Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 2** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.79251

A Modified Proximal Point Algorithm for a Hybrid Pair of Mappings in Geodesic Space

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ABSTRACT: In this paper, we propose a new modified proximal point algorithm for solving minimization problems and common fixed point problem in CAT(0) spaces. We prove Δ and strong convergence of the proposed algorithm. Our results extend and improve the corresponding recent results in the literature.

Key Words: Minimization problem, resolvent operator, CAT(0) space, proximal point algorithm, nonexpansive mappings.

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1. Introduction

Let $(\tilde{X}, \tilde{d})$ be a geodesic metric space and $\tilde{f} : \tilde{X} \to (-\infty, \infty]$ be a proper and convex function. One of the major problem in optimization is to find $\tilde{p} \in \tilde{X}$ such that

$$\tilde{f}(\tilde{p}) = \min_{\tilde{q} \in \tilde{X}} \tilde{f}(\tilde{q}). \tag{1.1}$$

We denote by

$$arg \min_{\tilde{q} \in \tilde{X}} \tilde{f}(\tilde{q}),$$

the set of a minimizer of a convex function. One of the most effective way of solving problem (1.1) is the Proximal Point Algorithm (for short term, PPA). Its origin goes back to Martinet [3], Rockafellar [4], and Brézis and Lions [5]. Martinet studied the PPA for variational inequalities whereas Rockafellar showed the weak convergence of the sequence generated by the proximal point algorithm to a zero of the maximal monotone operator in Hilbert spaces. Güler's counterexample [6] showed that the sequence generated by the proximal point algorithm does not necessarily converge strongly even if the maximal monotone operator is the subdifferential of a convex, proper, and lower semicontinuous function. Kamimura and Takahashi [7] combined the PPA with Halpern's algorithm [8] so that the strong convergence is guaranteed. The proximal point algorithm can be used in numerous problems such as equilibrium problems, saddle point problems, convex minimization problems, and variational inequality problems.
Recently, many convergence results for the PPA for solving optimization problems have been extended from the classical linear spaces such as Euclidean spaces, Hilbert spaces and Banach spaces to the setting of manifolds ([9,10,11,12]). The minimizers of the objective convex functionals in the spaces with nonlinearity play a crucial role in the branch of analysis and geometry. Numerous applications in computer vision, machine learning, electronic structure computation, system balancing and robot manipulation can

* Corresponding author.
2010 *Mathematics Subject Classification:* 47H09, 47H10.
Submitted September 29, 2025. Published December 19, 2025

be considered as solving optimization problems on manifolds (see [13,14,15,16]).
In 2014, Bačák [17] obtained few results using the proximal point algorithm in CAT(0) spaces. Also, he employed a splitting version of the PPA to find minimizer of a sum of convex functions, thereby extending the results of Bertsekas [18] into Hadamard spaces. Following this, many mathematicians have obtained numerous results involving the proximal point algorithm in the framework of CAT(0) spaces (see [19,20,21,22,23]).

Fascinated by the ongoing research, in this paper, we propose a new modified proximal point algorithm for finding a common element of the set of fixed points of three single-valued nonexpansive mappings, the set of fixed points of three multi-valued nonexpansive mappings and the set of minimizers of convex and lower semi-continuous functions. We prove few convergence results for the proposed algorithm under some mild conditions.

2. Preliminaries

In this section, we present some fundamental concepts, definitions, and some results, which will be used in the next section.
A metric space $(\tilde{X}, \tilde{d})$ is said to be a CAT(0) space if it is geodesically connected, and if every geodesic triangle in $\tilde{X}$ is at least as thin as its comparison triangle in the Euclidean plane (see more details in [24]). A complete CAT(0) space is then called a Hadamard space. Euclidean spaces, Hilbert spaces, the Hilbert ball [25], hyperbolic spaces [26], R-tress [27] and a complete, simply connected Riemannian manifold having non-positive sectional curvature are some examples of a CAT(0) space.

Definition 1 *A subset $\tilde{E}$ of a CAT(0) space $\tilde{X}$ is said to be convex if $\tilde{E}$ includes every geodesic segment joining ant two of its points, that is, for any $\tilde{p}, \tilde{q} \in \tilde{E}$, we have $[\tilde{p}, \tilde{q}] \subset \tilde{E}$, where $[\tilde{p}, \tilde{q}] := \{\alpha\tilde{p} \oplus (1-\alpha)\tilde{q} : 0 \le \alpha \le 1\}$ is the unique geodesic joining $\tilde{p}$ and $\tilde{q}$.*

Definition 2 *A single-valued mapping $\tilde{G} : \tilde{E} \to \tilde{E}$ is said to be*

(i) Nonexpansive if $\tilde{d}(\tilde{G}\tilde{p}, \tilde{G}\tilde{q}) \le \tilde{d}(\tilde{p}, \tilde{q})$ for all $\tilde{p}, \tilde{q} \in \tilde{E}$;

(ii) Semi-compact if for any sequence $\{\tilde{p}_t\}$ in $\tilde{E}$ such that

$$\lim_{t\to\infty} \tilde{d}(\tilde{G}\tilde{p}_t, \tilde{p}_t) = 0,$$

there exist a subsequence $\{\tilde{p}_{t_i}\}$ of $\{\tilde{p}_t\}$ such that $\{\tilde{p}_{t_i}\}$ converges strongly to $\tilde{p}^* \in \tilde{E}$.

We denote the set of all fixed points of $\tilde{G}$ is denoted by $F(\tilde{G})$. Now, we state the following Lemma to be used later on.

Lemma 2.1 *([28]) Let $(\tilde{X}, \tilde{d})$ be a CAT(0) space, then the following assertions hold:*

(i) For $\tilde{p}, \tilde{q} \in \tilde{X}$ and $n \in [0,1]$, there exists a unique $\tilde{r} \in [\tilde{p}, \tilde{q}]$ such that

$$\tilde{d}(\tilde{p}, \tilde{r}) = n\tilde{d}(\tilde{p}, \tilde{q}) \text{ and } \tilde{d}(\tilde{q}, \tilde{r}) = (1-n)\tilde{d}(\tilde{p}, \tilde{q}).$$

(ii) For $\tilde{p}, \tilde{q}, \tilde{r} \in \tilde{X}$ and $n \in [0,1]$, we have

$$\tilde{d}((1-n)\tilde{p} \oplus n\tilde{q}, \tilde{r}) \le (1-n)\tilde{d}(\tilde{p}, \tilde{r}) + n\tilde{d}(\tilde{q}, \tilde{r})$$

and

$$\tilde{d}^2((1-n)\tilde{p} \oplus n\tilde{q}, \tilde{r}) \le (1-n)\tilde{d}^2(\tilde{p}, \tilde{r}) + n\tilde{d}^2(\tilde{q}, \tilde{r}) - n(1-n)\tilde{d}^2(\tilde{p}, \tilde{q}).$$

We use the notation $(1-n)\tilde{p} \oplus n\tilde{q}$ for the unique point $\tilde{r}$ of the above Lemma.
Now, we collect some basic geometric properties which are instrumental throughout the discussions.
Let $\{\tilde{p}_t\}$ be a bounded sequence in a complete CAT(0) space $\tilde{X}$. For $\tilde{p} \in \tilde{X}$ we write:

$$r(\tilde{p}, \{\tilde{p}_t\}) = \limsup_{t\to\infty} \tilde{d}(\tilde{p}, \tilde{p}_t).$$

The asymptotic radius $r(\{\tilde{p}_t\})$ is given by

$$r(\{\tilde{p}_t\}) = \inf\{r(\tilde{p}, \tilde{p}_t) : \tilde{p} \in \tilde{X}\}$$

and the asymptotic center $A(\{\tilde{p}_t\})$ of $\{\tilde{p}_t\}$ is defined as:

$$A(\{\tilde{p}_t\}) = \{\tilde{p} \in \tilde{X} : r(\tilde{p}, \tilde{p}_t) = r(\{\tilde{p}_t\})\}.$$

It is well known that, in a complete CAT(0) space, $A(\{\tilde{p}_t\})$ consists of exactly one point [29]. We now present the definition and some basic properties of the Δ- convergence which will be fruitful for our subsequent discussion.

Definition 3 *([30]) A sequence $\{\tilde{p}_t\}$ in a CAT(0) space $\tilde{X}$ is said to be Δ-convergent to a point $\tilde{p} \in \tilde{X}$ if $\tilde{p}$ is the unique asymptotic center of $\{u_t\}$ for every subsequence $\{u_t\}$ of $\{\tilde{p}_t\}$. In this case, we write $\Delta - \lim_{t\to\infty} \tilde{p}_t = \tilde{p}$ and call $\tilde{p}$ the Δ-limit of $\{\tilde{p}_t\}$.*

Lemma 2.2 *([30]) Every bounded sequence in a complete CAT(0) space admits a Δ-convergent subsequence.*

Lemma 2.3 *([31]) If $\tilde{E}$ is a closed convex subset of a complete CAT(0) space $\tilde{X}$ and if $\{\tilde{p}_t\}$ is a bounded sequence in $\tilde{E}$, then the asymptotic center of $\{\tilde{p}_t\}$ is in $\tilde{E}$.*

Lemma 2.4 *([28]) Let $\tilde{E}$ be a nonempty closed convex subset of a complete CAT(0) space $(\tilde{X}, \tilde{d})$ and $\tilde{G} : \tilde{E} \to \tilde{E}$ be a nonexpansive mapping. If $\{\tilde{p}_t\}$ is a bounded sequence in $\tilde{E}$ such that $\Delta - \lim_t \tilde{p}_t = \tilde{p}$ and $\lim_{t\to\infty} \tilde{d}(\tilde{G}\tilde{p}_t, \tilde{p}_t) = 0$, then $\tilde{p}$ is a fixed point of $\tilde{G}$.*

Lemma 2.5 *([28]) If $\{\tilde{p}_t\}$ is a bounded sequence in a complete CAT(0) space with $A(\{\tilde{p}_t\}) = \{\tilde{p}\}$, $\{u_t\}$ is a subsequence of $\{\tilde{p}_t\}$ with $A(\{u_t\}) = \{u\}$ and the sequence $\{\tilde{d}(\tilde{p}_t, u)\}$ converges, then $\tilde{p} = u$.*

Lemma 2.6 *([32]) Let $\tilde{E}$ be a nonempty closed and convex subset of a CAT(0) space $\tilde{X}$. Then, for any $\{\tilde{p}_i\}_{i=1}^t \in \tilde{E}$ and $\alpha_i \in (0, 1)$, $i = 1, 2, ..., t$ with $\sum_{i=1}^t \alpha_i = 1$, we have the following inequalities:*

$$\tilde{d}(\oplus_{i=1}^t \alpha_i \tilde{p}_i, \tilde{r}) \leq \sum_{i=1}^t \alpha_i \tilde{d}(\tilde{p}_i, \tilde{r}), \quad \forall \ \tilde{r} \in \tilde{E} \tag{2.1}$$

and

$$\tilde{d}^2(\oplus_{i=1}^t \alpha_i \tilde{p}_i, \tilde{r}) \leq \sum_{i=1}^t \alpha_i \tilde{d}^2(\tilde{p}_i, \tilde{r}) - \sum_{i,j=1, i\neq j}^t \tilde{d}^2(\tilde{p}_i, \tilde{p}_j), \quad \forall \ \tilde{r} \in \tilde{E}. \tag{2.2}$$

A function $\tilde{f} : \tilde{E} \to (-\infty, \infty]$ defined on a convex subset $\tilde{E}$ of a CAT(0) space is convex if, for any geodesic $\gamma : [a, b] \to \tilde{E}$, the function $\tilde{f} o \gamma$ is convex, i.e., $\tilde{f}(\alpha\tilde{p} \oplus (1 - \alpha)\tilde{q}) \leq \alpha\tilde{f}(\tilde{p}) + (1 - \alpha)\tilde{f}(\tilde{q})$ for all $\tilde{p}, \tilde{q} \in \tilde{E}$. For some important examples one can refer [33]. Now, a function $\tilde{f}$ defined on $\tilde{E}$ is said to be lower semi-continuous at $\tilde{p} \in \tilde{E}$ if

$$\tilde{f}(\tilde{p}) \leq \lim \inf_{t\to\infty} \tilde{f}(\tilde{p}_t)$$

for each sequence $\{\tilde{p}_t\}$ such that $\tilde{p}_t \to \tilde{p}$ as $t \to \infty$. A function $\tilde{f}$ is said to be lower semi-continuous on $\tilde{E}$ if it is lower semi-continuous at any point in $\tilde{E}$.

For any $\tilde{\lambda} > 0$, define the Moreau-Yosida resolvent of $\tilde{f}$ in CAT(0) space as follows:

$$J_{\tilde{\lambda}}(\tilde{p}) = arg \min_{\tilde{q}\in\tilde{E}} [\tilde{f}(\tilde{q}) + \frac{1}{2\tilde{\lambda}} \tilde{d}^2(\tilde{q}, \tilde{p})]$$

for all $\tilde{p} \in \tilde{E}$. Now, we list few results which will be used in the sequel.

Lemma 2.7 *([33]) Let $(\tilde{X},\tilde{d})$ be a complete CAT(0) space and $\tilde{f}:\tilde{E}\to(-\infty,\infty]$ be a proper, convex and lower semi-continuous function, then the set $F(J_{\tilde{\lambda}})$ of the fixed point of the resolvent $J_{\tilde{\lambda}}$ associated with $\tilde{f}$ coincides with the set $arg\min_{\tilde{q}\in\tilde{E}}\tilde{f}(\tilde{q})$ of minimizers of $\tilde{f}$.*

Lemma 2.8 *([34]) For any $\tilde{\lambda}>0$, the resolvent $J_{\tilde{\lambda}}$ of f is nonexpansive.*

Lemma 2.9 *([35]) Let $(\tilde{X},\tilde{d})$ be a complete CAT(0) space and $\tilde{f}:\tilde{E}\to(-\infty,\infty]$ be a proper, convex and lower semi-continuous function, then for all $\tilde{p},\tilde{q}\in\tilde{X}$ and $\tilde{\lambda}>0$, we have*

$$\frac{1}{2\tilde{\lambda}}\tilde{d}^2(J_{\tilde{\lambda}}\tilde{p},\tilde{q})-\frac{1}{2\tilde{\lambda}}\tilde{d}^2(\tilde{p},\tilde{q})+\frac{1}{2\tilde{\lambda}}\tilde{d}^2(\tilde{p},J_{\tilde{\lambda}}\tilde{p})+\tilde{f}(J_{\tilde{\lambda}}\tilde{p})\leq\tilde{f}(\tilde{q}).$$

Lemma 2.10 *([34,36]) Let $(\tilde{X},\tilde{d})$ be a complete CAT(0) space and $\tilde{f}:\tilde{E}\to(-\infty,\infty]$ be a proper, convex and lower semi-continuous function. Then the following identity holds:*

$$J_{\tilde{\lambda}}\tilde{p}=J_{\mu}(\frac{\tilde{\lambda}-\mu}{\tilde{\lambda}}J_{\tilde{\lambda}}\tilde{p}\oplus\frac{\mu}{\tilde{\lambda}}\tilde{p})$$

for all $\tilde{p}\in\tilde{X}$ and $\tilde{\lambda}>\mu>0$.

Let CB($\tilde{E}$), CC($\tilde{E}$) and KC($\tilde{E}$) denote the families of nonempty closed bounded subsets, closed convex subsets and compact convex subsets of $\tilde{E}$, respectively. The Pompeiu-Hausdorff distance [37] on CB($\tilde{E}$) is defined by

$$H(A,B)=max\{\sup_{\tilde{p}\in A}dist(\tilde{p},B),\sup_{\tilde{q}\in B}dist(\tilde{q},A)\}$$

for $A,B\in CB(\tilde{E})$, where $dist(\tilde{p},\tilde{E})=inf\{d(\tilde{p},\tilde{q}):\tilde{q}\in\tilde{E}\}$ is the distance from a point $\tilde{p}$ to a subset $\tilde{E}$. An element $\tilde{p}\in\tilde{E}$ is said to be a fixed point of a multi-valued mapping $\tilde{S}:\tilde{E}\to CB(\tilde{E})$ if $\tilde{p}\in\tilde{S}\tilde{p}$. We denote the set of all fixed points of $\tilde{S}$ by $F(\tilde{S})$.

Definition 4 *A multi-valued mapping $\tilde{S}:\tilde{E}\to CB(\tilde{E})$ is said to be*

(i) Nonexpansive if $H(\tilde{S}\tilde{p},\tilde{S}\tilde{q})\leq\tilde{d}(\tilde{p},\tilde{q})$ for all $\tilde{p},\tilde{q}\in\tilde{E}$;

(ii) Hemi-compact if for any sequence $\{\tilde{p}_t\}$ in $\tilde{E}$ with $\lim_{t\to\infty}dist(\tilde{S}\tilde{p}_t,\tilde{p}_t)=0$, there exist a subsequence $\{\tilde{p}_{t_i}\}$ of $\{\tilde{p}_t\}$ such that $\{\tilde{p}_{t_i}\}$ converges strongly to $\tilde{p}^*\in\tilde{E}$.

3. Main Results

Theorem 3.1 *Let $\tilde{E}$ be a nonempty closed and convex subset of a complete CAT(0) space $\tilde{X}$. Let $\tilde{G}_n:\tilde{E}\to\tilde{E}$, $n=1,2,3$ be single-valued nonexpansive mappings, $\tilde{S}_n:\tilde{E}\to CB(\tilde{E})$, $n=1,2,3$ be multi-valued nonexpansive mappings and $\tilde{f}:\tilde{E}\to(-\infty,\infty]$ be a proper convex and lower semi-continuous function. Suppose that $\Omega=F(\tilde{G}_1)\cap F(\tilde{G}_2)\cap F(\tilde{G}_3)\cap F(\tilde{S}_1)\cap F(\tilde{S}_2)\cap F(\tilde{S}_3)\cap arg\min_{\tilde{q}\in\tilde{E}}\tilde{f}(\tilde{q})\neq\emptyset$ and $\tilde{S}_n\tilde{x}=\{\tilde{x}\}$, $n=1,2,3$ for $\tilde{x}\in\Omega$. For $\tilde{p}_1\in\tilde{E}$, let the sequence $\{\tilde{p}_t\}$ is generated in the following manner:*

$$\begin{cases} w_t=arg\min_{\tilde{q}\in\tilde{E}}[\tilde{f}(\tilde{q})+\frac{1}{2\tilde{\lambda}_t}\tilde{d}^2(\tilde{q},\tilde{p}_t)],\\ \tilde{r}_t=\alpha_t\tilde{p}_t\oplus\beta_t w_t'\oplus\gamma_t w_t'',\\ \tilde{q}_t=\psi_t\tilde{p}_t\oplus\kappa_t w_t'''\oplus\phi_t\tilde{G}_1\tilde{p}_t,\\ \tilde{p}_{t+1}=\delta_t\tilde{G}_1\tilde{p}_t\oplus\eta_t\tilde{G}_2\tilde{p}_t\oplus\xi_t\tilde{G}_3\tilde{q}_t,\quad for\quad all\quad t\in\mathbb{N}\end{cases}\tag{3.1}$$

where $\{\alpha_t\}$, $\{\beta_t\}$, $\{\gamma_t\}$, $\{\psi_t\}$, $\{\kappa_t\}$, $\{\phi_t\}$, $\{\delta_t\}$, $\{\eta_t\}$ and $\{\xi_t\}$ are sequences in $(0,1)$ such that

$$0<\tilde{a}\leq\alpha_t,\beta_t,\gamma_t,\psi_t,\kappa_t,\phi_t,\delta_t,\eta_t,\xi_t\leq\tilde{b}<1,$$

$$\alpha_t+\beta_t+\gamma_t=1,\psi_t+\kappa_t+\phi_t=1,\delta_t+\eta_t+\xi_t=1,$$

for all $t\in\mathbb{N}$ and $\{\tilde{\lambda}_t\}$ is a sequence such that $\tilde{\lambda}_t\geq\tilde{\lambda}>0$ for all $t\in\mathbb{N}$ and some $\tilde{\lambda}$. Then, the following statements hold:

(i) $\lim_{t\to\infty} \tilde{d}(\tilde{p}_t, \tilde{x})$ *exists for all* $\tilde{x} \in \Omega$*;*

(ii) $\lim_{t\to\infty} \tilde{d}(\tilde{p}_t, w_t) = 0$*;*

(iii) $\lim_{t\to\infty} dist(\tilde{p}_t, \tilde{S}_n\tilde{p}_t) = 0, n = 1, 2, 3$*;*

(iv) $\lim_{t\to\infty} \tilde{d}(\tilde{p}_t, \tilde{G}_n\tilde{p}_t) = 0, n = 1, 2, 3$*;*

(v) $\lim_{t\to\infty} \tilde{d}(\tilde{p}_t, J_{\tilde{\lambda}}\tilde{p}_t) = 0$.

Proof 1 *Let* $\tilde{x} \in \Omega$*, then*

$$\tilde{x} = \tilde{G}_1\tilde{x} = \tilde{G}_2\tilde{x} = \tilde{G}_3\tilde{x} \in (\tilde{S}_1\tilde{x} \cap \tilde{S}_2\tilde{x} \cap \tilde{S}_3\tilde{x})$$

and

$$\tilde{f}(\tilde{x}) \le \tilde{f}(\tilde{q}), \quad \forall \ \ \tilde{q} \in \tilde{E}.$$

Therefore, we have

$$\tilde{f}(\tilde{x}) + \frac{1}{2\tilde{\lambda}_t} d^2(\tilde{x}, \tilde{x}) \le \tilde{f}(\tilde{q}) + \frac{1}{2\tilde{\lambda}_t}\tilde{d}^2(\tilde{q}, \tilde{x}),$$

for all $\tilde{q} \in \tilde{E}$ *and hence* $\tilde{x} = J_{\tilde{\lambda}}\tilde{x}$*.*

(i) *Note that* $w_t = J_{\tilde{\lambda}_t}\tilde{p}_t$ *and* $J_{\tilde{\lambda}_t}$ *is nonexpansive map for each* $t \in \mathbb{N}$*. So, we have*

$$\tilde{d}(w_t, \tilde{x}) = \tilde{d}(J_{\tilde{\lambda}_t}\tilde{p}_t, J_{\tilde{\lambda}_t}\tilde{x}) \le \tilde{d}(\tilde{p}_t, \tilde{x}). \tag{3.2}$$

As $\tilde{x} \in \tilde{S}_n(\tilde{x})$ *for* $n = 1, 2, 3$*, using Lemma 2.6 and (3.2) we have*

$$\begin{aligned} \tilde{d}(\tilde{r}_t, \tilde{x}) &= \tilde{d}(\alpha_t\tilde{p}_t \oplus \beta_t w'_t \oplus \gamma_t w''_t, \tilde{x}) \\ &\le \alpha_t\tilde{d}(\tilde{p}_t, \tilde{x}) + \beta_t\tilde{d}(w'_t, \tilde{x}) + \gamma_t\tilde{d}(w''_t, \tilde{x}) \\ &\le \alpha_t\tilde{d}(\tilde{p}_t, \tilde{x}) + \beta_t\tilde{d}(\tilde{S}_1\tilde{p}_t, \tilde{S}_1\tilde{x}) + \gamma_t\tilde{d}(\tilde{S}_2 w_t, \tilde{S}_2\tilde{x}) \\ &\le \tilde{d}(\tilde{p}_t, \tilde{x}) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \tilde{d}(\tilde{q}_n, \tilde{x}) &= \tilde{d}(\psi_t\tilde{p}_t \oplus \kappa_t w'''_t \oplus \phi_t\tilde{G}_1\tilde{p}_t, \tilde{x}) \\ &\le \psi_t\tilde{d}(\tilde{p}_t, \tilde{x}) + \kappa_t\tilde{d}(w'''_t, \tilde{x}) + \phi_t\tilde{d}(\tilde{G}_1\tilde{p}_t, \tilde{x}) \\ &\le \psi_t\tilde{d}(\tilde{p}_t, \tilde{x}) + \kappa_t\tilde{d}(\tilde{S}_3\tilde{r}_t, \tilde{x}) + \phi_t\tilde{d}(\tilde{G}_1\tilde{p}_t, \tilde{x}) \\ &\le \tilde{d}(\tilde{p}_t, \tilde{x}). \end{aligned} \tag{3.4}$$

Now, consider

$$\begin{aligned} \tilde{d}(\tilde{p}_{t+1}, \tilde{x}) &= \tilde{d}(\delta_t\tilde{G}_1\tilde{p}_t \oplus \eta_t\tilde{G}_2\tilde{p}_t \oplus \xi_t\tilde{G}_3\tilde{q}_t, \tilde{x}) \\ &\le \delta_t\tilde{d}(\tilde{G}_1\tilde{p}_t, \tilde{x}) + \eta_t\tilde{d}(\tilde{G}_2\tilde{p}_t, \tilde{x}) + \xi_t\tilde{d}(\tilde{G}_3\tilde{q}_t) \\ &\le \tilde{d}(\tilde{p}_t, \tilde{x}). \end{aligned} \tag{3.5}$$

This shows that $\lim_{t\to\infty} \tilde{d}(\tilde{p}_t, \tilde{x})$ *exists and so we assume that*

$$\lim_{t\to\infty} \tilde{d}(\tilde{p}_t, \tilde{x}) = \tilde{y} \ge 0. \tag{3.6}$$

(ii) *Next, we show that* $\lim_{t\to\infty} \tilde{d}(\tilde{p}_t, w_t) = 0$*. By Lemma 2.9 , we get*

$$\frac{1}{2\tilde{\lambda}_t}\{\tilde{d}^2(w_t, \tilde{x}) - \tilde{d}^2(\tilde{p}_t, \tilde{x}) + \tilde{d}^2(\tilde{p}_t, w_t)\} \le \tilde{f}(\tilde{x}) - \tilde{f}(w_t).$$

Since $\tilde{f}(\tilde{x}) \leq \tilde{f}(w_t)$ for each $t \in \mathbb{N}$, it follows that

$$\tilde{d}^2(\tilde{p}_t, w_t) \leq \tilde{d}^2(\tilde{p}_t, \tilde{x}) - \tilde{d}^2(w_t, \tilde{x}). \tag{3.7}$$

So, in order to show that $\lim_{t\to\infty} \tilde{d}(\tilde{p}_t, w_t) = 0$, it is sufficient to show that

$$\lim_{t\to\infty} \tilde{d}(w_t, \tilde{x}) = \tilde{y}.$$

From (3.3), we have

$$\limsup_{t\to\infty} \tilde{d}(\tilde{r}_t, \tilde{x}) \leq \limsup_{t\to\infty} \tilde{d}(\tilde{p}_t, \tilde{x}) = \tilde{y}. \tag{3.8}$$

Also, using (3.4), we get

$$\limsup_{t\to\infty} \tilde{d}(\tilde{q}_t, \tilde{x}) \leq \limsup_{t\to\infty} \tilde{d}(\tilde{p}_t, \tilde{x}) = \tilde{y}. \tag{3.9}$$

Using (3.5) along with the fact that $\delta_t + \eta_t + \xi_t = 1$ for all $t \geq 1$, we obtain

$$\begin{aligned} \tilde{d}(\tilde{p}_{t+1}, \tilde{x}) &\leq \delta_t \tilde{d}(\tilde{G}_1 \tilde{p}_t, \tilde{x}) + \eta_t \tilde{d}(\tilde{G}_2 \tilde{p}_t, \tilde{x}) + \xi_t \tilde{d}(\tilde{G}_3 \tilde{q}_t, \tilde{x}) \\ &\leq (1 - \xi_t)\tilde{d}(\tilde{p}_t, \tilde{x}) + \xi_t \tilde{d}(\tilde{q}_t, \tilde{x}), \end{aligned}$$

which is same as

$$\begin{aligned} \tilde{d}(\tilde{p}_t, \tilde{x}) &\leq \frac{1}{\xi_t}[\tilde{d}(\tilde{p}_t, \tilde{x}) - \tilde{d}(\tilde{p}_{t+1}, \tilde{x})] + \tilde{d}(\tilde{q}_t, \tilde{x}) \\ &\leq \frac{1}{\tilde{a}}[\tilde{d}(\tilde{p}_t, \tilde{x}) - \tilde{d}(\tilde{p}_{t+1}, \tilde{x})] + \tilde{d}(\tilde{q}_t, \tilde{x}), \end{aligned}$$

which gives

$$\liminf_{t\to\infty} \tilde{d}(\tilde{p}_t, \tilde{x}) \leq \liminf_{t\to\infty} \{\frac{1}{\tilde{a}}[\tilde{d}(\tilde{p}_t, \tilde{x}) - \tilde{d}(\tilde{p}_{t+1}, \tilde{x})] + \tilde{d}(\tilde{q}_t, \tilde{x})\}.$$

On using (3.6), we get

$$\tilde{y} \leq \liminf_{t\to\infty} \tilde{d}(\tilde{q}_t, \tilde{x}). \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$\lim_{t\to\infty} \tilde{d}(\tilde{q}_t, \tilde{x}) = \tilde{y}. \tag{3.11}$$

Similarly, (3.4) yields

$$\begin{aligned} \tilde{d}(\tilde{q}_t, \tilde{x}) &\leq \psi_t \tilde{d}(\tilde{p}_t, \tilde{x}) + \kappa_t \tilde{d}(\tilde{r}_t, \tilde{x}) + \phi_t \tilde{d}(\tilde{p}_t, \tilde{x}) \\ &\leq \tilde{d}(\tilde{p}_t, \tilde{x}) - \kappa_t \tilde{d}(\tilde{p}_t, \tilde{x}) + \kappa_t \tilde{d}(\tilde{r}_t, \tilde{x}), \end{aligned}$$

which results into

$$\begin{aligned} \tilde{d}(\tilde{p}_t, \tilde{x}) &\leq \frac{1}{\kappa_t}[\tilde{d}(\tilde{p}_t, \tilde{x}) - \tilde{d}(\tilde{q}_t, \tilde{x})] + \tilde{d}(\tilde{r}_t, \tilde{x}) \\ &\leq \frac{1}{\tilde{a}}[\tilde{d}(\tilde{p}_t, \tilde{x}) - \tilde{d}(\tilde{q}_t, \tilde{x})] + \tilde{d}(\tilde{r}_t, \tilde{x}), \end{aligned}$$

which on using (3.6) and (3.11) gives

$$\tilde{y} \leq \liminf_{t\to\infty} \tilde{d}(\tilde{r}_t, \tilde{x}). \tag{3.12}$$

From (3.8) and (3.12), we get

$$\lim_{t\to\infty} \tilde{d}(\tilde{r}_t, \tilde{x}) = \tilde{y}. \tag{3.13}$$

Now, on using (3.3), we have

$$\tilde{d}(\tilde{p}_t, \tilde{x}) \leq \frac{1}{\tilde{a}}[\tilde{d}(\tilde{p}_t, \tilde{x}) - \tilde{d}(\tilde{r}_t, \tilde{x})] + \tilde{d}(w_t, \tilde{x}),$$

which along with (3.6) and (3.13) gives

$$\tilde{y} \le \liminf_{t\to\infty} \tilde{d}(w_t, \tilde{x}). \tag{3.14}$$

Also, (3.2) results into

$$\limsup_{t\to\infty} \tilde{d}(w_t, \tilde{x}) \le \limsup_{t\to\infty} \tilde{d}(\tilde{p}_t, \tilde{x}) = \tilde{y}. \tag{3.15}$$

On using (3.14) and (3.15), we obtain

$$\lim_{t\to\infty} \tilde{d}(w_t, \tilde{x}) = \tilde{y}. \tag{3.16}$$

From (3.6), (3.7) and (3.16), we get

$$\lim_{t\to\infty} \tilde{d}(\tilde{p}_t, w_t) = 0. \tag{3.17}$$

(iii) Now, we prove $\lim_{t\to\infty} \tilde{d}(\tilde{p}_t, \tilde{S}_n\tilde{p}_t) = 0$ *for* $n = 1, 2, 3$.
Consider

$$\begin{aligned}
\tilde{d}^2(\tilde{r}_t, \tilde{x}) &= \tilde{d}^2(\alpha_t\tilde{p}_t \oplus \beta_t w'_t \oplus \gamma_t w''_t, \tilde{x}) \\
&\le \alpha_t\tilde{d}^2(\tilde{p}_t, \tilde{x}) + \beta_t\tilde{d}^2(w'_t, \tilde{x}) + \gamma_t\tilde{d}^2(w''_t, \tilde{x}) \\
&\quad -\alpha_t\beta_t\tilde{d}^2(\tilde{p}_t, w'_t) - \alpha_t\gamma_t\tilde{d}^2(\tilde{p}_t, w''_t) - \beta_t\gamma_t\tilde{d}^2(w'_t, w''_t) \\
&\le \tilde{d}^2(\tilde{p}_t, \tilde{x}) - \alpha_t\beta_t\tilde{d}^2(\tilde{p}_t, w'_t) - \alpha_t\gamma_t\tilde{d}^2(\tilde{p}_t, w''_t) - \beta_t\gamma_t\tilde{d}^2(w'_t, w''_t),
\end{aligned}$$

which is equivalent to

$$\alpha_t\beta_t\tilde{d}^2(\tilde{p}_t, w'_t) + \alpha_t\gamma_t\tilde{d}^2(\tilde{p}_t, w''_t) + \beta_t\gamma_t\tilde{d}^2(w'_t, w''_t) \le \tilde{d}^2(\tilde{p}_t, \tilde{x}) - \tilde{d}^2(\tilde{r}_t, \tilde{x}).$$

On using (3.6) and (3.8), we obtain

$$\lim_{t\to\infty} \tilde{d}(\tilde{p}_t, w'_t) = 0, \tag{3.18}$$

$$\lim_{t\to\infty} \tilde{d}(\tilde{p}_t, w''_t) = 0 \tag{3.19}$$

and

$$\lim_{t\to\infty} \tilde{d}(w'_t, w''_t) = 0. \tag{3.20}$$

Now, triangle inequality gives

$$dist(\tilde{p}_t, \tilde{S}_1\tilde{p}_t) \le \tilde{d}(\tilde{p}_t, w'_t) + dist(w'_t, \tilde{S}_1\tilde{p}_t),$$

which on using (3.18) results into

$$\lim_{t\to\infty} dist(\tilde{p}_t, \tilde{S}_1\tilde{p}_t) = 0. \tag{3.21}$$

Again, consider

$$\begin{aligned}
dist(\tilde{p}_t, \tilde{S}_2\tilde{p}_t) &\le \tilde{d}(\tilde{p}_t, w''_t) + dist(w''_t, \tilde{S}_2\tilde{p}_t) \\
&\le \tilde{d}(\tilde{p}_t, w''_t) + \tilde{d}(w_t, \tilde{p}_t),
\end{aligned}$$

which on using (3.17) and (3.19) gives

$$\lim_{t\to\infty} dist(\tilde{p}_t, \tilde{S}_2\tilde{p}_t) = 0. \tag{3.22}$$

Now, we have

$$\begin{aligned}
\tilde{d}^2(\tilde{q}_t, \tilde{x}) &\le \psi_t\tilde{d}^2(\tilde{p}_t, \tilde{x}) + \kappa_t\tilde{d}^2(w'''_t, \tilde{x}) + \phi_t\tilde{d}^2(\tilde{G}_1\tilde{p}_t, \tilde{x}) \\
&\quad -\psi_t\kappa_t\tilde{d}^2(\tilde{p}_t, w'''_t) - \psi_t\phi_t\tilde{d}^2(\tilde{p}_t, \tilde{G}_1\tilde{p}_t) - \kappa_t\phi_t\tilde{d}^2(w'''_t, \tilde{G}_1\tilde{p}_t) \\
&\le \tilde{d}^2(\tilde{p}_t, \tilde{x}) - \psi_t\kappa_t\tilde{d}^2(\tilde{p}_t, w'''_t) - \psi_t\phi_t\tilde{d}^2(\tilde{p}_t, \tilde{G}_1\tilde{p}_t) - \kappa_t\phi_t\tilde{d}^2(w'''_t, \tilde{G}_1\tilde{p}_t),
\end{aligned}$$

which is equivalent to

$$\psi_t\kappa_t\tilde{d}^2(\tilde{p}_t, w'''_t) + \psi_t\phi_t\tilde{d}^2(\tilde{p}_t, \tilde{G}_1\tilde{p}_t) + \kappa_t\phi_t\tilde{d}^2(w'''_t, \tilde{G}_1\tilde{p}_t) \le \tilde{d}^2(\tilde{p}_t, \tilde{x}) - \tilde{d}^2(\tilde{q}_t, \tilde{x}),$$

this on using (3.6) and (3.11) gives

$$\lim_{t\to\infty} \tilde{d}(\tilde{p}_t, w_t''') = 0, \tag{3.23}$$

$$\lim_{t\to\infty} \tilde{d}(\tilde{p}_t, \tilde{G}_1\tilde{p}_t) = 0 \tag{3.24}$$

and

$$\lim_{t\to\infty} \tilde{d}(\tilde{G}_1\tilde{p}_t, w_t''') = 0. \tag{3.25}$$

On using (3.18) and (3.19), we have

$$\begin{aligned} \tilde{d}(\tilde{r}_t, \tilde{p}_t) &\leq \alpha_t\tilde{d}(\tilde{p}_t, \tilde{p}_t) + \beta_t\tilde{d}(w_t', \tilde{p}_t) + \gamma_t\tilde{d}(w_t'', \tilde{p}_t) \\ &\to 0 \quad as \quad t\to\infty. \end{aligned} \tag{3.26}$$

Thus, with the help of (3.23) and (3.26), we obtain

$$\begin{aligned} dist(\tilde{p}_t, \tilde{S}_3\tilde{p}_t) &\leq \tilde{d}(\tilde{p}_t, w_t''') + dist(w_n''', \tilde{S}_3\tilde{p}_t) \\ &\leq \tilde{d}(\tilde{p}_t, w_t''') + \tilde{d}(\tilde{r}_t, \tilde{p}_t) \\ &\to 0 \quad as \quad t\to\infty. \end{aligned} \tag{3.27}$$

(iv) Next, we show that $\lim_{t\to\infty} \tilde{d}(\tilde{p}_t, \tilde{G}_1\tilde{p}_t) = \lim_{t\to\infty} \tilde{d}(\tilde{p}_t, \tilde{G}_2\tilde{p}_t) = \lim_{t\to\infty} \tilde{d}(\tilde{p}_t, \tilde{G}_3\tilde{p}_t) = 0.$
In (3.24), we have already proved that $\lim_{t\to\infty} \tilde{d}(\tilde{p}_t, \tilde{G}_1\tilde{p}_t) = 0.$
So, consider

$$\tilde{d}^2(\tilde{p}_{t+1}, \tilde{x}) \leq \tilde{d}^2(\tilde{p}_t, \tilde{x}) - \delta_t\eta_t\tilde{d}^2(\tilde{G}_1\tilde{p}_t, \tilde{G}_2\tilde{p}_t) - \delta_t\xi_t\tilde{d}^2(\tilde{G}_1\tilde{p}_t, \tilde{G}_3\tilde{q}_t) - \eta_t\xi_t\tilde{d}^2(\tilde{G}_2\tilde{p}_t, \tilde{G}_3\tilde{q}_t),$$

which results into

$$\lim_{t\to\infty} \tilde{d}(\tilde{G}_1\tilde{p}_t, \tilde{G}_2\tilde{p}_t) = 0, \tag{3.28}$$

$$\lim_{t\to\infty} \tilde{d}(\tilde{G}_1\tilde{p}_t, \tilde{G}_3\tilde{q}_t) = 0 \tag{3.29}$$

and

$$\lim_{t\to\infty} \tilde{d}(\tilde{G}_2\tilde{p}_t, \tilde{G}_3\tilde{q}_t) = 0. \tag{3.30}$$

Now, from (3.24) and (3.28), we have

$$\begin{aligned} \tilde{d}(\tilde{p}_t, \tilde{G}_2\tilde{p}_t) &\leq \tilde{d}(\tilde{p}_t, \tilde{G}_1\tilde{p}_t) + \tilde{d}(\tilde{G}_1\tilde{p}_t, \tilde{G}_2\tilde{p}_t) \\ &\to 0 \quad as \quad t\to\infty. \end{aligned} \tag{3.31}$$

On using (3.23) and (3.24), we obtain

$$\begin{aligned} \tilde{d}(\tilde{q}_t, \tilde{p}_t) &\leq \psi_t\tilde{d}(\tilde{p}_t, \tilde{p}_t) + \kappa_t\tilde{d}(w_t''', \tilde{p}_t) + \phi_t\tilde{d}(\tilde{G}_1\tilde{p}_t, \tilde{p}_t) \\ &\to 0 \quad as \quad t\to\infty. \end{aligned} \tag{3.32}$$

Now, (3.30), (3.31) and (3.32) yields

$$\begin{aligned} \tilde{d}(\tilde{p}_t, \tilde{G}_3\tilde{p}_t) &\leq \tilde{d}(\tilde{p}_t, \tilde{G}_2\tilde{p}_t) + \tilde{d}(\tilde{G}_2\tilde{p}_t, \tilde{G}_3\tilde{q}_t) + \tilde{d}(\tilde{G}_3\tilde{q}_t, \tilde{G}_3\tilde{p}_t) \\ &\to 0 \quad as \quad t\to\infty. \end{aligned} \tag{3.33}$$

(v) Now, as $w_t = J_{\tilde{\lambda}_t}\tilde{p}_t$, from Lemma 2.10 we have

$$\begin{aligned}
\tilde{d}(J_{\tilde{\lambda}}\tilde{p}_t,\tilde{p}_t) &\leq \tilde{d}(J_{\tilde{\lambda}}\tilde{p}_t,w_t)+\tilde{d}(w_t,\tilde{p}_t)\\
&= \tilde{d}(J_{\tilde{\lambda}}\tilde{p}_t,J_{\tilde{\lambda}_t}\tilde{p}_t)+\tilde{d}(w_t,\tilde{p}_t)\\
&= \tilde{d}(J_{\tilde{\lambda}}\tilde{p}_t,J_{\tilde{\lambda}}(\frac{\tilde{\lambda}_t-\tilde{\lambda}}{\tilde{\lambda}_t}J_{\tilde{\lambda}_t}\tilde{p}_t\oplus\frac{\tilde{\lambda}}{\tilde{\lambda}_t}\tilde{p}_t))+\tilde{d}(w_t,\tilde{p}_t)\\
&\leq \tilde{d}(\tilde{p}_t,(1-\frac{\tilde{\lambda}}{\tilde{\lambda}_t})J_{\tilde{\lambda}_t}\tilde{p}_t\oplus\frac{\tilde{\lambda}}{\tilde{\lambda}_t}\tilde{p}_t)+\tilde{d}(w_t,\tilde{p}_t)\\
&\leq (1-\frac{\tilde{\lambda}}{\tilde{\lambda}_t})\tilde{d}(\tilde{p}_t,J_{\tilde{\lambda}_t}\tilde{p}_t)+\frac{\tilde{\lambda}}{\tilde{\lambda}_t}\tilde{d}(\tilde{p}_t,\tilde{p}_t)+\tilde{d}(w_t,\tilde{p}_t)\\
&= (1-\frac{\tilde{\lambda}}{\tilde{\lambda}_t})\tilde{d}(\tilde{p}_t,w_t)+\tilde{d}(w_t,\tilde{p}_t)\\
&\to 0 \quad as \quad t\to\infty.
\end{aligned}$$

Theorem 3.2 *Let $\tilde{E}$ be a nonempty closed and convex subset of a complete CAT(0) space $\tilde{X}$. Let $\tilde{G}_n:\tilde{E}\to\tilde{E}$, $n=1,2,3$ be single-valued nonexpansive mappings, $\tilde{S}_n:\tilde{E}\to KC(\tilde{E})$, $n=1,2,3$ be multi-valued nonexpansive mappings, and $\tilde{f}:\tilde{E}\to(-\infty,\infty]$ be a proper convex and lower semi-continuous function. Suppose that $\Omega=F(\tilde{G}_1)\cap F(\tilde{G}_2)\cap F(\tilde{G}_3)\cap F(\tilde{S}_1)\cap F(\tilde{S}_2)\cap F(\tilde{S}_3)\cap arg\min_{\tilde{q}\in\tilde{E}}\tilde{f}(\tilde{q})\neq\emptyset$ and $\tilde{S}_n\tilde{x}=\{\tilde{x}\}$, $n=1,2,3$ for $\tilde{x}\in\Omega$. For $\tilde{p}_1\in\tilde{E}$, let the sequence $\{\tilde{p}_t\}$ is generated by (3.1), where $\{\alpha_t\}$, $\{\beta_t\}$, $\{\gamma_t\}$, $\{\psi_t\}$, $\{\kappa_t\}$, $\{\phi_t\}$, $\{\delta_t\}$, $\{\eta_t\}$ and $\{\xi_t\}$ are sequences in $(0,1)$ such that*

$$0<\tilde{a}\leq\alpha_t,\beta_t,\gamma_t,\psi_t,\kappa_t,\phi_t,\delta_t,\eta_t,\xi_t\leq\tilde{b}<1,$$

$$\alpha_t+\beta_t+\gamma_t=1,\psi_t+\kappa_t+\phi_t=1,\delta_t+\eta_t+\xi_t=1,$$

for all $t\in\mathbb{N}$ and $\{\tilde{\lambda}_t\}$ is a sequence such that $\tilde{\lambda}_t\geq\tilde{\lambda}>0$ for all $t\in\mathbb{N}$ and some $\tilde{\lambda}$. Then, the sequence $\{\tilde{p}_t\}$ Δ-converges to a point in Ω.

Proof 2 *Let $W_\omega(\{\tilde{p}_t\})=\cup A(\{u_t\})$, where union is taken over all subsequences $\{u_t\}$ over $\{\tilde{p}_t\}$. In order to show the Δ-convergence of $\{\tilde{p}_t\}$ to a point of Ω, firstly we will prove $W_\omega(\{\tilde{p}_t\})\subset\Omega$ and thereafter argue that $W_\omega(\{\tilde{p}_t\})$ is a singleton set.*
To show $W_\omega(\{\tilde{p}_t\})\subset\Omega$, let $\tilde{x}\in W_\omega(\{\tilde{p}_t\})$. Then, there exists a subsequence $\{u_t\}$ of $\{\tilde{p}_t\}$ such that $A(\{u_t\})=\tilde{x}$. By Lemma 2.2 and 2.3, there exists a subsequence $\{v_t\}$ of $\{u_t\}$ such that $\Delta-\lim_t v_t=v$ and $v\in\tilde{E}$. From Theorem 3.1, we have

$$\lim_{t\to\infty}\tilde{d}(v_t,\tilde{G}_nv_t)=0,\qquad n=1,2,3$$

and

$$\lim_{t\to\infty}\tilde{d}(v_t,J_{\tilde{\lambda}}v_t)=0.$$

Since $\tilde{G}_n$, $n=1,2,3$ and $J_{\tilde{\lambda}}$ are nonexpansive mappings, with the use of Lemma 2.4, we obtain

$$v=\tilde{G}_1v=\tilde{G}_2v=\tilde{G}_3v=J_{\tilde{\lambda}}v.$$

So, we have

$$v\in F(\tilde{G}_1)\cap F(\tilde{G}_2)\cap F(\tilde{G}_3)\cap arg\min_{\tilde{q}\in\tilde{E}}\tilde{f}(\tilde{q}). \tag{3.34}$$

Since $\tilde{S}_n$, $n=1,2,3$ is compact valued for each $n\in\mathbb{N}$, there exist $r_t^n\in\tilde{S}_nv_t$ and $p_t^n\in\tilde{S}_nv$, $n=1,2,3$ such that

$$\tilde{d}(v_t,r_t^n)=dist(v_t,\tilde{S}_nv_t),\quad n=1,2,3$$

and

$$\tilde{d}(r_t^n,p_t^n)=dist(r_t^n,\tilde{S}_nv),\quad n=1,2,3.$$

From Theorem 3.1, we get

$$\lim_{t\to\infty} \tilde{d}(v_t, r_t^n) = 0, \quad n = 1,2,3.$$

By using the compactness of $\tilde{S}_n v$, $n = 1,2,3$, *there exists a subsequence* $\{p_{t_j}^n\}$ *of* $\{p_t^n\}$ *such that* $\lim_{j\to\infty} p_{t_j}^n = p^n \in \tilde{S}_n v$, $n = 1,2,3$. *With the help of Opial condition, we have*

$$\begin{aligned}
\limsup_{j\to\infty} \tilde{d}(v_{t_j}, p^n) &\le \limsup_{j\to\infty}(\tilde{d}(v_{t_j}, r_{t_j}^n) + \tilde{d}(r_{t_j}^n, p_{t_j}^n) + \tilde{d}(p_{t_j}^n, p^n)) \\
&\le \limsup_{j\to\infty}(\tilde{d}(v_{t_j}, r_{t_j}^n) + dist(r_{t_j}^n, \tilde{S}_n v) + \tilde{d}(p_{t_j}^n, p^n)) \\
&\le \limsup_{j\to\infty}(\tilde{d}(v_{t_j}, r_{t_j}^n) + H(\tilde{S}_n v_{t_j}, \tilde{S}_n v) + \tilde{d}(p_{t_j}^n, p^n)) \\
&\le \limsup_{j\to\infty}(\tilde{d}(v_{t_j}, r_{t_j}^n) + \tilde{d}(v_{t_j}, v) + \tilde{d}(p_{t_j}^n, p^n)) \\
&= \limsup_{j\to\infty} \tilde{d}(v_{t_j}, v).
\end{aligned}$$

Since asymptotic center is unique, we get $v = p^n \in \tilde{S}_n v$, $n = 1,2,3$. *By using (3.34), we obtain*

$$v \in F(\tilde{G}_1) \cap F(\tilde{G}_2) \cap F(\tilde{G}_3) \cap F(\tilde{S}_1) \cap F(\tilde{S}_2) \cap F(\tilde{S}_3) \cap arg \min_{\tilde{q}\in\tilde{E}} \tilde{f}(\tilde{q}) = \Omega.$$

From Theorem 3.1 and Lemma 2.5 , we get $\tilde{q} = v$, *and* $W_\omega(\{\tilde{p}_t\}) \subset \Omega$.
Now it is left to show that $W_\omega(\{\tilde{p}_t\})$ *consists of single element only. For this, let* $\{u_t\}$ *be a subsequence of* $\{\tilde{p}_t\}$. *Again, by using Lemma 2.2, we can find a subsequence* $\{v_t\}$ *of* $\{u_t\}$ *such that* $\Delta - \lim_t v_t = v$. *Let* $A(\{u_t\}) = u$ *and* $A(\{\tilde{p}_t\}) = \tilde{p}$. *It is enough to show that* $v = \tilde{p}$. *Since* $v \in \Omega$, *by Theorem 3.1,* $\{\tilde{d}(\tilde{p}_t, v)\}$ *is convergent. Again, by Lemma 2.5, we have* $v = \tilde{p}$ *which proves that* $W_\omega(\{\tilde{p}_t\}) = \{\tilde{p}\}$. *Hence the conclusion follows.*

Next, we establish the strong convergence theorems of our iteration.

Theorem 3.3 *Under the hypothesis of Theorem 2, the sequence* $\{\tilde{p}_t\}$ *converges to an element of* Ω *if* $J_{\tilde{\lambda}}$ *is semi-compact or* $\tilde{G}_1$ *is semi-compact or* $\tilde{G}_2$ *is semi-compact or* $\tilde{G}_3$ *is semi-compact or* $\tilde{S}_1$ *is hemi-compact or* $\tilde{S}_2$ *is hemi-compact or* $\tilde{S}_3$ *is hemi-compact.*

Proof 3 *Without loss of generality, we assume that* $\tilde{S}_1$ *is hemi-compact. Therefore, there exist a subsequence* $\{v_t\}$ *of* $\{\tilde{p}_t\}$ *which is having a strong limit* $\tilde{p}$ *in* $\tilde{E}$. *From Theorem 3.1, we get*

$$\lim_{t\to\infty} \tilde{d}(\tilde{G}_n u_t, u_t) = 0, \quad n = 1,2,3,$$

$$\lim_{t\to\infty} \tilde{d}(J_{\tilde{\lambda}} u_t, u_t) = 0$$

and

$$\lim_{t\to\infty} dist(\tilde{S}_n u_t, u_t) = 0, \quad n = 1,2,3.$$

From Lemma 2.4 , we obtain

$$\tilde{p} \in F(\tilde{G}_1) \cap F(\tilde{G}_2) \cap F(\tilde{G}_3) \cap arg \min_{\tilde{q}\in\tilde{E}} \tilde{f}(\tilde{q}). \tag{3.35}$$

By using nonexpansiveness of $\tilde{S}_1$, *we have*

$$\begin{aligned}
dist(\tilde{p}, \tilde{S}_1\tilde{p}) &\le \tilde{d}(\tilde{p}, u_t) + dist(u_t, \tilde{S}_1 u_t) + H(\tilde{S}_1 u_t, \tilde{S}_1\tilde{p}) \\
&\le 2\tilde{d}(\tilde{p}, u_t) + dist(u_t, \tilde{S}_1 u_t) \\
&\to 0 \ \ as \ \ t \ \to \ \infty.
\end{aligned}$$

This results into $dist(\tilde{p}, \tilde{S}_1\tilde{p}) = 0$, *which is same as* $\tilde{p} \in \tilde{S}_1\tilde{p}$. *Thus,* $\tilde{p} \in F(\tilde{S}_1)$. *Similarly, we can show that* $\tilde{p} \in F(\tilde{S}_2)$ *and* $\tilde{p} \in F(\tilde{S}_3)$. *Therefore, from (3.35), we get*

$$\tilde{p} \in F(\tilde{G}_1) \cap F(\tilde{G}_2) \cap F(\tilde{G}_3) \cap F(\tilde{S}_1) \cap F(\tilde{S}_2) \cap F(\tilde{S}_3) \cap arg \min_{\tilde{q}\in\tilde{E}} \tilde{f}(\tilde{q}) = \Omega.$$

By using double extract subsequence principle, we get that the sequence $\{\tilde{p}_t\}$ *converges strongly to* $\tilde{p} \in \Omega$.

Since every multi-valued mapping $\tilde{S}:\tilde{E}\to CB(\tilde{E})$ is hemi-compact if $\tilde{E}$ is a compact subset of $\tilde{X}$. So, the following result can be obtained from Theorem 3.3 immediately.

Theorem 3.4 *Let $\tilde{E}$ be a nonempty compact and convex subset of a complete CAT(0) space $\tilde{X}$. Let $\tilde{G}_n:\tilde{E}\to\tilde{E}$, $n=1,2,3$ be single-valued nonexpansive mappings, $\tilde{S}_n:\tilde{E}\to KC(\tilde{E})$, $n=1,2,3$ be multi-valued nonexpansive mappings, and $\tilde{f}:\tilde{E}\to(-\infty,\infty]$ be a proper convex and lower semi-continuous function. Suppose that $\Omega=F(\tilde{G}_1)\cap F(\tilde{G}_2)\cap F(\tilde{G}_3)\cap F(\tilde{S}_1)\cap F(\tilde{S}_2)\cap F(\tilde{S}_3)\cap arg\min\limits_{\tilde{q}\in\tilde{E}}\tilde{f}(\tilde{q})\neq\emptyset$ and $\tilde{S}_n\tilde{x}=\{\tilde{x}\}$, $n=1,2,3$ for $\tilde{x}\in\Omega$. For $\tilde{p}_1\in\tilde{E}$, let the sequence $\{\tilde{p}_t\}$ is generated by (3.1), where $\{\alpha_t\}$, $\{\beta_t\}$, $\{\gamma_t\}$, $\{\psi_t\}$, $\{\kappa_t\}$, $\{\phi_t\}$, $\{\delta_t\}$, $\{\eta_t\}$ and $\{\xi_t\}$ are sequences in $(0,1)$ such that*

$$0<\tilde{a}\leq\alpha_t,\beta_t,\gamma_t,\psi_t,\kappa_t,\phi_t,\delta_t,\eta_t,\xi_t\leq\tilde{b}<1,$$

$$\alpha_t+\beta_t+\gamma_t=1,\psi_t+\kappa_t+\phi_t=1,\delta_t+\eta_t+\xi_t=1,$$

for all $t\in\mathbb{N}$ and $\{\tilde{\lambda}_t\}$ is a sequence such that $\tilde{\lambda}_t\geq\tilde{\lambda}>0$ for all $t\in\mathbb{N}$ and some $\tilde{\lambda}$. Then, the sequence $\{\tilde{p}_t\}$ converges strongly to a point in Ω.

Since every real Hilbert space H is a complete CAT(0) space, so we have the following convergence results which can be obtained from Theorem 3.2 and 3.3.

Corollary 3.5 *Let $\tilde{E}$ be a nonempty closed and convex subset of a real Hilbert space X. Let $\tilde{G}_n:\tilde{E}\to\tilde{E}$, $n=1,2,3$ be single-valued nonexpansive mappings, $\tilde{S}_n:\tilde{E}\to CB(\tilde{E})$, $n=1,2,3$ be multi-valued nonexpansive mappings and $\tilde{f}:\tilde{E}\to(-\infty,\infty]$ be a proper convex and lower semi-continuous function. Suppose that $\Omega=F(\tilde{G}_1)\cap F(\tilde{G}_2)\cap F(\tilde{G}_3)\cap F(\tilde{S}_1)\cap F(\tilde{S}_2)\cap F(\tilde{S}_3)\cap arg\min\limits_{\tilde{q}\in\tilde{E}}\tilde{f}(\tilde{q})\neq\emptyset$ and $\tilde{S}_n\tilde{x}=\{\tilde{x}\}$, $n=1,2,3$ for $\tilde{x}\in\Omega$. For $\tilde{p}_1\in\tilde{E}$, let the sequence $\{\tilde{p}_t\}$ is generated in the following manner:*

$$\begin{cases} w_t=arg\min\limits_{\tilde{q}\in\tilde{E}}[\tilde{f}(\tilde{q})+\frac{1}{2\tilde{\lambda}_t}\|\tilde{q}-\tilde{p}_t\|^2],\\ \tilde{r}_t=\alpha_t\tilde{p}_t+\beta_t w_t'+\gamma_t w_t'',\\ \tilde{q}_t=\psi_t\tilde{p}_t+\kappa_t w_t'''+\phi_t\tilde{G}_1\tilde{p}_t,\\ \tilde{p}_{t+1}=\delta_t\tilde{G}_1\tilde{p}_t+\eta_t\tilde{G}_2\tilde{p}_t+\xi_t\tilde{G}_3\tilde{q}_t,\quad for\quad all\quad t\in\mathbb{N}\end{cases}\tag{3.36}$$

where $\{\alpha_t\}$, $\{\beta_t\}$, $\{\gamma_t\}$, $\{\psi_t\}$, $\{\kappa_t\}$, $\{\phi_t\}$, $\{\delta_t\}$, $\{\eta_t\}$ and $\{\xi_t\}$ are sequences in $(0,1)$ such that

$$0<\tilde{a}\leq\alpha_t,\beta_t,\gamma_t,\psi_t,\kappa_t,\phi_t,\delta_t,\eta_t,\xi_t\leq\tilde{b}<1,$$

$$\alpha_t+\beta_t+\gamma_t=1,\psi_t+\kappa_t+\phi_t=1,\delta_t+\eta_t+\xi_t=1,$$

for all $t\in\mathbb{N}$ and $\{\tilde{\lambda}_t\}$ is a sequence such that $\tilde{\lambda}_t\geq\tilde{\lambda}>0$ for all $t\in\mathbb{N}$ and some $\tilde{\lambda}$. Then, the sequence $\{\tilde{p}_t\}$ converges weakly to a point in Ω.

Corollary 3.6 *Let $\tilde{E}$ be a nonempty closed and convex subset of a real Hilbert space H. Let $\tilde{G}_n:\tilde{E}\to\tilde{E}$, $n=1,2,3$ be single-valued nonexpansive mappings, $\tilde{S}_n:\tilde{E}\to CB(\tilde{E})$, $n=1,2,3$ be multi-valued nonexpansive mappings, and $\tilde{f}:\tilde{E}\to(-\infty,\infty]$ be a proper convex and lower semi-continuous function. Suppose that $\Omega=F(\tilde{G}_1)\cap F(\tilde{G}_2)\cap F(\tilde{G}_3)\cap F(\tilde{S}_1)\cap F(\tilde{S}_2)\cap F(\tilde{S}_3)\cap arg\min\limits_{\tilde{q}\in\tilde{E}}\tilde{f}(\tilde{q})\neq\emptyset$ and $\tilde{S}_n\tilde{x}=\{\tilde{x}\}$, $n=1,2,3$ for $\tilde{x}\in\Omega$. For $\tilde{p}_1\in\tilde{E}$, let the sequence $\{\tilde{p}_t\}$ is generated by (3.36), where $\{\alpha_t\}$, $\{\beta_t\}$, $\{\gamma_t\}$, $\{\psi_t\}$, $\{\kappa_t\}$, $\{\phi_t\}$, $\{\delta_t\}$, $\{\eta_t\}$ and $\{\xi_t\}$ are sequences in $(0,1)$ such that*

$$0<\tilde{a}\leq\alpha_t,\beta_t,\gamma_t,\psi_t,\kappa_t,\phi_t,\delta_t,\eta_t,\xi_t\leq\tilde{b}<1,$$

$$\alpha_t+\beta_t+\gamma_t=1,\psi_t+\kappa_t+\phi_t=1,\delta_t+\eta_t+\xi_t=1,$$

for all $t\in\mathbb{N}$ and $\{\tilde{\lambda}_t\}$ is a sequence such that $\tilde{\lambda}_t\geq\tilde{\lambda}>0$ for all $t\in\mathbb{N}$ and some $\tilde{\lambda}$. Then, the sequence $\{\tilde{p}_t\}$ converges to an element of Ω if $J_{\tilde{\lambda}}$ is semi-compact or $\tilde{G}_1$ is semi-compact or $\tilde{G}_2$ is semi-compact or $\tilde{G}_3$ is semi-compact or $\tilde{S}_1$ is hemi-compact or $\tilde{S}_2$ is hemi-compact or $\tilde{S}_3$ is hemi-compact.

4. Conclusion

In this article, we present a new proximal point algorithm for solving the constrained convex minimization problem as well as the fixed point problem of nonexpansive single-valued and multi-valued mappings in CAT(0) spaces. Theorem 2-4 are the main convergence results of the paper. We also obtained some corollaries in the class of Hilbert spaces. Our results extend and improves the corresponding results of Cholamjiak [20], Suantai and Phuengrattana [38], Kumam et al. [39], Weng et al. [40] and Weng et al. [41].

5. Declarations

Funding : Not applicable.
Conflicts of Interest : No conflicts of interest are associated with this article.
Availability of data and material : Not applicable.
Code availability : Not applicable.
Author's contribution : Not applicable.

Acknowledgments

We think the referee by your suggestions.

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