilde{\tau}) = (X \setminus A) \cup \text{Accu}(A, X, \tilde{\tau})$, then $\text{Cond}(A, X, \tilde{\tau}) \subseteq X \setminus \text{Isol}(A, X, \tilde{\tau})$ from Proposition 4.5 *i*.

iv. From the definition of Cond ,

$$\begin{aligned} a \in \text{Cond}(\text{Cond}(A, X, \tilde{\tau}), X, \tilde{\lambda}) &\Rightarrow \tilde{\lambda}(a) \cap \text{Cond}(A, X, \tilde{\tau}) \text{ is an uncountable set} \\ &\Rightarrow \exists x \in X \exists x \in \tilde{\lambda}(a) \wedge x \in \text{Cond}(A, X, \tilde{\tau}) \\ &\Rightarrow \exists x \in X \exists x \in \tilde{\lambda}(a) \wedge \tilde{\tau}(x) \cap A \text{ is an uncountable set} \\ &\Rightarrow \tilde{\tau}(\tilde{\lambda}(a)) \cap A \text{ is an uncountable set} \\ &\Rightarrow (\tilde{\tau} \circ \tilde{\lambda})(a) \cap A \text{ is an uncountable set} \\ &\Rightarrow a \in \text{Cond}(A, X, \tilde{\tau} \circ \tilde{\lambda}) \end{aligned}$$

Hence, $\text{Cond}(\text{Cond}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq \text{Cond}(A, X, \tilde{\tau} \circ \tilde{\lambda})$.

v. Assume that the set A is a countable set. Since $\tilde{\tau}(a) \cap A \subseteq A$, for all $a \in X$, then $\tilde{\tau}(a) \cap A$ is a countable set. Therefore, $\text{Cond}(A, X, \tilde{\tau}) = \emptyset$.

□

Proposition 4.6 *Let I be an index set and $A_i \subseteq X$, for all $i \in I$. Then, the following hold:*

- i.* $\bigcup_{i \in I} \text{Cond}(A_i, X, \tilde{\tau}) \subseteq \text{Cond}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right)$
- ii.* $\text{Cond}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \subseteq \bigcap_{i \in I} \text{Cond}(A_i, X, \tilde{\tau})$

Proof: Let I be an index set and $A_i \subseteq X$, for all $i \in I$.

i. From the definition of Cond,

$$\begin{aligned}
a \in \bigcup_{i \in I} \text{Cond}(A_i, X, \tilde{\tau}) &\Rightarrow \exists i \in I \ni a \in \text{Cond}(A_i, X, \tilde{\tau}) \\
&\Rightarrow \exists i \in I \ni \tilde{\tau}(a) \cap A_i \text{ is an uncountable set} \\
&\Rightarrow \tilde{\tau}(a) \cap \left(\bigcup_{i \in I} A_i \right) \text{ is an uncountable set} \\
&\Rightarrow a \in \text{Cond}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right)
\end{aligned}$$

Hence, $\bigcup_{i \in I} \text{Cond}(A_i, X, \tilde{\tau}) \subseteq \text{Cond}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right)$.

ii. From the definition of Cond,

$$\begin{aligned}
a \in \text{Cond}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) &\Rightarrow \tilde{\tau}(a) \cap \left(\bigcap_{i \in I} A_i\right) \text{ is an uncountable set} \\
&\Rightarrow \bigcap_{i \in I} (\tilde{\tau}(a) \cap A_i) \text{ is an uncountable set} \\
&\Rightarrow \forall i \in I, \tilde{\tau}(a) \cap A_i \text{ is an uncountable set} \\
&\Rightarrow \forall i \in I, a \in \text{Cond}(A_i, X, \tilde{\tau}) \\
&\Rightarrow a \in \bigcap_{i \in I} \text{Cond}(A_i, X, \tilde{\tau})
\end{aligned}$$

Hence, $\text{Cond}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \subseteq \bigcap_{i \in I} \text{Cond}(A_i, X, \tilde{\tau})$.

□

Theorem 4.2 *Let $A, B \subseteq X$. If $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, and $A \subseteq B$, then $\text{Cond}(A, X, \tilde{\tau}) \subseteq \text{Cond}(B, X, \tilde{\lambda})$.*

Proof: Let $A, B \subseteq X$, $\tilde{\tau}(x) \subseteq \tilde{\lambda}(x)$, for all $x \in X$, and $A \subseteq B$. From the definition of Cond,

$$\begin{aligned}
a \in \text{Cond}(A, X, \tilde{\tau}) &\Rightarrow \tilde{\tau}(a) \cap A \text{ is an uncountable set} \\
&\Rightarrow \tilde{\lambda}(a) \cap B \text{ is an uncountable set} \\
&\Rightarrow a \in \text{Cond}(B, X, \tilde{\lambda})
\end{aligned}$$

Hence, $\text{Cond}(A, X, \tilde{\tau}) \subseteq \text{Cond}(B, X, \tilde{\lambda})$.

□

5. Soft Topologies

This section defines soft closed sets, soft open sets, and soft topologies via Kuratowski's approach [8] and establishes their fundamental properties.

Note 2 [6] *The operators Close and Touch satisfy Kuratowski's axioms except for $\text{cl}(\text{cl}(A)) = \text{cl}(A)$, for all $A \subseteq X$, where cl denotes the closure operator in a classical topology (X, τ) . Similarly, the operators Int and Bow hold the properties $\text{int}(X) = X$, $\text{int}(A) \subseteq A$, and $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$, for all $A, B \subseteq X$, where int indicates the interior operator in a classical topology (X, τ) .*

5.1. $\tilde{\tau}$ - C - and $\tilde{\tau}$ - B -Soft Topologies

This subsection first introduces $\tilde{\tau}$ - C -closed and $\tilde{\tau}$ - B -open sets and derives their key properties.

Definition 5.1 Let $A \subseteq X$. If $\text{Close}(A, X, \tilde{\tau}) = A$, then the set A is called a $\tilde{\tau}$ - C -closed set in X (briefly a $\tilde{\tau}$ - C -closed). If the set $X \setminus A$ is $\tilde{\tau}$ - C -closed, then the set A is called a $\tilde{\tau}$ - B -open set in X (briefly a $\tilde{\tau}$ - B -open). The set of all $\tilde{\tau}$ - C -closed sets is denoted by $C(X, \tilde{\tau})$. Similarly, the set of all $\tilde{\tau}$ - B -open sets is denoted by $B(X, \tilde{\tau})$.

Definition 5.2 Let $A \subseteq X$. If $A \in C(X, \tilde{\tau})$ and $A \in B(X, \tilde{\tau})$, then the set A is called a $\tilde{\tau}$ - CB -closed-open set in X (briefly a $\tilde{\tau}$ - CB -clopen). Moreover, if $A \notin C(X, \tilde{\tau})$ and $A \notin B(X, \tilde{\tau})$, then the set A is called a $\tilde{\tau}$ - CB -neither-closed-nor-open set in X (briefly a $\tilde{\tau}$ - CB -nclnopen).

Remark 5.1 Let $A \subseteq X$. Then, it can be observed that the set A is $\tilde{\tau}$ - C -closed if and only if the set $X \setminus A$ is $\tilde{\tau}$ - B -open.

Corollary 5.1 X and \emptyset are $\tilde{\tau}$ - CB -clopen.

Theorem 5.1 Let $A \subseteq X$. Then, the set A is $\tilde{\tau}$ - C -closed if and only if $\text{Bow}(X \setminus A, X, \tilde{\tau}) = X \setminus A$.

Proof: Let $A \subseteq X$. From Proposition 3.1 *xii*,

$$\begin{aligned} A \text{ is } \tilde{\tau}\text{-}C\text{-closed} &\Leftrightarrow \text{Close}(A, X, \tilde{\tau}) = A \\ &\Leftrightarrow X \setminus \text{Close}(A, X, \tilde{\tau}) = X \setminus A \\ &\Leftrightarrow \text{Bow}(X \setminus A, X, \tilde{\tau}) = X \setminus A \end{aligned}$$

□

From Remark 5.1 and Theorem 5.1, the following corollary is observed.

Corollary 5.2 Let $A \subseteq X$. Then, the set A is $\tilde{\tau}$ - B -open if and only if $\text{Bow}(A, X, \tilde{\tau}) = A$.

Theorem 5.2 Let $A \subseteq X$. Then, the following hold:

- i.* If the set A is $\tilde{\tau}$ - B -open, then $A \subseteq \text{Bow}(\text{Close}(A, X, \tilde{\lambda}), X, \tilde{\tau})$.
- ii.* If the set A is $\tilde{\tau}$ - B -open, then $A \subseteq \text{Close}(\text{Bow}(A, X, \tilde{\tau}), X, \tilde{\lambda})$.
- iii.* If the set A is $\tilde{\tau}$ - C -closed, then $\text{Close}(\text{Bow}(A, X, \tilde{\lambda}), X, \tilde{\tau}) \subseteq A$.
- iv.* If the set A is $\tilde{\tau}$ - C -closed, then $\text{Bow}(\text{Close}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq A$.
- v.* If the set A is $\tilde{\tau}$ - C -closed, then $\text{Close}(\text{Close}(A, X, \tilde{\tau}), X, \tilde{\tau}) = A$.
- vi.* If the set A is $\tilde{\tau}$ - B -open, then $\text{Bow}(\text{Bow}(A, X, \tilde{\tau}), X, \tilde{\tau}) = A$.

Proof: Let $A \subseteq X$.

- i.* Assume that the set A is $\tilde{\tau}$ - B -open. From Theorem 2.1 *vi* and Proposition 2.2 *ii*,

$$\text{Bow}(A, X, \tilde{\tau}) \subseteq \text{Bow}(\text{Close}(A, X, \tilde{\lambda}), X, \tilde{\tau})$$

Hence, $A \subseteq \text{Bow}(\text{Close}(A, X, \tilde{\lambda}), X, \tilde{\tau})$ from Corollary 5.2.

- ii.* Assume that the set A is $\tilde{\tau}$ - B -open. From Corollary 5.2, $A \subseteq \text{Bow}(A, X, \tilde{\tau})$. From Theorem 2.1 *i*,

$$\text{Close}(A, X, \tilde{\lambda}) \subseteq \text{Close}(\text{Bow}(A, X, \tilde{\tau}), X, \tilde{\lambda})$$

Hence, $A \subseteq \text{Close}(\text{Bow}(A, X, \tilde{\tau}), X, \tilde{\lambda})$ from Proposition 2.2 *ii*.

iii. Assume that the set A is $\tilde{\tau}$ - C -closed. From Theorem 2.1 *i* and Proposition 2.2 *xxi*,

$$\text{Close}(\text{Bow}(A, X, \tilde{\lambda}), X, \tilde{\tau}) \subseteq \text{Close}(A, X, \tilde{\tau})$$

Hence, $\text{Close}(\text{Bow}(A, X, \tilde{\lambda}), X, \tilde{\tau}) \subseteq A$ from Definition 5.1.

iv. Assume that the set A is $\tilde{\tau}$ - C -closed. Then, $\text{Close}(A, X, \tilde{\tau}) \subseteq A$. From Theorem 2.1 *vi*,

$$\text{Bow}(\text{Close}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq \text{Bow}(A, X, \tilde{\lambda})$$

Hence, $\text{Bow}(\text{Close}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq A$ from Proposition 2.2 *xxi*.

v. Assume that the set A is $\tilde{\tau}$ - C -closed. Then, $\text{Close}(A, X, \tilde{\tau}) = A$. Thus,

$$\text{Close}(\text{Close}(A, X, \tilde{\tau}), X, \tilde{\tau}) = \text{Close}(A, X, \tilde{\tau}) = A$$

vi. Assume that the set A is $\tilde{\tau}$ - B -open. From Corollary 5.2, $\text{Bow}(A, X, \tilde{\tau}) = A$. Thus,

$$\text{Bow}(\text{Bow}(A, X, \tilde{\tau}), X, \tilde{\tau}) = \text{Bow}(A, X, \tilde{\tau}) = A$$

□

Corollary 5.3 *Let $A \subseteq X$. If the set A is $\tilde{\tau}$ - C -closed, then the set $\text{Close}(A, X, \tilde{\tau})$ is $\tilde{\tau}$ - C -closed. Similarly, if the set A is $\tilde{\tau}$ - B -open, then the set $\text{Bow}(A, X, \tilde{\tau})$ is $\tilde{\tau}$ - B -open.*

Corollary 5.4 *Let $A \subseteq X$. If the set A is $\tilde{\tau}$ - CB -clopen, then $\text{Bow}(\text{Close}(A, X, \tilde{\tau}), X, \tilde{\tau}) = A$ and $\text{Close}(\text{Bow}(A, X, \tilde{\tau}), X, \tilde{\tau}) = A$.*

Theorem 5.3 *Let $A, B \subseteq X$. Then, the following hold:*

- i. If the set A is $\tilde{\tau}$ - C -closed and $B \subseteq A$, then $\text{Close}(B, X, \tilde{\tau}) \subseteq A$.*
- ii. If the set B is $\tilde{\tau}$ - B -open and $B \subseteq A$, then $B \subseteq \text{Bow}(A, X, \tilde{\tau})$.*
- iii. If $A \cap \text{Close}(B, X, \tilde{\tau}) = \emptyset$ and the set $A \cup B$ is $\tilde{\tau}$ - C -closed, then the set B is $\tilde{\tau}$ - C -closed.*
- iv. If $A \cap \text{Close}(B, X, \tilde{\tau}) = \emptyset$ and the set $A \cup B$ is $\tilde{\tau}$ - B -open, then the set A is $\tilde{\tau}$ - B -open.*
- v. If the set A is $\tilde{\tau}$ - B -open, then $A \cap \text{Close}(B, X, \tilde{\tau}) \subseteq \text{Close}(A \cap B, X, \tilde{\tau})$.*

Proof: Let $A, B \subseteq X$.

i. Assume that the set A is $\tilde{\tau}$ - C -closed and $B \subseteq A$. Then, $\text{Close}(B, X, \tilde{\tau}) \subseteq \text{Close}(A, X, \tilde{\tau})$ from Theorem 2.1 *i*. Hence, $\text{Close}(B, X, \tilde{\tau}) \subseteq A$.

ii. Assume that the set B is $\tilde{\tau}$ - B -open and $B \subseteq A$. Then, $\text{Bow}(B, X, \tilde{\tau}) \subseteq \text{Bow}(A, X, \tilde{\tau})$ from Theorem 2.1 *vi*. Hence, $B \subseteq \text{Bow}(A, X, \tilde{\tau})$ from Corollary 5.2.

iii. Assume that $A \cap \text{Close}(B, X, \tilde{\tau}) = \emptyset$ and the set $A \cup B$ is $\tilde{\tau}$ - C -closed. From Theorem 2.1 *i*,

$$\begin{aligned} a \in \text{Close}(B, X, \tilde{\tau}) &\Rightarrow a \in \text{Close}(A \cup B, X, \tilde{\tau}) \wedge a \notin A \\ &\Rightarrow a \in A \cup B \wedge a \notin A \\ &\Rightarrow a \in B \end{aligned}$$

Thus, $\text{Close}(B, X, \tilde{\tau}) \subseteq B$. Hence, the set B is $\tilde{\tau}$ - C -closed from Proposition 2.2 *ii*.

iv. Assume that $A \cap \text{Close}(B, X, \tilde{\tau}) = \emptyset$ and the set $A \cup B$ is $\tilde{\tau}$ - B -open. From Theorem 2.1 *vi*, Propositions 2.1 *xii* and 3.1 *xii*, and Corollary 5.2,

$$\begin{aligned} a \in A &\Rightarrow a \in A \cup B \wedge a \notin \text{Close}(B, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Bow}(A \cup B, X, \tilde{\tau}) \wedge a \in X \setminus \text{Close}(B, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Bow}(A \cup B, X, \tilde{\tau}) \wedge a \in \text{Bow}(X \setminus B, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Bow}((A \cup B) \cap (X \setminus B), X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Bow}(A \setminus B, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Bow}(A, X, \tilde{\tau}) \end{aligned}$$

Thus, $A \subseteq \text{Bow}(A, X, \tilde{\tau})$. Hence, the set A is $\tilde{\tau}$ - B -open from Proposition 2.2 *xxi* and Corollary 5.2.

v. Assume that the set A is $\tilde{\tau}$ - B -open. From Propositions 3.1 *xiii* and 3.2 *i* and Corollary 5.2,

$$\begin{aligned} a \in A \cap \text{Close}(B, X, \tilde{\tau}) &\Rightarrow a \in A \wedge a \in \text{Close}(B, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Bow}(A, X, \tilde{\tau}) \wedge a \in \text{Close}(B, X, \tilde{\tau}) \\ &\Rightarrow a \notin X \setminus \text{Bow}(A, X, \tilde{\tau}) \wedge a \in \text{Close}(B, X, \tilde{\tau}) \\ &\Rightarrow a \notin \text{Close}(X \setminus A, X, \tilde{\tau}) \wedge a \in \text{Close}(B, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Close}(B, X, \tilde{\tau}) \setminus \text{Close}(X \setminus A, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Close}(B \setminus (X \setminus A), X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Close}(A \cap B, X, \tilde{\tau}) \end{aligned}$$

Hence, $A \cap \text{Close}(B, X, \tilde{\tau}) \subseteq \text{Close}(A \cap B, X, \tilde{\tau})$.

□

Theorem 5.4 *Let $A \subseteq X$. Then, the following hold:*

- i.* The set A is $\tilde{\tau}$ - C -closed if and only if $\text{Bord}(A, X, \tilde{\tau}) \subseteq A$.
- ii.* The set A is $\tilde{\tau}$ - B -open if and only if $\text{Bord}(A, X, \tilde{\tau}) \cap A = \emptyset$.
- iii.* The set A is $\tilde{\tau}$ - C -closed if and only if $\text{Ext}(A, X, \tilde{\tau}) = X \setminus A$.
- iv.* The set A is $\tilde{\tau}$ - C -closed if and only if $\text{Crowd}(A, X, \tilde{\tau}) \subseteq A$.

Proof: Let $A \subseteq X$.

i. (\Rightarrow): Assume that the set A is $\tilde{\tau}$ - C -closed. Then,

$$\text{Bord}(A, X, \tilde{\tau}) = \text{Close}(A, X, \tilde{\tau}) \cap \text{Close}(X \setminus A, X, \tilde{\tau}) = A \cap \text{Close}(X \setminus A, X, \tilde{\tau}) \subseteq A$$

(\Leftarrow): Assume that $\text{Bord}(A, X, \tilde{\tau}) \subseteq A$. Then, $\text{Close}(A, X, \tilde{\tau}) \subseteq \text{Bow}(A, X, \tilde{\tau}) \cup A$ from Proposition 2.2 *xxvi*. Thus, $\text{Close}(A, X, \tilde{\tau}) \subseteq A$ from Proposition 2.2 *xxi*. Therefore, $\text{Close}(A, X, \tilde{\tau}) = A$ from Proposition 2.2 *ii*. Hence, the set A is $\tilde{\tau}$ - C -closed.

ii. (\Rightarrow): Assume that the set A is $\tilde{\tau}$ - B -open. From Proposition 3.1 *xiv* and Corollary 5.2,

$$\text{Bord}(A, X, \tilde{\tau}) \cap A = \text{Bord}(A, X, \tilde{\tau}) \cap \text{Bow}(A, X, \tilde{\tau}) = \emptyset$$

(\Leftarrow): Assume that $\text{Bord}(A, X, \tilde{\tau}) \cap A = \emptyset$. Then, $\text{Bow}(A, X, \tilde{\tau}) = A \setminus \text{Bord}(A, X, \tilde{\tau}) = A$. Hence, the set A is $\tilde{\tau}$ - B -open from Corollary 5.2.

iii. (\Rightarrow): Assume that the set A is $\tilde{\tau}$ - C -closed. Then, $\text{Close}(A, X, \tilde{\tau}) = A$. From Proposition 3.1 *xii*,

$$\text{Ext}(A, X, \tilde{\tau}) = \text{Bow}(X \setminus A, X, \tilde{\tau}) = X \setminus \text{Close}(A, X, \tilde{\tau}) = X \setminus A$$

(\Leftarrow): Assume that $\text{Ext}(A, X, \tilde{\tau}) = X \setminus A$. Then, $\text{Bow}(X \setminus A, X, \tilde{\tau}) = X \setminus A$. Thus, the set A is $\tilde{\tau}$ - C -closed from Theorem 5.1.

iv. (\Rightarrow): Assume that the set A is $\tilde{\tau}$ - C -closed. Thus, $\text{Close}(A, X, \tilde{\tau}) = A$. Then, $\text{Crowd}(A, X, \tilde{\tau}) \subseteq A$ from Proposition 3.5 v.

(\Leftarrow): Assume that $\text{Crowd}(A, X, \tilde{\tau}) \subseteq A$. Then, $\text{Close}(A, X, \tilde{\tau}) = A \cup \text{Crowd}(A, X, \tilde{\tau}) = A$ from Proposition 3.5 ix. Thus, the set A is $\tilde{\tau}$ - C -closed. □

Corollary 5.5 *Let $A \subseteq X$. Then, the set A is $\tilde{\tau}$ - CB -clopen if and only if $\text{Bord}(A, X, \tilde{\tau}) = \emptyset$.*

Afterward, this subsection proposes $\tilde{\tau}$ - C - and $\tilde{\tau}$ - B -soft topologies by $\tilde{\tau}$ - C -closed and $\tilde{\tau}$ - B -open sets, respectively.

Definition 5.3 *Let $X \neq \emptyset$ and $\tilde{\tau} : X \rightarrow P(X)$ be a proximity mapping. Then, the set $C(X, \tilde{\tau})$ is called a $\tilde{\tau}$ - C -soft topology on X .*

Proposition 5.1 *The following are valid:*

- i. \emptyset and X are $\tilde{\tau}$ - C -closed.
- ii. Any union of elements in the set $C(X, \tilde{\tau})$ is an element of the set $C(X, \tilde{\tau})$.
- iii. Any intersection of elements in the set $C(X, \tilde{\tau})$ is an element of the set $C(X, \tilde{\tau})$.

Proof: Let I be an index set and $A_i \subseteq X$, for all $i \in I$.

i. The proof follows from the corresponding definition.

ii. Assume that $A_i \in C(X, \tilde{\tau})$, for all $i \in I$. Then, $\text{Close}(A_i, X, \tilde{\tau}) = A_i$, for all $i \in I$. From Proposition 2.1 i,

$$\text{Close}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) = \bigcup_{i \in I} \text{Close}(A_i, X, \tilde{\tau}) = \bigcup_{i \in I} A_i$$

Therefore, any union of elements in the set $C(X, \tilde{\tau})$ is an element of the set $C(X, \tilde{\tau})$.

iii. Assume that $A_i \in C(X, \tilde{\tau})$, for all $i \in I$. Then, $\text{Close}(A_i, X, \tilde{\tau}) = A_i$, for all $i \in I$. From Proposition 2.1 ii,

$$\text{Close}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \subseteq \bigcap_{i \in I} \text{Close}(A_i, X, \tilde{\tau}) = \bigcap_{i \in I} A_i$$

Thus, $\text{Close}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) = \bigcap_{i \in I} A_i$ from Proposition 2.2 ii. Therefore, any intersection of elements in the set $C(X, \tilde{\tau})$ is an element of the set $C(X, \tilde{\tau})$. □

Definition 5.4 *Let $X \neq \emptyset$ and $\tilde{\tau} : X \rightarrow P(X)$ be a proximity mapping. Then, the set $B(X, \tilde{\tau})$ is called a $\tilde{\tau}$ - B -soft topology on X .*

Proposition 5.2 *The following are valid:*

- i. \emptyset and X are $\tilde{\tau}$ - B -open.
- ii. Any union of elements in the set $B(X, \tilde{\tau})$ is an element of the set $B(X, \tilde{\tau})$.
- iii. Any intersection of elements in the set $B(X, \tilde{\tau})$ is an element of the set $B(X, \tilde{\tau})$.

The proof is similar to that of Proposition 5.1.

Corollary 5.6 *The following hold:*

- i. The set $\{X \setminus U : U \in B(X, \tilde{\tau})\}$ is a $\tilde{\tau}$ - C -soft topology on X .
- ii. The set $\{X \setminus K : K \in C(X, \tilde{\tau})\}$ is a $\tilde{\tau}$ - B -soft topology on X .

Theorem 5.5 *Let $U, K \subseteq X$. Then, the following hold:*

- i. If $U \in B(X, \tilde{\tau})$ and $K \in C(X, \tilde{\tau})$, then $K \setminus U \in C(X, \tilde{\tau})$.*
- ii. If $U \in B(X, \tilde{\tau})$ and $K \in C(X, \tilde{\tau})$, then $U \setminus K \in B(X, \tilde{\tau})$.*

Proof: Let $U, K \subseteq X$.

- i. Assume that $U \in B(X, \tilde{\tau})$ and $K \in C(X, \tilde{\tau})$. From Remark 5.1, $X \setminus U \in C(X, \tilde{\tau})$. Thus, $K \setminus U = K \cap (X \setminus U) \in C(X, \tilde{\tau})$ from Proposition 5.1 iii.*
- ii. From Remark 5.1 and Proposition 5.2 iii, the proof is similar to that of i.*

□

Definition 5.5 *Let $X \neq \emptyset$ and $\tilde{\tau} : X \rightarrow P(X)$ be a proximity mapping. If $C(X, \tilde{\tau}) = P(X)$, then the set $C(X, \tilde{\tau})$ is called a $\tilde{\tau}$ -C-soft discrete topology on X . Similarly, if $B(X, \tilde{\tau}) = P(X)$, then the set $B(X, \tilde{\tau})$ is called a $\tilde{\tau}$ -B-soft discrete topology on X . Moreover, if $C(X, \tilde{\tau}) = \{\emptyset, X\}$, then the set $C(X, \tilde{\tau})$ is called a $\tilde{\tau}$ -C-soft indiscrete (or a $\tilde{\tau}$ -C-soft trivial) topology on X . Similarly, if $B(X, \tilde{\tau}) = \{\emptyset, X\}$, then the set $B(X, \tilde{\tau})$ is called a $\tilde{\tau}$ -B-soft indiscrete (or a $\tilde{\tau}$ -B-soft trivial) topology on X .*

5.2. $\tilde{\tau}$ -T- and $\tilde{\tau}$ -I-Soft Topologies

This subsection first introduces $\tilde{\tau}$ -T-closed and $\tilde{\tau}$ -I-open sets and explores their main properties.

Definition 5.6 *Let $A \subseteq X$. If $\text{Touch}(A, X, \tilde{\tau}) = A$, then the set A is called a $\tilde{\tau}$ -T-closed set in X (briefly a $\tilde{\tau}$ -T-closed). If the set $X \setminus A$ is $\tilde{\tau}$ -T-closed, then the set A is called a $\tilde{\tau}$ -I-open set in X (briefly a $\tilde{\tau}$ -I-open). The set of all $\tilde{\tau}$ -T-closed sets is denoted by $T(X, \tilde{\tau})$. Similarly, the set of all $\tilde{\tau}$ -I-open sets is denoted by $I(X, \tilde{\tau})$.*

Definition 5.7 *Let $A \subseteq X$. If $A \in T(X, \tilde{\tau})$ and $A \in I(X, \tilde{\tau})$, then the set A is called a $\tilde{\tau}$ -TI-closed-open set in X (briefly a $\tilde{\tau}$ -TI-clopen). Moreover, if $A \notin T(X, \tilde{\tau})$ and $A \notin I(X, \tilde{\tau})$, then the set A is called a $\tilde{\tau}$ -TI-neither-closed-nor-open set in X (briefly a $\tilde{\tau}$ -TI-nclnopen).*

Remark 5.2 *Let $A \subseteq X$. Then, it can be observed that the set A is $\tilde{\tau}$ -T-closed if and only if the set $X \setminus A$ is $\tilde{\tau}$ -I-open.*

Corollary 5.7 *X and \emptyset are $\tilde{\tau}$ -TI-clopen.*

Theorem 5.6 *Let $A \subseteq X$. Then, the set A is $\tilde{\tau}$ -T-closed if and only if $\text{Int}(X \setminus A, X, \tilde{\tau}) = X \setminus A$.*

Proof: Let $A \subseteq X$. From Proposition 2.2 xii,

$$\begin{aligned} A \text{ is } \tilde{\tau}\text{-T-closed} &\Leftrightarrow \text{Touch}(A, X, \tilde{\tau}) = A \\ &\Leftrightarrow X \setminus \text{Int}(X \setminus A, X, \tilde{\tau}) = A \\ &\Leftrightarrow \text{Int}(X \setminus A, X, \tilde{\tau}) = X \setminus A \end{aligned}$$

□

From Remark 5.2 and Theorem 5.6, the following corollary is observed.

Corollary 5.8 *Let $A \subseteq X$. Then, the set A is $\tilde{\tau}$ -I-open if and only if $\text{Int}(A, X, \tilde{\tau}) = A$.*

Theorem 5.7 *Let $A \subseteq X$. Then, the following hold:*

- i. If the set A is $\tilde{\tau}$ -I-open, then $A \subseteq \text{Int}(\text{Touch}(A, X, \tilde{\lambda}), X, \tilde{\tau})$.*
- ii. If the set A is $\tilde{\tau}$ -I-open, then $A \subseteq \text{Touch}(\text{Int}(A, X, \tilde{\tau}), X, \tilde{\lambda})$.*
- iii. If the set A is $\tilde{\tau}$ -T-closed, then $\text{Touch}(\text{Int}(A, X, \tilde{\lambda}), X, \tilde{\tau}) \subseteq A$.*
- iv. If the set A is $\tilde{\tau}$ -T-closed, then $\text{Int}(\text{Touch}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq A$.*

v. If the set A is $\tilde{\tau}$ - T -closed, then $\text{Touch}(\text{Touch}(A, X, \tilde{\tau}), X, \tilde{\tau}) = A$.

vi. If the set A is $\tilde{\tau}$ - I -open, then $\text{Int}(\text{Int}(A, X, \tilde{\tau}), X, \tilde{\tau}) = A$.

Proof: Let $A \subseteq X$.

i. Assume that the set A is $\tilde{\tau}$ - I -open. From Theorem 2.1 iii and Proposition 2.2 iv,

$$\text{Int}(A, X, \tilde{\tau}) \subseteq \text{Int}(\text{Touch}(A, X, \tilde{\lambda}), X, \tilde{\tau})$$

Hence, $A \subseteq \text{Int}(\text{Touch}(A, X, \tilde{\lambda}), X, \tilde{\tau})$ from Corollary 5.8.

ii. Assume that the set A is $\tilde{\tau}$ - I -open. From Corollary 5.8, $A \subseteq \text{Int}(A, X, \tilde{\tau})$. From Theorem 2.1 ii,

$$\text{Touch}(A, X, \tilde{\lambda}) \subseteq \text{Touch}(\text{Int}(A, X, \tilde{\tau}), X, \tilde{\lambda})$$

Hence, $A \subseteq \text{Touch}(\text{Int}(A, X, \tilde{\tau}), X, \tilde{\lambda})$ from Proposition 2.2 iv.

iii. Assume that the set A is $\tilde{\tau}$ - T -closed. From Theorem 2.1 ii and Proposition 2.2 viii,

$$\text{Touch}(\text{Int}(A, X, \tilde{\lambda}), X, \tilde{\tau}) \subseteq \text{Touch}(A, X, \tilde{\tau})$$

Hence, $\text{Touch}(\text{Int}(A, X, \tilde{\lambda}), X, \tilde{\tau}) \subseteq A$ from Definition 5.6.

iv. Assume that the set A is $\tilde{\tau}$ - T -closed. Then, $\text{Touch}(A, X, \tilde{\tau}) \subseteq A$. From Theorem 2.1 iii,

$$\text{Int}(\text{Touch}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq \text{Int}(A, X, \tilde{\lambda})$$

Hence, $\text{Int}(\text{Touch}(A, X, \tilde{\tau}), X, \tilde{\lambda}) \subseteq A$ from Proposition 2.2 viii.

v. Assume that the set A is $\tilde{\tau}$ - T -closed. Then, $\text{Touch}(A, X, \tilde{\tau}) = A$. Thus,

$$\text{Touch}(\text{Touch}(A, X, \tilde{\tau}), X, \tilde{\tau}) = \text{Touch}(A, X, \tilde{\tau}) = A$$

vi. Assume that the set A is $\tilde{\tau}$ - I -open. From Corollary 5.8, $\text{Int}(A, X, \tilde{\tau}) = A$. Thus,

$$\text{Int}(\text{Int}(A, X, \tilde{\tau}), X, \tilde{\tau}) = \text{Int}(A, X, \tilde{\tau}) = A$$

□

Corollary 5.9 Let $A \subseteq X$. If the set A is $\tilde{\tau}$ - T -closed, then the set $\text{Touch}(A, X, \tilde{\tau})$ is $\tilde{\tau}$ - T -closed. Similarly, if the set A is $\tilde{\tau}$ - I -open, then the set $\text{Int}(A, X, \tilde{\tau})$ is $\tilde{\tau}$ - I -open.

Corollary 5.10 Let $A \subseteq X$. If the set A is $\tilde{\tau}$ - TI -clopen, then $\text{Int}(\text{Touch}(A, X, \tilde{\tau}), X, \tilde{\tau}) = A$ and $\text{Touch}(\text{Int}(A, X, \tilde{\tau}), X, \tilde{\tau}) = A$.

Theorem 5.8 Let $A, B \subseteq X$. Then, the following hold:

i. If the set A is $\tilde{\tau}$ - T -closed and $B \subseteq A$, then $\text{Touch}(B, X, \tilde{\tau}) \subseteq A$.

ii. If the set B is $\tilde{\tau}$ - I -open and $B \subseteq A$, then $B \subseteq \text{Int}(A, X, \tilde{\tau})$.

iii. If $A \cap \text{Touch}(B, X, \tilde{\tau}) = \emptyset$ and the set $A \cup B$ is $\tilde{\tau}$ - T -closed, then the set B is $\tilde{\tau}$ - T -closed.

iv. If $A \cap \text{Touch}(B, X, \tilde{\tau}) = \emptyset$ and the set $A \cup B$ is $\tilde{\tau}$ - I -open, then the set A is $\tilde{\tau}$ - I -open.

v. If the set A is $\tilde{\tau}$ - I -open, then $A \cap \text{Touch}(B, X, \tilde{\tau}) \subseteq \text{Touch}(A \cap B, X, \tilde{\tau})$.

Proof: Let $A, B \subseteq X$.

i. Assume that the set A is $\tilde{\tau}$ - T -closed and $B \subseteq A$. Then, $\text{Touch}(B, X, \tilde{\tau}) \subseteq \text{Touch}(A, X, \tilde{\tau})$ from Theorem 2.1 ii. Hence, $\text{Touch}(B, X, \tilde{\tau}) \subseteq A$.

ii. Assume that the set B is $\tilde{\tau}$ - I -open and $B \subseteq A$. Then, $\text{Int}(B, X, \tilde{\tau}) \subseteq \text{Int}(A, X, \tilde{\tau})$ from Theorem 2.1 iii. Hence, $B \subseteq \text{Int}(A, X, \tilde{\tau})$ from Corollary 5.8.

iii. Assume that $A \cap \text{Touch}(B, X, \tilde{\tau}) = \emptyset$ and the set $A \cup B$ is $\tilde{\tau}$ - T -closed. From Theorem 2.1 ii,

$$\begin{aligned} a \in \text{Touch}(B, X, \tilde{\tau}) &\Rightarrow a \in \text{Touch}(A \cup B, X, \tilde{\tau}) \wedge a \notin A \\ &\Rightarrow a \in A \cup B \wedge a \notin A \\ &\Rightarrow a \in B \end{aligned}$$

Thus, $\text{Touch}(B, X, \tilde{\tau}) \subseteq B$. Hence, the set B is $\tilde{\tau}$ - T -closed from Proposition 2.2 iv.

iv. Assume that $A \cap \text{Touch}(B, X, \tilde{\tau}) = \emptyset$ and the set $A \cup B$ is $\tilde{\tau}$ - I -open. From the relevant definitions and Corollary 5.8,

$$\begin{aligned} a \in A &\Rightarrow a \in A \cup B \wedge a \notin \text{Touch}(B, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Int}(A \cup B, X, \tilde{\tau}) \wedge \tilde{\tau}(a) \cap B = \emptyset \\ &\Rightarrow \tilde{\tau}(a) \subseteq A \cup B \wedge \tilde{\tau}(a) \cap B = \emptyset \\ &\Rightarrow \tilde{\tau}(a) \subseteq A \\ &\Rightarrow a \in \text{Int}(A, X, \tilde{\tau}) \end{aligned}$$

Thus, $A \subseteq \text{Int}(A, X, \tilde{\tau})$. Hence, the set A is $\tilde{\tau}$ - I -open from Proposition 2.2 viii and Corollary 5.8.

v. Assume that the set A is $\tilde{\tau}$ - I -open. From the relevant definitions and Corollary 5.8,

$$\begin{aligned} a \in A \cap \text{Touch}(B, X, \tilde{\tau}) &\Rightarrow a \in A \wedge a \in \text{Touch}(B, X, \tilde{\tau}) \\ &\Rightarrow a \in \text{Int}(A, X, \tilde{\tau}) \wedge a \in \text{Touch}(B, X, \tilde{\tau}) \\ &\Rightarrow \tilde{\tau}(a) \subseteq A \wedge \tilde{\tau}(a) \cap B \neq \emptyset \\ &\Rightarrow \tilde{\tau}(a) \cap (A \cap B) \neq \emptyset \\ &\Rightarrow a \in \text{Touch}(A \cap B, X, \tilde{\tau}) \end{aligned}$$

Hence, $A \cap \text{Touch}(B, X, \tilde{\tau}) \subseteq \text{Touch}(A \cap B, X, \tilde{\tau})$.

□

Theorem 5.9 *Let $A \subseteq X$. Then, the following hold:*

- i. *The set A is $\tilde{\tau}$ - T -closed if and only if $\text{Fr}(A, X, \tilde{\tau}) \subseteq A$.*
- ii. *The set A is $\tilde{\tau}$ - I -open if and only if $\text{Fr}(A, X, \tilde{\tau}) \cap A = \emptyset$.*
- iii. *The set A is $\tilde{\tau}$ - T -closed if and only if $\text{Disj}(A, X, \tilde{\tau}) = X \setminus A$.*
- iv. *The set A is $\tilde{\tau}$ - T -closed if and only if $\text{Accu}(A, X, \tilde{\tau}) \subseteq A$.*

Proof: Let $A \subseteq X$.

i. (\Rightarrow): Assume that the set A is $\tilde{\tau}$ - T -closed. From Proposition 2.2 xv,

$$\text{Fr}(A, X, \tilde{\tau}) = \text{Touch}(A, X, \tilde{\tau}) \cap \text{Touch}(X \setminus A, X, \tilde{\tau}) = A \cap \text{Touch}(X \setminus A, X, \tilde{\tau}) \subseteq A$$

(\Leftarrow): Assume that $\text{Fr}(A, X, \tilde{\tau}) \subseteq A$. Then, $\text{Touch}(A, X, \tilde{\tau}) \subseteq \text{Int}(A, X, \tilde{\tau}) \cup A$ from Proposition 2.2 xviii. Thus, $\text{Touch}(A, X, \tilde{\tau}) \subseteq A$ from Proposition 2.2 viii. Therefore, $\text{Touch}(A, X, \tilde{\tau}) = A$ from Proposition 2.2 iv. Hence, the set A is $\tilde{\tau}$ - T -closed.

ii. (\Rightarrow): Assume that the set A is $\tilde{\tau}$ - I -open. From Proposition 2.2 iv, xii, and xv,

$$\begin{aligned} \text{Fr}(A, X, \tilde{\tau}) \cap A &= (\text{Touch}(A, X, \tilde{\tau}) \cap \text{Touch}(X \setminus A, X, \tilde{\tau})) \cap A \\ &= (A \cap \text{Touch}(A, X, \tilde{\tau})) \cap \text{Touch}(X \setminus A, X, \tilde{\tau}) \\ &= A \cap \text{Touch}(X \setminus A, X, \tilde{\tau}) \\ &= A \cap (X \setminus \text{Int}(A, X, \tilde{\tau})) \\ &= A \cap (X \setminus A) \\ &= \emptyset \end{aligned}$$

(\Leftarrow): Assume that $\text{Fr}(A, X, \tilde{\tau}) \cap A = \emptyset$. From Propositions 2.2 iv and xv and 3.1 vii, $A \cap (X \setminus \text{Int}(A, X, \tilde{\tau})) = \emptyset$. Therefore, $A \subseteq \text{Int}(A, X, \tilde{\tau})$. Thereby, $\text{Int}(A, X, \tilde{\tau}) = A$ from Proposition 2.2 viii. Hence, the set A is $\tilde{\tau}$ - I -open from Corollary 5.8.

iii. (\Rightarrow): Assume that the set A is $\tilde{\tau}$ - T -closed. Then, $\text{Touch}(A, X, \tilde{\tau}) = A$. From Proposition 2.2 xii,

$$\text{Disj}(A, X, \tilde{\tau}) = \text{Int}(X \setminus A, X, \tilde{\tau}) = X \setminus \text{Touch}(A, X, \tilde{\tau}) = X \setminus A$$

(\Leftarrow): Assume that $\text{Disj}(A, X, \tilde{\tau}) = X \setminus A$. Then, $\text{Int}(X \setminus A, X, \tilde{\tau}) = X \setminus A$. Thus, the set A is $\tilde{\tau}$ - T -closed from Theorem 5.6.

iv. (\Rightarrow): Assume that the set A is $\tilde{\tau}$ - T -closed. Thus, $\text{Touch}(A, X, \tilde{\tau}) = A$. Then, $\text{Accu}(A, X, \tilde{\tau}) \subseteq A$ from Proposition 3.5 iv.

(\Leftarrow): Assume that $\text{Accu}(A, X, \tilde{\tau}) \subseteq A$. Then, $\text{Touch}(A, X, \tilde{\tau}) = A \cup \text{Accu}(A, X, \tilde{\tau}) = A$ from Proposition 3.5 viii. Thus, the set A is $\tilde{\tau}$ - T -closed. □

Corollary 5.11 *Let $A \subseteq X$. Then, the set A is $\tilde{\tau}$ - TI -clopen if and only if $\text{Fr}(A, X, \tilde{\tau}) = \emptyset$.*

Afterward, this subsection proposes $\tilde{\tau}$ - T - and $\tilde{\tau}$ - I -soft topologies by $\tilde{\tau}$ - T -closed and $\tilde{\tau}$ - I -open sets, respectively.

Definition 5.8 *Let $X \neq \emptyset$ and $\tilde{\tau} : X \rightarrow P(X)$ be a proximity mapping. Then, the set $T(X, \tilde{\tau})$ is called a $\tilde{\tau}$ - T -soft topology on X .*

Proposition 5.3 *The following are valid:*

- i. \emptyset and X are $\tilde{\tau}$ - T -closed.
- ii. Any union of elements in the set $T(X, \tilde{\tau})$ is an element of the set $T(X, \tilde{\tau})$.
- iii. Any intersection of elements in the set $T(X, \tilde{\tau})$ is an element of the set $T(X, \tilde{\tau})$.

Proof: Let I be an index set and $A_i \subseteq X$, for all $i \in I$.

i. The proof follows from the corresponding definition.

ii. Assume that $A_i \in T(X, \tilde{\tau})$, for all $i \in I$. Then, $\text{Touch}(A_i, X, \tilde{\tau}) = A_i$, for all $i \in I$. From Proposition 2.1 iii,

$$\text{Touch}\left(\bigcup_{i \in I} A_i, X, \tilde{\tau}\right) = \bigcup_{i \in I} \text{Touch}(A_i, X, \tilde{\tau}) = \bigcup_{i \in I} A_i$$

Therefore, any union of elements in the set $T(X, \tilde{\tau})$ is an element of the set $T(X, \tilde{\tau})$.

iii. Assume that $A_i \in T(X, \tilde{\tau})$, for all $i \in I$. Then, $\text{Touch}(A_i, X, \tilde{\tau}) = A_i$, for all $i \in I$. From Proposition 2.1 iv,

$$\text{Touch}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) \subseteq \bigcap_{i \in I} \text{Touch}(A_i, X, \tilde{\tau}) = \bigcap_{i \in I} A_i$$

Thus, $\text{Touch}\left(\bigcap_{i \in I} A_i, X, \tilde{\tau}\right) = \bigcap_{i \in I} A_i$ from Proposition 2.2 *iv*. Therefore, any intersection of elements in the set $T(X, \tilde{\tau})$ is an element of the set $T(X, \tilde{\tau})$. \square

Definition 5.9 Let $X \neq \emptyset$ and $\tilde{\tau} : X \rightarrow P(X)$ be a proximity mapping. Then, the set $I(X, \tilde{\tau})$ is called a $\tilde{\tau}$ -*I*-soft topology on X .

Proposition 5.4 The following are valid:

- i.* \emptyset and X are $\tilde{\tau}$ -*I*-open.
- ii.* Any union of elements in the set $I(X, \tilde{\tau})$ is an element of the set $I(X, \tilde{\tau})$.
- iii.* Any intersection of elements in the set $I(X, \tilde{\tau})$ is an element of the set $I(X, \tilde{\tau})$.

The proof is similar to that of Proposition 5.3.

Corollary 5.12 The following hold:

- i.* The set $\{X \setminus U : U \in I(X, \tilde{\tau})\}$ is a $\tilde{\tau}$ -*T*-soft topology on X .
- ii.* The set $\{X \setminus K : K \in T(X, \tilde{\tau})\}$ is a $\tilde{\tau}$ -*I*-soft topology on X .

Theorem 5.10 Let $U, K \subseteq X$. Then, the following hold:

- i.* If $U \in I(X, \tilde{\tau})$ and $K \in T(X, \tilde{\tau})$, then $K \setminus U \in T(X, \tilde{\tau})$.
- ii.* If $U \in I(X, \tilde{\tau})$ and $K \in T(X, \tilde{\tau})$, then $U \setminus K \in I(X, \tilde{\tau})$.

Proof: Let $U, K \subseteq X$.

- i.* Assume that $U \in I(X, \tilde{\tau})$ and $K \in T(X, \tilde{\tau})$. From Remark 5.2, $X \setminus U \in T(X, \tilde{\tau})$. Thus, $K \setminus U = K \cap (X \setminus U) \in T(X, \tilde{\tau})$ from Proposition 5.3 *iii*.
- ii.* From Remark 5.2 and Proposition 5.4 *iii*, the proof is similar to that of *i*. \square

Definition 5.10 Let $X \neq \emptyset$ and $\tilde{\tau} : X \rightarrow P(X)$ be a proximity mapping. If $T(X, \tilde{\tau}) = P(X)$, then the set $T(X, \tilde{\tau})$ is called a $\tilde{\tau}$ -*T*-soft discrete topology on X . Similarly, if $I(X, \tilde{\tau}) = P(X)$, then the set $I(X, \tilde{\tau})$ is called a $\tilde{\tau}$ -*I*-soft discrete topology on X . Moreover, if $T(X, \tilde{\tau}) = \{\emptyset, X\}$, then the set $T(X, \tilde{\tau})$ is called a $\tilde{\tau}$ -*T*-soft indiscrete (or a $\tilde{\tau}$ -*T*-soft trivial) topology on X . Similarly, if $I(X, \tilde{\tau}) = \{\emptyset, X\}$, then the set $I(X, \tilde{\tau})$ is called a $\tilde{\tau}$ -*I*-soft indiscrete (or a $\tilde{\tau}$ -*I*-soft trivial) topology on X .

5.3. Relations between the Introduced Soft Topologies

This subsection analyzes the relations between $\tilde{\tau}$ -*C*-closed and $\tilde{\tau}$ -*I*-open sets, between $\tilde{\tau}$ -*T*-closed and $\tilde{\tau}$ -*B*-open sets, and between the corresponding pairs of soft topologies.

Theorem 5.11 Let $A \subseteq X$. Then, the following hold:

- i.* The set A is $\tilde{\tau}$ -*C*-closed if and only if the set A is $\tilde{\tau}$ -*I*-open.
- ii.* The set A is $\tilde{\tau}$ -*T*-closed if and only if the set A is $\tilde{\tau}$ -*B*-open.

Proof: Let $A \subseteq X$.

- i.* From Proposition 2.2 *ii* and *viii* and Corollary 5.8,

$$\begin{aligned}
 A \text{ is } \tilde{\tau}\text{-}C\text{-closed} &\Leftrightarrow \text{Close}(A, X, \tilde{\tau}) \subseteq A \\
 &\Leftrightarrow \bigcup_{x \in A} \tilde{\tau}(x) \subseteq A \\
 &\Leftrightarrow \forall x \in A, \tilde{\tau}(x) \subseteq A \\
 &\Leftrightarrow \forall x \in A, x \in \text{Int}(A, X, \tilde{\tau}) \\
 &\Leftrightarrow A \subseteq \text{Int}(A, X, \tilde{\tau}) \\
 &\Leftrightarrow A \text{ is } \tilde{\tau}\text{-}I\text{-open}
 \end{aligned}$$

ii. From Remarks 5.1 and 5.2 and Theorem 5.11 i,

$$\begin{aligned} A \text{ is } \tilde{\tau}\text{-}T\text{-closed} &\Leftrightarrow X \setminus A \text{ is } \tilde{\tau}\text{-}I\text{-open} \\ &\Leftrightarrow X \setminus A \text{ is } \tilde{\tau}\text{-}C\text{-closed} \\ &\Leftrightarrow A \text{ is } \tilde{\tau}\text{-}B\text{-open} \end{aligned}$$

□

Corollary 5.13 *Every $\tilde{\tau}$ -C-soft topology on X is also a $\tilde{\tau}$ -I-soft topology on X , and vice versa. In other words, $C(X, \tilde{\tau}) = I(X, \tilde{\tau})$.*

Corollary 5.14 *Every $\tilde{\tau}$ -T-soft topology on X is also a $\tilde{\tau}$ -B-soft topology on X , and vice versa. In other words, $T(X, \tilde{\tau}) = B(X, \tilde{\tau})$.*

Corollary 5.15 *Let $A \subseteq X$. Then, the following hold:*

- i. *The set A is $\tilde{\tau}$ -C-closed if and only if the set $X \setminus A$ is $\tilde{\tau}$ -T-closed.*
- ii. *The set A is $\tilde{\tau}$ -B-open if and only if the set $X \setminus A$ is $\tilde{\tau}$ -I-open.*

6. Illustrative Examples

This section provides examples and counterexamples that illustrate the concepts considered throughout the paper and demonstrate that certain properties do not hold in general. Throughout this section, let $I_n := \{1, 2, 3, \dots, n\}$ and $I_n^* := \{0, 1, 2, 3, \dots, n\}$, for all $n \in \mathbb{N}$, where \mathbb{N} denotes the set of all nonnegative integers. Examples 6.1-6.19 illustrate various proximity mappings.

Example 6.1 *Let $\tilde{\tau}, \tilde{\lambda} : X \rightarrow P(X)$ be two proximity mappings. Then, the mapping $\tilde{\tau} \cap \tilde{\lambda} : X \rightarrow P(X)$ defined by $(\tilde{\tau} \cap \tilde{\lambda})(x) = \tilde{\tau}(x) \cap \tilde{\lambda}(x)$ is a proximity mapping.*

Example 6.2 *Let $\tilde{\tau}, \tilde{\lambda} : X \rightarrow P(X)$ be two proximity mappings. Then, the mapping $\tilde{\tau} \cup \tilde{\lambda} : X \rightarrow P(X)$ defined by $(\tilde{\tau} \cup \tilde{\lambda})(x) = \tilde{\tau}(x) \cup \tilde{\lambda}(x)$ is a proximity mapping.*

Example 6.3 [6] *Let $\tilde{\tau}, \tilde{\lambda} : X \rightarrow P(X)$ be two proximity mappings. Then, the mapping $\tilde{\tau} \circ \tilde{\lambda} : X \rightarrow P(X)$ defined by $(\tilde{\tau} \circ \tilde{\lambda})(x) = \tilde{\tau}(\tilde{\lambda}(x))$ is a proximity mapping.*

Example 6.4 [7] *Let $\tilde{\tau} : \mathbb{N} \rightarrow P(\mathbb{N})$ be a function defined by $\tilde{\tau}(0) = \{0\}$ and $\tilde{\tau}(n) = \{0, n\}$, for all $n \in \mathbb{N} \setminus \{0\}$. Then, $\tilde{\tau}$ is a proximity mapping on \mathbb{N} .*

Example 6.5 *Let $n \in \mathbb{N}$ and $\tilde{\tau} : I_n^* \rightarrow P(I_n^*)$ be a function defined by $\tilde{\tau}(0) = \{0\}$ and $\tilde{\tau}(m) = \{0, m\}$, for all $m \in I_n$. Then, $\tilde{\tau}$ is a proximity mapping on I_n^* .*

Example 6.6 *Let $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ be a function. Then, the function $\tilde{\tau}_\delta^+ : \mathbb{R} \rightarrow P(\mathbb{R})$ defined by $\tilde{\tau}_\delta^+(x) = [x, x + \delta(x)]$ is a proximity mapping.*

Example 6.7 [7] *Let $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ be a function. Then, the function $\tilde{\tau}_{\delta_r}^+ : \mathbb{R} \rightarrow P(\mathbb{R})$ defined by $\tilde{\tau}_{\delta_r}^+(x) = [x, x + \delta(x))$ is a proximity mapping. Here, the subscript r in the notation $\tilde{\tau}_{\delta_r}^+$ represents that the interval $[x, x + \delta(x))$ is right half-open interval.*

Example 6.8 *Let $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ be a function. Then, the function $\tilde{\tau}_\delta^- : \mathbb{R} \rightarrow P(\mathbb{R})$ defined by $\tilde{\tau}_\delta^-(x) = [x - \delta(x), x]$ is a proximity mapping.*

Example 6.9 *Let $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ be a function. Then, the function $\tilde{\tau}_{\delta_l}^- : \mathbb{R} \rightarrow P(\mathbb{R})$ defined by $\tilde{\tau}_{\delta_l}^-(x) = (x - \delta(x), x]$ is a proximity mapping. Here, the subscript l in the notation $\tilde{\tau}_{\delta_l}^-$ represents that the interval $(x - \delta(x), x]$ is left half-open interval.*

Example 6.10 [7] *Let $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ be a function. Then, the function $\tilde{\tau}_\delta : \mathbb{R} \rightarrow P(\mathbb{R})$ defined by $\tilde{\tau}_\delta(x) = [x - \delta(x), x + \delta(x)]$ is a proximity mapping.*

Example 6.11 Let $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ be a function. Then, the function $\tilde{\tau}_{\delta_l} : \mathbb{R} \rightarrow P(\mathbb{R})$ defined by $\tilde{\tau}_{\delta_l}(x) = [x - \delta(x), x + \delta(x)]$ is a proximity mapping.

Example 6.12 Let $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ be a function. Then, the function $\tilde{\tau}_{\delta_r} : \mathbb{R} \rightarrow P(\mathbb{R})$ defined by $\tilde{\tau}_{\delta_r}(x) = [x - \delta(x), x + \delta(x))$ is a proximity mapping.

Example 6.13 [7] Let $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ be a function. Then, the function $\tilde{\tau}_{\delta_{lr}} : \mathbb{R} \rightarrow P(\mathbb{R})$ defined by $\tilde{\tau}_{\delta_{lr}}(x) = (x - \delta(x), x + \delta(x))$ is a proximity mapping.

Example 6.14 Let $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ be a function. Then, the function $\tilde{\tau}_{\delta}^{-\infty} : \mathbb{R} \rightarrow P(\mathbb{R})$ defined by $\tilde{\tau}_{\delta}^{-\infty}(x) = (-\infty, x + \delta(x)]$ is a proximity mapping.

Example 6.15 [7] Let $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ be a function. Then, the function $\tilde{\tau}_{\delta_r}^{-\infty} : \mathbb{R} \rightarrow P(\mathbb{R})$ defined by $\tilde{\tau}_{\delta_r}^{-\infty}(x) = (-\infty, x + \delta(x))$ is a proximity mapping.

Example 6.16 Let $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ be a function. Then, the function $\tilde{\tau}_{\delta}^{\infty} : \mathbb{R} \rightarrow P(\mathbb{R})$ defined by $\tilde{\tau}_{\delta}^{\infty}(x) = [x - \delta(x), \infty)$ is a proximity mapping.

Example 6.17 Let $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ be a function. Then, the function $\tilde{\tau}_{\delta_l}^{\infty} : \mathbb{R} \rightarrow P(\mathbb{R})$ defined by $\tilde{\tau}_{\delta_l}^{\infty}(x) = (x - \delta(x), \infty)$ is a proximity mapping.

Example 6.18 Let (X, \mathcal{T}) be a topological space and $\mathcal{T}(x)$ denote the collection of all open neighborhoods of x , for all $x \in X$, where an open neighborhood of x is an open set including x . Then, the mapping $\tilde{\tau} : X \rightarrow P(X)$ defined by $\tilde{\tau}(x) = \bigcap_{U \in \mathcal{T}(x)} U$ is a proximity mapping.

Example 6.19 The function $\tilde{\tau} : X \rightarrow P(X)$ defined by $\tilde{\tau}(x) = \{x\}$ is a proximity mapping. Moreover, the set $\{\tilde{\tau}(x) \mid x \in X\}$ is a basis for the classical discrete topology on X .

Example 6.20 illustrates the notions of the $\tilde{\tau}$ -neighborhood of a point, $\tilde{\tau}$ -neighborhood of a set, $\tilde{\tau}$ -closure, $\tilde{\tau}$ -interior, $\tilde{\tau}$ -boundary, $\tilde{\tau}$ -border set, and $\tilde{\tau}$ -hidden set.

Example 6.20 Consider the proximity mapping $\tilde{\tau}_{\delta}^+$ in Example 6.6 such that $\delta(x) = 1$, for all $x \in \mathbb{R}$. Let $A = [-1, 1]$, $B = [0, 2] \cup \{5\}$, $C = \{-1, 1\}$, and $D = (-\infty, -3]$.

- The $\tilde{\tau}_{\delta}^+$ -neighborhood of the point 0 is $\tilde{\tau}_{\delta}^+(0) = [0, 1]$.
- The $\tilde{\tau}_{\delta}^+$ -neighborhoods of the sets A , B , C , and D are as follows, respectively:

$$\text{Close}(A, \mathbb{R}, \tilde{\tau}_{\delta}^+) = \bigcup_{x \in A} \tilde{\tau}_{\delta}^+(x) = \bigcup_{x \in [-1, 1]} [x, x + 1] = [-1, 2]$$

$$\text{Close}(B, \mathbb{R}, \tilde{\tau}_{\delta}^+) = [0, 3] \cup [5, 6]$$

$$\text{Close}(C, \mathbb{R}, \tilde{\tau}_{\delta}^+) = [-1, 0] \cup [1, 2]$$

and

$$\text{Close}(D, \mathbb{R}, \tilde{\tau}_{\delta}^+) = (-\infty, -2]$$

- The $\tilde{\tau}_{\delta}^+$ -closures of the sets A , B , C , and D are as follows, respectively:

$$\text{Touch}(A, \mathbb{R}, \tilde{\tau}_{\delta}^+) = \left\{ x \in \mathbb{R} : \tilde{\tau}_{\delta}^+(x) \cap A \neq \emptyset \right\} = \left\{ x \in \mathbb{R} : [x, x + 1] \cap [-1, 1] \neq \emptyset \right\} = [-2, 1]$$

$$\text{Touch}(B, \mathbb{R}, \tilde{\tau}_{\delta}^+) = [-1, 2] \cup [4, 5]$$

$$\text{Touch}(C, \mathbb{R}, \tilde{\tau}_{\delta}^+) = [-2, -1] \cup [0, 1]$$

and

$$\text{Touch}(D, \mathbb{R}, \tilde{\tau}_{\delta}^+) = (-\infty, -3]$$

- The $\tilde{\tau}_\delta^+$ -interiors of the sets A , B , C , and D are as follows, respectively:

$$\text{Int}\left(A, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \left\{x \in \mathbb{R} : \tilde{\tau}_\delta^+(x) \subseteq A\right\} = \left\{x \in \mathbb{R} : [x, x+1] \subseteq [-1, 1]\right\} = [-1, 0]$$

$$\text{Int}\left(B, \mathbb{R}, \tilde{\tau}_\delta^+\right) = [0, 1]$$

$$\text{Int}\left(C, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \emptyset$$

and

$$\text{Int}\left(D, \mathbb{R}, \tilde{\tau}_\delta^+\right) = (-\infty, -4]$$

- The $\tilde{\tau}_\delta^+$ -boundaries of the sets A , B , C , and D are as follows, respectively:

$$\text{Fr}\left(A, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \text{Touch}\left(A, \mathbb{R}, \tilde{\tau}_\delta^+\right) \setminus \text{Int}\left(A, \mathbb{R}, \tilde{\tau}_\delta^+\right) = [-2, -1) \cup (0, 1]$$

$$\text{Fr}\left(B, \mathbb{R}, \tilde{\tau}_\delta^+\right) = [-1, 0) \cup (1, 2] \cup [4, 5]$$

$$\text{Fr}\left(C, \mathbb{R}, \tilde{\tau}_\delta^+\right) = [-2, -1] \cup [0, 1]$$

and

$$\text{Fr}\left(D, \mathbb{R}, \tilde{\tau}_\delta^+\right) = (-4, -3]$$

- The $\tilde{\tau}_\delta^+$ -border sets of the sets A , B , C , and D are as follows, respectively:

$$\text{Bord}\left(A, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \text{Close}\left(A, \mathbb{R}, \tilde{\tau}_\delta^+\right) \cap \text{Close}\left(\mathbb{R} \setminus A, \mathbb{R}, \tilde{\tau}_\delta^+\right) = [-1, 0) \cup (1, 2]$$

$$\text{Bord}\left(B, \mathbb{R}, \tilde{\tau}_\delta^+\right) = [0, 1) \cup (2, 3] \cup [5, 6]$$

$$\text{Bord}\left(C, \mathbb{R}, \tilde{\tau}_\delta^+\right) = [-1, 0] \cup [1, 2]$$

and

$$\text{Bord}\left(D, \mathbb{R}, \tilde{\tau}_\delta^+\right) = (-3, -2]$$

- The $\tilde{\tau}_\delta^+$ -hidden sets of the sets A , B , C , and D are as follows, respectively:

$$\text{Bow}\left(A, \mathbb{R}, \tilde{\tau}_\delta^+\right) = A \setminus \text{Bord}\left(A, \mathbb{R}, \tilde{\tau}_\delta^+\right) = [0, 1]$$

$$\text{Bow}\left(B, \mathbb{R}, \tilde{\tau}_\delta^+\right) = [1, 2]$$

$$\text{Bow}\left(C, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \emptyset$$

and

$$\text{Bow}\left(D, \mathbb{R}, \tilde{\tau}_\delta^+\right) = (-\infty, -3]$$

Example 6.21 demonstrates that the operators Fr and Bord are neither monotone nor anti-monotone with respect to the first variable.

Example 6.21 Consider the proximity mapping $\tilde{\tau}_\delta^+$ in Example 6.6 such that $\delta(x) = 1$, for all $x \in \mathbb{R}$, and the sets A and C in Example 6.20.

- For Theorem 2.1 iv, although $C \subseteq A$, $\text{Fr}\left(C, \mathbb{R}, \tilde{\tau}_\delta^+\right) \not\subseteq \text{Fr}\left(A, \mathbb{R}, \tilde{\tau}_\delta^+\right)$. Similarly, $\text{Fr}\left(A, \mathbb{R}, \tilde{\tau}_\delta^+\right) \not\subseteq \text{Fr}\left(\{1\}, \mathbb{R}, \tilde{\tau}_\delta^+\right) = [0, 1]$ though $\{1\} \subseteq A$. Thus, the operator Fr satisfies neither monotonicity nor anti-monotonicity properties with respect to the first variable.

- For Theorem 2.1 v, although $C \subseteq A$, $\text{Bord}(C, \mathbb{R}, \tilde{\tau}_\delta^+) \not\subseteq \text{Bord}(A, \mathbb{R}, \tilde{\tau}_\delta^+)$. Similarly, $\text{Bord}(A, \mathbb{R}, \tilde{\tau}_\delta^+) \not\subseteq \text{Bord}(\{1\}, \mathbb{R}, \tilde{\tau}_\delta^+) = [1, 2]$ though $\{1\} \subseteq A$. Hence, the operator Bord satisfies neither monotonicity nor anti-monotonicity properties with respect to the first variable.

Example 6.22 indicates that the converses of the inclusions in Proposition 2.1 are invalid.

Example 6.22 Consider the proximity mapping $\tilde{\tau}_\delta^+$ in Example 6.6 such that $\delta(x) = 1$, for all $x \in \mathbb{R}$, the sets A, B, C , and D in Example 6.20, and the set $E = [1, 3]$.

- For Proposition 2.1 ii, consider the following:

$$\text{Close}(B, \mathbb{R}, \tilde{\tau}_\delta^+) \cap \text{Close}(C, \mathbb{R}, \tilde{\tau}_\delta^+) = \{0\} \cup [1, 2]$$

and

$$\text{Close}(B \cap C, \mathbb{R}, \tilde{\tau}_\delta^+) = \text{Close}(\{1\}, \mathbb{R}, \tilde{\tau}_\delta^+) = [1, 2]$$

Hence, $\text{Close}(B, \mathbb{R}, \tilde{\tau}_\delta^+) \cap \text{Close}(C, \mathbb{R}, \tilde{\tau}_\delta^+) \not\subseteq \text{Close}(B \cap C, \mathbb{R}, \tilde{\tau}_\delta^+)$.

- For Proposition 2.1 iv, consider the following:

$$\text{Touch}(B, \mathbb{R}, \tilde{\tau}_\delta^+) \cap \text{Touch}(C, \mathbb{R}, \tilde{\tau}_\delta^+) = \{-1\} \cup [0, 1]$$

and

$$\text{Touch}(B \cap C, \mathbb{R}, \tilde{\tau}_\delta^+) = \text{Touch}(\{1\}, \mathbb{R}, \tilde{\tau}_\delta^+) = [0, 1]$$

Hence, $\text{Touch}(B, \mathbb{R}, \tilde{\tau}_\delta^+) \cap \text{Touch}(C, \mathbb{R}, \tilde{\tau}_\delta^+) \not\subseteq \text{Touch}(B \cap C, \mathbb{R}, \tilde{\tau}_\delta^+)$.

- For Proposition 2.1 v, consider the following:

$$\text{Int}(E, \mathbb{R}, \tilde{\tau}_\delta^+) = [1, 2]$$

$$\text{Int}(A \cup E, \mathbb{R}, \tilde{\tau}_\delta^+) = \text{Int}([-1, 3], \mathbb{R}, \tilde{\tau}_\delta^+) = [-1, 2]$$

and

$$\text{Int}(A, \mathbb{R}, \tilde{\tau}_\delta^+) \cup \text{Int}(E, \mathbb{R}, \tilde{\tau}_\delta^+) = [-1, 0] \cup [1, 2]$$

Hence, $\text{Int}(A \cup E, \mathbb{R}, \tilde{\tau}_\delta^+) \not\subseteq \text{Int}(A, \mathbb{R}, \tilde{\tau}_\delta^+) \cup \text{Int}(E, \mathbb{R}, \tilde{\tau}_\delta^+)$.

- For Proposition 2.1 vii, consider the following:

$$\text{Fr}(A, \mathbb{R}, \tilde{\tau}_\delta^+) \cup \text{Fr}(C, \mathbb{R}, \tilde{\tau}_\delta^+) = [-2, -1] \cup [0, 1]$$

and

$$\text{Fr}(A \cup C, \mathbb{R}, \tilde{\tau}_\delta^+) = \text{Fr}(A, \mathbb{R}, \tilde{\tau}_\delta^+) = [-2, 1] \cup (0, 1]$$

Hence, $\text{Fr}(A, \mathbb{R}, \tilde{\tau}_\delta^+) \cup \text{Fr}(C, \mathbb{R}, \tilde{\tau}_\delta^+) \not\subseteq \text{Fr}(A \cup C, \mathbb{R}, \tilde{\tau}_\delta^+)$.

- For Proposition 2.1 viii, consider the following:

$$\text{Fr}(A, \mathbb{R}, \tilde{\tau}_\delta^+) \cup \text{Fr}(D, \mathbb{R}, \tilde{\tau}_\delta^+) = (-4, -3] \cup [-2, -1] \cup (0, 1]$$

and

$$\text{Fr}(A \cap D, \mathbb{R}, \tilde{\tau}_\delta^+) = \text{Fr}(\emptyset, \mathbb{R}, \tilde{\tau}_\delta^+) = \emptyset$$

Hence, $\text{Fr}(A, \mathbb{R}, \tilde{\tau}_\delta^+) \cup \text{Fr}(D, \mathbb{R}, \tilde{\tau}_\delta^+) \not\subseteq \text{Fr}(A \cap D, \mathbb{R}, \tilde{\tau}_\delta^+)$.

- For Proposition 2.1 ix, consider the following:

$$\text{Bord}(A, \mathbb{R}, \tilde{\tau}_\delta^+) \cup \text{Bord}(C, \mathbb{R}, \tilde{\tau}_\delta^+) = [-1, 0] \cup [1, 2]$$

and

$$\text{Bord}(A \cup C, \mathbb{R}, \tilde{\tau}_\delta^+) = \text{Bord}(A, \mathbb{R}, \tilde{\tau}_\delta^+) = [-1, 0] \cup (1, 2]$$

Hence, $\text{Bord}(A, \mathbb{R}, \tilde{\tau}_\delta^+) \cup \text{Bord}(C, \mathbb{R}, \tilde{\tau}_\delta^+) \not\subseteq \text{Bord}(A \cup C, \mathbb{R}, \tilde{\tau}_\delta^+)$.

- For Proposition 2.1 x, consider the following:

$$\text{Bord}(A, \mathbb{R}, \tilde{\tau}_\delta^+) \cup \text{Bord}(D, \mathbb{R}, \tilde{\tau}_\delta^+) = (-3, -2] \cup [-1, 0] \cup (1, 2]$$

and

$$\text{Bord}(A \cap D, \mathbb{R}, \tilde{\tau}_\delta^+) = \text{Bord}(\emptyset, \mathbb{R}, \tilde{\tau}_\delta^+) = \emptyset$$

Hence, $\text{Bord}(A, \mathbb{R}, \tilde{\tau}_\delta^+) \cup \text{Bord}(D, \mathbb{R}, \tilde{\tau}_\delta^+) \not\subseteq \text{Bord}(A \cap D, \mathbb{R}, \tilde{\tau}_\delta^+)$.

- For Proposition 2.1 xi, consider the following:

$$\text{Bow}(E, \mathbb{R}, \tilde{\tau}_\delta^+) = [2, 3]$$

$$\text{Bow}(A \cup E, \mathbb{R}, \tilde{\tau}_\delta^+) = \text{Bow}([-1, 3], \mathbb{R}, \tilde{\tau}_\delta^+) = [0, 3]$$

and

$$\text{Bow}(A, \mathbb{R}, \tilde{\tau}_\delta^+) \cup \text{Bow}(E, \mathbb{R}, \tilde{\tau}_\delta^+) = [0, 1] \cup [2, 3]$$

Hence, $\text{Bow}(A \cup E, \mathbb{R}, \tilde{\tau}_\delta^+) \not\subseteq \text{Bow}(A, \mathbb{R}, \tilde{\tau}_\delta^+) \cup \text{Bow}(E, \mathbb{R}, \tilde{\tau}_\delta^+)$.

Example 6.23 provides examples for $\tilde{\tau}$ -accumulation, $\tilde{\tau}$ -cluster, $\tilde{\tau}$ -isolated, and $\tilde{\tau}$ -separated points of a set.

Example 6.23 Consider the proximity mapping $\tilde{\tau}_\delta^+$ in Example 6.6 such that $\delta(x) = 1$, for all $x \in \mathbb{R}$, and the sets A , B , C , and D in Example 6.20.

- The sets of all $\tilde{\tau}_\delta^+$ -accumulation points of the sets A , B , C , and D are as follows, respectively:

$$\text{Accu}(A, \mathbb{R}, \tilde{\tau}_\delta^+) = \left\{ a \in \mathbb{R} : \left(\tilde{\tau}_\delta^+(a) \setminus \{a\} \right) \cap A \neq \emptyset \right\} = [-2, 1)$$

$$\text{Accu}(B, \mathbb{R}, \tilde{\tau}_\delta^+) = [-1, 2) \cup [4, 5)$$

$$\text{Accu}(C, \mathbb{R}, \tilde{\tau}_\delta^+) = [-2, -1) \cup [0, 1)$$

and

$$\text{Accu}(D, \mathbb{R}, \tilde{\tau}_\delta^+) = (-\infty, -3)$$

- The sets of all $\tilde{\tau}_\delta^+$ -cluster points of the sets A , B , C , and D are as follows, respectively:

$$\text{Crowd}(A, \mathbb{R}, \tilde{\tau}_\delta^+) = \bigcup_{x \in A} \left(\tilde{\tau}_\delta^+(x) \setminus \{x\} \right) = (-1, 2]$$

$$\text{Crowd}(B, \mathbb{R}, \tilde{\tau}_\delta^+) = (0, 3] \cup (5, 6]$$

$$\text{Crowd}(C, \mathbb{R}, \tilde{\tau}_\delta^+) = (-1, 0] \cup (1, 2]$$

and

$$\text{Crowd}(D, \mathbb{R}, \tilde{\tau}_\delta^+) = (-\infty, -2]$$

- The sets of all $\tilde{\tau}_\delta^+$ -isolated points of the sets A , B , C , and D are as follows, respectively:

$$\text{Isol}\left(A, \mathbb{R}, \tilde{\tau}_\delta^+\right) = A \setminus \text{Accu}\left(A, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \{1\}$$

$$\text{Isol}\left(B, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \{2, 5\}$$

$$\text{Isol}\left(C, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \{-1, 1\}$$

and

$$\text{Isol}\left(D, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \{-3\}$$

- The sets of all $\tilde{\tau}_\delta^+$ -separated points of the sets A , B , C , and D are as follows, respectively:

$$\text{Sep}\left(A, \mathbb{R}, \tilde{\tau}_\delta^+\right) = A \setminus \text{Crowd}\left(A, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \{-1\}$$

$$\text{Sep}\left(B, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \{0, 5\}$$

$$\text{Sep}\left(C, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \{-1, 1\}$$

and

$$\text{Sep}\left(D, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \emptyset$$

Example 6.24 demonstrates that the operators Isol and Sep are neither monotone nor anti-monotone with respect to the first variable.

Example 6.24 Consider the proximity mapping $\tilde{\tau}_\delta^+$ in Example 6.6 such that $\delta(x) = 1$, for all $x \in \mathbb{R}$, and the sets A and C in Example 6.20.

- For Theorem 3.1 ii, although $C \subseteq A$, $\text{Isol}\left(C, \mathbb{R}, \tilde{\tau}_\delta^+\right) \not\subseteq \text{Isol}\left(A, \mathbb{R}, \tilde{\tau}_\delta^+\right)$. Similarly, $\text{Isol}\left(C, \mathbb{R}, \tilde{\tau}_\delta^+\right) \not\subseteq \text{Isol}\left(\{1\}, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \{1\}$ though $\{1\} \subseteq C$. Thus, the operator Isol satisfies neither monotonicity nor anti-monotonicity properties with respect to the first variable.
- For Theorem 3.1 iii, although $C \subseteq A$, $\text{Sep}\left(C, \mathbb{R}, \tilde{\tau}_\delta^+\right) \not\subseteq \text{Sep}\left(A, \mathbb{R}, \tilde{\tau}_\delta^+\right)$. Similarly, $\text{Sep}\left(C, \mathbb{R}, \tilde{\tau}_\delta^+\right) \not\subseteq \text{Sep}\left(\{1\}, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \{1\}$ though $\{1\} \subseteq C$. Hence, the operator Sep satisfies neither monotonicity nor anti-monotonicity properties with respect to the first variable.

Example 6.25 indicates that the converses of the inclusions in Propositions 2.3 and 3.3 are invalid.

Example 6.25 Consider the proximity mapping $\tilde{\tau}_\delta^+$ in Example 6.6 such that $\delta(x) = 1$, for all $x \in \mathbb{R}$, the sets A and C in Example 6.20, and the set E in Example 6.22.

- For Proposition 2.3 iii, consider the following:

$$\text{Crowd}\left(A \setminus \{1\}, \mathbb{R}, \tilde{\tau}_\delta^+\right) \cap \text{Crowd}\left(E, \mathbb{R}, \tilde{\tau}_\delta^+\right) = (-1, 2) \cap (1, 4] = (1, 2)$$

and

$$\text{Crowd}\left((A \setminus \{1\}) \cap E, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \text{Crowd}\left(\emptyset, \mathbb{R}, \tilde{\tau}_\delta^+\right) = \emptyset$$

Hence, $\text{Crowd}\left(A \setminus \{1\}, \mathbb{R}, \tilde{\tau}_\delta^+\right) \cap \text{Crowd}\left(E, \mathbb{R}, \tilde{\tau}_\delta^+\right) \not\subseteq \text{Crowd}\left((A \setminus \{1\}) \cap E, \mathbb{R}, \tilde{\tau}_\delta^+\right)$.

- For Proposition 3.3 ii, consider the following:

$$\text{Accu} \left(A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \cap \text{Accu} \left(E \setminus \{1\}, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = [-2, 1) \cap (0, 3) = (0, 1)$$

and

$$\text{Accu} \left(A \cap (E \setminus \{1\}), \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \text{Accu} \left(\emptyset, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \emptyset$$

Hence, $\text{Accu} \left(A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \cap \text{Accu} \left(E \setminus \{1\}, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \not\subseteq \text{Accu} \left(A \cap (E \setminus \{1\}), \mathbb{R}, \tilde{\tau}_\delta^+ \right)$.

- For Proposition 3.3 iii, consider the following:

$$\text{Isol} \left(A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \cup \text{Isol} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \{-1, 1\}$$

and

$$\text{Isol} \left(A \cup C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \text{Isol} \left(A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \{1\}$$

Hence, $\text{Isol} \left(A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \cup \text{Isol} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \not\subseteq \text{Isol} \left(A \cup C, \mathbb{R}, \tilde{\tau}_\delta^+ \right)$.

- For Proposition 3.3 iv, consider the following:

$$\text{Isol} \left(A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \cap \text{Isol} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \{1\}$$

and

$$\text{Isol} \left(A \cap C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \text{Isol} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \{-1, 1\}$$

Hence, $\text{Isol} \left(A \cap C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \not\subseteq \text{Isol} \left(A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \cap \text{Isol} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right)$.

- For Proposition 3.3 v, consider the following:

$$\text{Sep} \left(A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \cup \text{Sep} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \{-1, 1\}$$

and

$$\text{Sep} \left(A \cup C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \text{Sep} \left(A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \{-1\}$$

Hence, $\text{Sep} \left(A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \cup \text{Sep} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \not\subseteq \text{Sep} \left(A \cup C, \mathbb{R}, \tilde{\tau}_\delta^+ \right)$.

- For Proposition 3.3 vi, consider the following:

$$\text{Sep} \left(A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \cap \text{Sep} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \{-1\}$$

and

$$\text{Sep} \left(A \cap C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \text{Sep} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \{-1, 1\}$$

Hence, $\text{Sep} \left(A \cap C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \not\subseteq \text{Sep} \left(A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \cap \text{Sep} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right)$.

Example 6.26 illustrates $\tilde{\tau}$ -disjointness, $\tilde{\tau}$ -exterior, and $\tilde{\tau}$ -condensation points of a set.

Example 6.26 Consider the proximity mapping $\tilde{\tau}_\delta^+$ in Example 6.6 such that $\delta(x) = 1$, for all $x \in \mathbb{R}$, and the sets A , B , C , and D in Example 6.20.

- The $\tilde{\tau}$ -disjointnesses of the sets A , B , C , and D are as follows, respectively:

$$\text{Disj} \left(A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \text{Int} \left(\mathbb{R} \setminus A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \text{Int} \left((-\infty, -1) \cup (1, \infty), \mathbb{R}, \tilde{\tau}_\delta^+ \right) = (-\infty, -2) \cup (1, \infty)$$

$$\text{Disj} \left(B, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = (-\infty, -1) \cup (2, 4) \cup (5, \infty)$$

$$\text{Disj} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = (-\infty, -2) \cup (-1, 0) \cup (1, \infty)$$

and

$$\text{Disj} \left(D, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = (-3, \infty)$$

- The $\tilde{\tau}$ -exteriors of the sets A , B , C , and D are as follows, respectively:

$$\text{Ext} \left(A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \text{Bow} \left(\mathbb{R} \setminus A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \text{Bow} \left((-\infty, -1) \cup (1, \infty), \mathbb{R}, \tilde{\tau}_\delta^+ \right) = (-\infty, -1) \cup (2, \infty)$$

$$\text{Ext} \left(B, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = (-\infty, 0) \cup (3, 5) \cup (6, \infty)$$

$$\text{Ext} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = (-\infty, -1) \cup (0, 1) \cup (2, \infty)$$

and

$$\text{Ext} \left(D, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = (-2, \infty)$$

- The $\tilde{\tau}$ -condensation sets of the sets A , B , C , and D are as follows, respectively:

$$\text{Cond} \left(A, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \left\{ x \in \mathbb{R} : \tilde{\tau}_\delta^+(x) \cap A \text{ is an uncountable set} \right\} = (-2, 1)$$

$$\text{Cond} \left(B, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = (-1, 2)$$

$$\text{Cond} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \emptyset$$

and

$$\text{Cond} \left(D, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = (-\infty, -3)$$

Example 6.27 indicates that the converses of the inclusions in Propositions 4.2 and 4.6 are invalid.

Example 6.27 Consider the proximity mapping $\tilde{\tau}_\delta^+$ in Example 6.6 such that $\delta(x) = 1$, for all $x \in \mathbb{R}$, the sets A , B , and C in Example 6.20, the set E in Example 6.22, and $A_x = \{x\}$, for all $x \in I$, where $I = [0, 1]$.

- For Proposition 4.2 ii, consider the following:

$$\text{Disj} \left(B, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \cup \text{Disj} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = (-\infty, -1) \cup (-1, 0) \cup (1, \infty)$$

and

$$\text{Disj} \left(B \cap C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \text{Disj} \left(\{1\}, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = (-\infty, 0) \cup (1, \infty)$$

Hence, $\text{Disj} \left(B \cap C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \not\subseteq \text{Disj} \left(B, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \cup \text{Disj} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right)$.

- For Proposition 4.2 iv, consider the following:

$$\text{Ext} \left(B, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \cup \text{Ext} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = (-\infty, 0) \cup (0, 1) \cup (2, \infty)$$

and

$$\text{Ext} \left(B \cap C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \text{Ext} \left(\{1\}, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = (-\infty, 1) \cup (2, \infty)$$

Hence, $\text{Ext} \left(B \cap C, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \not\subseteq \text{Ext} \left(B, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \cup \text{Ext} \left(C, \mathbb{R}, \tilde{\tau}_\delta^+ \right)$.

- For Proposition 4.6 i, consider the following:

$$\text{Cond} \left(\bigcup_{x \in I} A_x, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \text{Cond} \left([0, 1], \mathbb{R}, \tilde{\tau}_\delta^+ \right) = (-1, 1)$$

and

$$\bigcup_{x \in I} \text{Cond} \left(A_x, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \bigcup_{x \in I} \text{Cond} \left(\{x\}, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \bigcup_{x \in I} \emptyset = \emptyset$$

Hence, $\text{Cond} \left(\bigcup_{x \in I} A_x, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \not\subseteq \bigcup_{x \in I} \text{Cond} \left(A_x, \mathbb{R}, \tilde{\tau}_\delta^+ \right)$.

- For Proposition 4.6 ii, consider the following:

$$\text{Cond} \left(A \cap B, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \cap \text{Cond} \left(E, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = (-1, 1) \cap (0, 3) = (0, 1)$$

and

$$\text{Cond} \left(A \cap B \cap E, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \text{Cond} \left(\{1\}, \mathbb{R}, \tilde{\tau}_\delta^+ \right) = \emptyset$$

$$\text{Hence, } \text{Cond} \left(A \cap B, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \cap \text{Cond} \left(E, \mathbb{R}, \tilde{\tau}_\delta^+ \right) \not\subseteq \text{Cond} \left(A \cap B \cap E, \mathbb{R}, \tilde{\tau}_\delta^+ \right).$$

Example 6.28 provides examples of $\tilde{\tau}$ -C-closed, $\tilde{\tau}$ -B-open, $\tilde{\tau}$ -T-closed, and $\tilde{\tau}$ -I-open sets and $\tilde{\tau}$ -C-, $\tilde{\tau}$ -B-, $\tilde{\tau}$ -T-, and $\tilde{\tau}$ -I-soft topologies.

Example 6.28 Consider the proximity mapping $\tilde{\tau}$ in Example 6.5, for $n = 3$.

- Consider the following:

$$\begin{aligned} \text{Close} \left(\emptyset, I_3^*, \tilde{\tau} \right) &= \emptyset & \text{Close} \left(\{0\}, I_3^*, \tilde{\tau} \right) &= \{0\} & \text{Close} \left(\{1\}, I_3^*, \tilde{\tau} \right) &= \{0, 1\} \\ \text{Close} \left(\{2\}, I_3^*, \tilde{\tau} \right) &= \{0, 2\} & \text{Close} \left(\{3\}, I_3^*, \tilde{\tau} \right) &= \{0, 3\} & \text{Close} \left(\{0, 1\}, I_3^*, \tilde{\tau} \right) &= \{0, 1\} \\ \text{Close} \left(\{0, 2\}, I_3^*, \tilde{\tau} \right) &= \{0, 2\} & \text{Close} \left(\{0, 3\}, I_3^*, \tilde{\tau} \right) &= \{0, 3\} & \text{Close} \left(\{1, 2\}, I_3^*, \tilde{\tau} \right) &= \{0, 1, 2\} \\ \text{Close} \left(\{1, 3\}, I_3^*, \tilde{\tau} \right) &= \{0, 1, 3\} & \text{Close} \left(\{2, 3\}, I_3^*, \tilde{\tau} \right) &= \{0, 2, 3\} & \text{Close} \left(\{0, 1, 2\}, I_3^*, \tilde{\tau} \right) &= \{0, 1, 2\} \\ \text{Close} \left(\{0, 1, 3\}, I_3^*, \tilde{\tau} \right) &= \{0, 1, 3\} & \text{Close} \left(\{0, 2, 3\}, I_3^*, \tilde{\tau} \right) &= \{0, 2, 3\} & \text{Close} \left(\{1, 2, 3\}, I_3^*, \tilde{\tau} \right) &= I_3^* \\ & & \text{Close} \left(I_3^*, I_3^*, \tilde{\tau} \right) &= I_3^* & & \end{aligned}$$

Hence, the sets $\emptyset, I_3^*, \{0\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 1, 2\}, \{0, 1, 3\}$, and $\{0, 2, 3\}$ are $\tilde{\tau}$ -C-closed. Thus,

$$C \left(I_3^*, \tilde{\tau} \right) = \left\{ \emptyset, I_3^*, \{0\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\} \right\}$$

is a $\tilde{\tau}$ -C-soft topology on I_3^* .

- Consider the following:

$$\begin{aligned} \text{Bow} \left(\emptyset, I_3^*, \tilde{\tau} \right) &= \emptyset & \text{Bow} \left(\{0\}, I_3^*, \tilde{\tau} \right) &= \emptyset & \text{Bow} \left(\{1\}, I_3^*, \tilde{\tau} \right) &= \{1\} \\ \text{Bow} \left(\{2\}, I_3^*, \tilde{\tau} \right) &= \{2\} & \text{Bow} \left(\{3\}, I_3^*, \tilde{\tau} \right) &= \{3\} & \text{Bow} \left(\{0, 1\}, I_3^*, \tilde{\tau} \right) &= \{1\} \\ \text{Bow} \left(\{0, 2\}, I_3^*, \tilde{\tau} \right) &= \{2\} & \text{Bow} \left(\{0, 3\}, I_3^*, \tilde{\tau} \right) &= \{3\} & \text{Bow} \left(\{1, 2\}, I_3^*, \tilde{\tau} \right) &= \{1, 2\} \\ \text{Bow} \left(\{1, 3\}, I_3^*, \tilde{\tau} \right) &= \{1, 3\} & \text{Bow} \left(\{2, 3\}, I_3^*, \tilde{\tau} \right) &= \{2, 3\} & \text{Bow} \left(\{0, 1, 2\}, I_3^*, \tilde{\tau} \right) &= \{1, 2\} \\ \text{Bow} \left(\{0, 1, 3\}, I_3^*, \tilde{\tau} \right) &= \{1, 3\} & \text{Bow} \left(\{0, 2, 3\}, I_3^*, \tilde{\tau} \right) &= \{2, 3\} & \text{Bow} \left(\{1, 2, 3\}, I_3^*, \tilde{\tau} \right) &= \{1, 2, 3\} \\ & & \text{Bow} \left(I_3^*, I_3^*, \tilde{\tau} \right) &= I_3^* & & \end{aligned}$$

Hence, the sets $\emptyset, I_3^*, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$, and $\{1, 2, 3\}$ are $\tilde{\tau}$ -B-open. Thus,

$$B \left(I_3^*, \tilde{\tau} \right) = \left\{ \emptyset, I_3^*, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \right\}$$

is a $\tilde{\tau}$ -B-soft topology on I_3^* .

- Consider the following:

$$\begin{array}{lll}
\text{Touch}(\emptyset, I_3^*, \tilde{\tau}) = \emptyset & \text{Touch}(\{0\}, I_3^*, \tilde{\tau}) = I_3^* & \text{Touch}(\{1\}, I_3^*, \tilde{\tau}) = \{1\} \\
\text{Touch}(\{2\}, I_3^*, \tilde{\tau}) = \{2\} & \text{Touch}(\{3\}, I_3^*, \tilde{\tau}) = \{3\} & \text{Touch}(\{0, 1\}, I_3^*, \tilde{\tau}) = I_3^* \\
\text{Touch}(\{0, 2\}, I_3^*, \tilde{\tau}) = I_3^* & \text{Touch}(\{0, 3\}, I_3^*, \tilde{\tau}) = I_3^* & \text{Touch}(\{1, 2\}, I_3^*, \tilde{\tau}) = \{1, 2\} \\
\text{Touch}(\{1, 3\}, I_3^*, \tilde{\tau}) = \{1, 3\} & \text{Touch}(\{2, 3\}, I_3^*, \tilde{\tau}) = \{2, 3\} & \text{Touch}(\{0, 1, 2\}, I_3^*, \tilde{\tau}) = I_3^* \\
\text{Touch}(\{0, 1, 3\}, I_3^*, \tilde{\tau}) = I_3^* & \text{Touch}(\{0, 2, 3\}, I_3^*, \tilde{\tau}) = I_3^* & \text{Touch}(\{1, 2, 3\}, I_3^*, \tilde{\tau}) = \{1, 2, 3\} \\
& \text{Touch}(I_3^*, I_3^*, \tilde{\tau}) = I_3^* &
\end{array}$$

Hence, the sets $\emptyset, I_3^*, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$, and $\{1, 2, 3\}$ are $\tilde{\tau}$ - T -closed. Thus,

$$T(I_3^*, \tilde{\tau}) = \{\emptyset, I_3^*, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

is a $\tilde{\tau}$ - T -soft topology on I_3^* .

- Consider the following:

$$\begin{array}{lll}
\text{Int}(\emptyset, I_3^*, \tilde{\tau}) = \emptyset & \text{Int}(\{0\}, I_3^*, \tilde{\tau}) = \{0\} & \text{Int}(\{1\}, I_3^*, \tilde{\tau}) = \emptyset \\
\text{Int}(\{2\}, I_3^*, \tilde{\tau}) = \emptyset & \text{Int}(\{3\}, I_3^*, \tilde{\tau}) = \emptyset & \text{Int}(\{0, 1\}, I_3^*, \tilde{\tau}) = \{0, 1\} \\
\text{Int}(\{0, 2\}, I_3^*, \tilde{\tau}) = \{0, 2\} & \text{Int}(\{0, 3\}, I_3^*, \tilde{\tau}) = \{0, 3\} & \text{Int}(\{1, 2\}, I_3^*, \tilde{\tau}) = \emptyset \\
\text{Int}(\{1, 3\}, I_3^*, \tilde{\tau}) = \emptyset & \text{Int}(\{2, 3\}, I_3^*, \tilde{\tau}) = \emptyset & \text{Int}(\{0, 1, 2\}, I_3^*, \tilde{\tau}) = \{0, 1, 2\} \\
\text{Int}(\{0, 1, 3\}, I_3^*, \tilde{\tau}) = \{0, 1, 3\} & \text{Int}(\{0, 2, 3\}, I_3^*, \tilde{\tau}) = \{0, 2, 3\} & \text{Int}(\{1, 2, 3\}, I_3^*, \tilde{\tau}) = \emptyset \\
& \text{Int}(I_3^*, I_3^*, \tilde{\tau}) = I_3^* &
\end{array}$$

Hence, the sets $\emptyset, I_3^*, \{0\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 1, 2\}, \{0, 1, 3\}$, and $\{0, 2, 3\}$ are $\tilde{\tau}$ - I -open. Thus,

$$I(I_3^*, \tilde{\tau}) = \{\emptyset, I_3^*, \{0\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}\}$$

is a $\tilde{\tau}$ - I -soft topology on I_3^* .

Example 6.29 Consider the proximity mapping $\tilde{\tau}$ in Example 6.19. Then, $\text{Close}(A, X, \tilde{\tau}) = A$ and $\text{Touch}(A, X, \tilde{\tau}) = A$, for all $A \subseteq X$. Thus, $C(X, \tilde{\tau}) = P(X)$ and $T(X, \tilde{\tau}) = P(X)$. Thereby, they are $\tilde{\tau}$ - C - and $\tilde{\tau}$ - T -soft discrete topologies on X , respectively. Moreover, from Corollaries 5.6 and 5.12, $B(X, \tilde{\tau}) = P(X)$ and $I(X, \tilde{\tau}) = P(X)$. Hence, they are $\tilde{\tau}$ - B - and $\tilde{\tau}$ - I -soft discrete topologies on X , respectively. Therefore, for all $A \subseteq X$, the set A is $\tilde{\tau}$ - CB - and $\tilde{\tau}$ - TI -clopen.

Example 6.30 Consider the proximity mapping $\tilde{\tau}_\delta^+$ in Example 6.6 such that $\delta(x) = 1$, for all $x \in \mathbb{R}$. Then,

$$C(\mathbb{R}, \tilde{\tau}_\delta^+) = \{(a, \infty) : a \in \mathbb{R}\} \cup \{[a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$

and

$$T(\mathbb{R}, \tilde{\tau}_\delta^+) = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(-\infty, a] : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$

From Corollaries 5.6 and 5.12,

$$B(\mathbb{R}, \tilde{\tau}_\delta^+) = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(-\infty, a] : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$

and

$$I(\mathbb{R}, \tilde{\tau}_\delta^+) = \{(a, \infty) : a \in \mathbb{R}\} \cup \{[a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$

Thus, $C(\mathbb{R}, \tilde{\tau}_\delta^+) = I(\mathbb{R}, \tilde{\tau}_\delta^+)$ and $T(\mathbb{R}, \tilde{\tau}_\delta^+) = B(\mathbb{R}, \tilde{\tau}_\delta^+)$. Hence, the following can be remarked on:

- Every $\tilde{\tau}_\delta^+$ - C -closed set is $\tilde{\tau}_\delta^+$ - I -open, and vice versa.
- Every $\tilde{\tau}_\delta^+$ - T -closed set is $\tilde{\tau}_\delta^+$ - B -open, and vice versa.

Moreover, the set $(3, 5)$ is not in $C(\mathbb{R}, \tilde{\tau}_\delta^+)$, $B(\mathbb{R}, \tilde{\tau}_\delta^+)$, $T(\mathbb{R}, \tilde{\tau}_\delta^+)$, and $I(\mathbb{R}, \tilde{\tau}_\delta^+)$ because

$$\text{Close}((3, 5), \mathbb{R}, \tilde{\tau}_\delta^+) = (3, 6)$$

$$\text{Bow}((3, 5), \mathbb{R}, \tilde{\tau}_\delta^+) = (4, 5)$$

$$\text{Touch}((3, 5), \mathbb{R}, \tilde{\tau}_\delta^+) = (2, 5)$$

and

$$\text{Int}((3, 5), \mathbb{R}, \tilde{\tau}_\delta^+) = (3, 4)$$

i.e., $\text{Close}((3, 5), \mathbb{R}, \tilde{\tau}_\delta^+)$, $\text{Bow}((3, 5), \mathbb{R}, \tilde{\tau}_\delta^+)$, $\text{Touch}((3, 5), \mathbb{R}, \tilde{\tau}_\delta^+)$, and $\text{Int}((3, 5), \mathbb{R}, \tilde{\tau}_\delta^+)$ are not equal to $(3, 5)$. Thus, the set $(3, 5)$ is $\tilde{\tau}$ - CB - and $\tilde{\tau}$ - TI -nclnopen.

7. Conclusion

This study makes an original, comprehensive, and systematic contribution to soft topological structures introduced by Molodtsov in [6,7]. In particular, it provides a detailed structural analysis of the following concepts, many of which have not been thoroughly investigated and clarified in the relevant literature:

- $\tilde{\tau}$ -neighborhood of a point
- $\tilde{\tau}$ -neighborhood of a set
- $\tilde{\tau}$ -closure of a set
- $\tilde{\tau}$ -interior of a set
- $\tilde{\tau}$ -boundary of a set
- $\tilde{\tau}$ -border set of a set
- $\tilde{\tau}$ -hidden set of a set
- $\tilde{\tau}$ -accumulation points of a set
- $\tilde{\tau}$ -cluster points of a set
- $\tilde{\tau}$ -isolated points of a set
- $\tilde{\tau}$ -separated points of a set

Beyond consolidating existing notions, another original contribution of this paper is introducing novel concepts within the framework and exploring their structural properties: $\tilde{\tau}$ -disjointness, $\tilde{\tau}$ -exterior, and $\tilde{\tau}$ -condensation points of a set, together with $\tilde{\tau}$ - C -closed, $\tilde{\tau}$ - B -open, $\tilde{\tau}$ - T -closed, and $\tilde{\tau}$ - I -open sets and the corresponding $\tilde{\tau}$ - C -, $\tilde{\tau}$ - B -, $\tilde{\tau}$ - T -, and $\tilde{\tau}$ - I -soft topologies. Finally, this study illustrates the aforementioned concepts through detailed examples.

In this context, the study's main contributions can be summarized as follows: This study demonstrates that

- A set is $\tilde{\tau}$ - C -closed if and only if its complement is $\tilde{\tau}$ - B -open,
- A set is $\tilde{\tau}$ - CB -clopen if and only if its $\tilde{\tau}$ -border set is empty,
- A set is $\tilde{\tau}$ - C -closed if and only if it contains its $\tilde{\tau}$ -cluster set,
- Any union of $\tilde{\tau}$ - C -closed sets is $\tilde{\tau}$ - C -closed,
- Any intersection of $\tilde{\tau}$ - C -closed sets is $\tilde{\tau}$ - C -closed,

- Any union of $\tilde{\tau}$ - B -open sets is $\tilde{\tau}$ - B -open,
- Any intersection of $\tilde{\tau}$ - B -open sets is $\tilde{\tau}$ - B -open,
- A set is $\tilde{\tau}$ - T -closed if and only if its complement is $\tilde{\tau}$ - I -open,
- A set is $\tilde{\tau}$ - TI -clopen if and only if its $\tilde{\tau}$ -boundary is empty,
- A set is $\tilde{\tau}$ - T -closed if and only if it contains its $\tilde{\tau}$ -accumulation set,
- Any union of $\tilde{\tau}$ - T -closed sets is $\tilde{\tau}$ - T -closed,
- Any intersection of $\tilde{\tau}$ - T -closed sets is $\tilde{\tau}$ - T -closed,
- Any union of $\tilde{\tau}$ - I -open sets is $\tilde{\tau}$ - I -open,
- Any intersection of $\tilde{\tau}$ - I -open sets is $\tilde{\tau}$ - I -open,
- Every $\tilde{\tau}$ - C -soft topology is also a $\tilde{\tau}$ - I -soft topology, and vice versa, and
- Every $\tilde{\tau}$ - T -soft topology is also a $\tilde{\tau}$ - B -soft topology, and vice versa,

It also explores

- The relationships between the concepts proposed by Molodtsov [6],
- The monotonicity of the concepts of $\tilde{\tau}$ -accumulation, $\tilde{\tau}$ -isolated, and $\tilde{\tau}$ -separated points,
- The properties related to the intersection and union of families of sets regarding the concepts of $\tilde{\tau}$ -accumulation, $\tilde{\tau}$ -isolated, and $\tilde{\tau}$ -separated points,
- The properties related to the concepts of $\tilde{\tau}$ -disjointness, $\tilde{\tau}$ -exterior, and $\tilde{\tau}$ -condensation points,
- The monotonicity of the concepts of $\tilde{\tau}$ -disjointness, $\tilde{\tau}$ -exterior, and $\tilde{\tau}$ -condensation points,
- The properties of the intersection and union of families of sets concerning the concepts of $\tilde{\tau}$ -disjointness, $\tilde{\tau}$ -exterior, and $\tilde{\tau}$ -condensation points,
- The relationships between $\tilde{\tau}$ - C -closed and $\tilde{\tau}$ - B -open sets and their $\tilde{\tau}$ -boundaries and $\tilde{\tau}$ -disjointness sets, and
- The relationships between $\tilde{\tau}$ - T -closed and $\tilde{\tau}$ - I -open sets and their $\tilde{\tau}$ -border and $\tilde{\tau}$ -exterior sets.

The concepts discussed herein constitute fundamental components of Molodtsov's soft topological structures. Consequently, they provide a natural framework for further investigations. In particular, the properties established in this study can be analyzed through the proximity mappings presented in Section 6. Moreover, since [6,7] introduce numerous soft topological notions, future research can systematically study separable, divisible, dense, bounded, and connected sets within the structural framework developed herein.

Conflict of Interest

All authors declare no conflict of interest.

Data Availability Statement

Data sharing does not apply to this paper as no new data were created or analyzed in this study.

Author Contributions

All authors equally contributed to this paper. They have all read and approved the final version of the paper.

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