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Fixed Point Results for Interpolative $\mathbb{F}$-Contraction in $\mathcal{S}$-Metric Spaces

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ABSTRACT: This paper introduces the concepts of extended interpolative Kannan-type $\mathbb{F}$-contraction and Reich-Rus-Ćirić type $\mathbb{F}$-contraction in $\mathcal{S}$-metric spaces. We develop a comprehensive theoretical framework for these new interpolative contractions and establish several fixed point results under suitable conditions. To validate our main results, illustrative examples are provided. Furthermore, we give an application to integral equations.

Key Words: Interpolative contraction, fixed point, $\mathcal{S}$-metric space, Kannan contraction, Reich-Rus-Ćirić contraction, $\mathbb{F}$-contraction.

Contents

1. Introduction and Preliminaries

Several researchers have extended the notion of metric spaces into various interesting generalizations. Among them, Sedghi et al. [1] introduced the concept of $\mathcal{S}$-metric space and established fixed point results within this framework. Some important definitions related to the new space were defined and some properties were proved. The authors established fixed point theorems for self-mappings on complete $\mathcal{S}$-metric space, demonstrating that under certain contractive conditions, such mappings have unique fixed point. Further, Sedghi et.al [2] investigated fixed point results in the framework of $\mathcal{S}$-metric space, which generalize [1]. The authors extended several fixed point theorems (like Banach Contraction Principle) to these generalized spaces, establishing conditions under which fixed point exists and are unique. Some generalizations and related developments are discusssed in [3], [4], [5].

On the otherhand, some researchers focused on extending Banach contraction and their generalizations. $\mathbb{F}$-contraction is one of the important generalization. It was introduced by Wardowski [6], which drew significant attention and was further developed by other authors who obtained important results. Chaipornjareansri [7] gave the definition of $\mathbb{F}$-contraction in the context of $\mathcal{S}$-metric space. D. Bajovic et al. [8] connected some known contractive conditions namely- Hardy-Rogers contraction, Wardowski $\mathbb{F}$-contraction, interpolations and Geraghty type contraction in complete metric space. They made a remark on the results of interpolative contractions. For additional details, one can refer to [3], [4], [9], [10].

Among the various generalizations of classical contraction mapping, interpolative contraction have recently attracted significant attention. The concept of interpolation was introduced by Karapinar [11] in 2018. He extended Kannan - type contraction by employing an interpolative concept. Karapinar's result has been extended in various ways. He also employed interpolative concept in Reich-Rus-Ćirić type contraction in 2018. Further these contractions has been extended to extended interpolative Kannan and Reich-Rus-Ćirić type contractions in metric space and its different generlized spaces ([12], [13], [14]). For advancements in interpolative contractions, one can refer to [8], [11], [14], [15], [16].

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This paper discusses the notions of extended interpolative Kannan-type $\mathbb{F}$-contraction and extended interpolative Reich-Rus-Ćirić-type $\mathbb{F}$-contraction.
We begin by recalling some basic definitions and results necessary for our discussion.

Definition 1.1 *[1] Let $\mathcal{H} \neq \phi$. A mapping $\mathcal{S} : \mathcal{H}^3 \to [0, +\infty)$ is called an $\mathcal{S}$- metric on $\mathcal{H}$ if*

(i) $\mathcal{S}(\mathtt{u}, \mathtt{v}, \mathtt{w}) = 0$ *iff* $\mathtt{u} = \mathtt{v} = \mathtt{w}$;

(ii) $\mathcal{S}(\mathtt{u}, \mathtt{v}, \mathtt{w}) \leq \mathcal{S}(\mathtt{u}, \mathtt{u}, \mathtt{b}) + \mathcal{S}(\mathtt{v}, \mathtt{v}, \mathtt{b}) + \mathcal{S}(\mathtt{w}, \mathtt{w}, \mathtt{b}) \ \forall \ \mathtt{u}, \mathtt{v}, \mathtt{w}, \mathtt{b} \in \mathcal{H}$.

Then, $(\mathcal{H}, \mathcal{S})$ is called an $\mathcal{S}$- metric space.

Example 1.1 *[1] Let $\mathcal{H} \neq \phi$, $\mathtt{d}$ is a metric on $\mathcal{H}$, then $\mathcal{S}(\mathtt{u}, \mathtt{v}, \mathtt{w}) = \mathtt{d}(\mathtt{u}, \mathtt{w}) + \mathtt{d}(\mathtt{v}, \mathtt{w})$ is an $\mathcal{S}-$ metric on $\mathcal{H}$.*

Definition 1.2 *[1] Let $(\mathcal{H}, \mathcal{S})$ be an $\mathcal{S}$- metric space.*

(i) A sequence $\{\mathtt{u}_\mathtt{p}\}$ in $\mathcal{H}$ is called a Cauchy sequence iff $\mathcal{S}(\mathtt{u}_\mathtt{p}, \mathtt{u}_\mathtt{p}, \mathtt{u}_\mathtt{m}) \to 0$ as $\mathtt{p}, \mathtt{m} \to +\infty$.

(ii) A sequence $\{\mathtt{u}_\mathtt{p}\}$ in $\mathcal{H}$ converges to $\mathtt{u} \in \mathcal{H}$ iff $\mathcal{S}(\mathtt{u}_\mathtt{p}, \mathtt{u}_\mathtt{p}, \mathtt{u}) \to 0$ as $\mathtt{p} \to +\infty$.

(iii) An $\mathcal{S}$- metric space $(\mathcal{H}, \mathcal{S})$ is complete if each Cauchy sequence in it is convergent.

Wardowski [6] gave the following definition of an auxiliary function that was used to define a new type of contraction.

Definition 1.3 *[6] Consider a mapping $\mathbb{F} : (0, +\infty) \to (-\infty, +\infty)$ satisfying:*
(F_1) $\mathtt{u} < \mathtt{v}$ *iff* $\mathbb{F}(\mathtt{u}) < \mathbb{F}(\mathtt{v}) \ \forall \ \mathtt{u}, \ \mathtt{v} \in \mathcal{H}$;
(F_2) *for every $\{\mathtt{u}_\mathtt{p}\}$ of positive numbers, $\lim_{\mathtt{p}\to+\infty} \mathtt{u}_\mathtt{p} = 0$ iff $\lim_{\mathtt{p}\to+\infty} \mathbb{F}(\mathtt{u}_\mathtt{p}) = -\infty$;*
(F_3) $\exists \ k \in (0, 1)$ *such that* $\lim_{\mathtt{u}\to 0^+} \mathtt{u}^k \mathbb{F}(\mathtt{u}) = 0$.

$\mathfrak{F}$ denotes the collection of all mappings $\mathbb{F}$ that meet the requirements specified in (F_1), (F_2) and (F_3).

Example 1.2 *([6]) The functions $\log \mathtt{u}$, $-\dfrac{1}{\sqrt{\mathtt{u}}}$ $(\mathtt{u} > 0)$, $\mathtt{u} + \log \mathtt{u}$ $(\mathtt{u} > 0)$, belong to the family $\mathfrak{F}$.*

Wardowski [6] introduced the definition of $\mathbb{F}$-contraction.

Definition 1.4 *[6] Let $(\mathcal{H}, \mathtt{d})$ be a metric space. A self-map $\mathbb{T}$ on $\mathcal{H}$ is an $\mathbb{F}$-contraction if $\exists \ \mathbb{F} \in \mathfrak{F}$ and a positive number τ such that $\forall \ \mathtt{u}, \mathtt{v} \in \mathcal{H}$,*

$$\mathtt{d}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}) > 0 \Rightarrow \tau + \mathbb{F}(\mathtt{d}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v})) \leq \mathbb{F}(\mathtt{d}(\mathtt{u}, \mathtt{v})). \tag{1.1}$$

The concept of $\mathbb{F}$- contraction in an $\mathcal{S}$-metric space was introduced by Chaipornjareansri [7].

Definition 1.5 *[7] Let $(\mathcal{H}, \mathcal{S})$ be an $\mathcal{S}$-metric space. A self-map $\mathbb{T}$ on $\mathcal{H}$ is called an $\mathbb{F}$- contraction if $\exists$ a positive number τ such that*

$$\mathcal{S}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}, \mathbb{T}\mathtt{w}) > 0$$
$$\Rightarrow \tau + \mathbb{F}(\mathcal{S}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}, \mathbb{T}\mathtt{w})) \leq \mathbb{F}(\mathcal{S}(\mathtt{u}, \mathtt{v}, \mathtt{w}), \forall \ \mathtt{u}, \mathtt{v}, \mathtt{w} \in \mathcal{H}.$$

Karapinar [11] gave the following definition.

Definition 1.6 *[11] Let $(\mathcal{H}, \mathtt{d})$ be a metric space. A self-map $\mathbb{T}$ on $\mathcal{H}$ is called an interpolative Kannan-type contraction if $\exists \ L \in [0, 1)$ and $c_1 \in (0, 1)$ such that*

$$\mathtt{d}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}) \leq L[\mathtt{d}(\mathtt{u}, \mathbb{T}\mathtt{u})]^{c_1}.[\mathtt{d}(\mathtt{v}, \mathbb{T}\mathtt{v})]^{1-c_1},$$
$$\forall \ \mathtt{u}, \mathtt{v} \in \mathcal{H} \setminus Fix(\mathbb{T}) \ where \ Fix(\mathbb{T}) = \{\mathtt{u} \in \mathcal{H} : \mathbb{T}\mathtt{u} = \mathtt{u}\}.$$

Konwar et al. [12] gave the following definitions of contractions in the context of b-metric space .

Definition 1.7 *[12] Consider the b-metric space $(\mathcal{H}, \mathtt{d}, s)$ and a self-map $\mathbb{T}$ on $\mathcal{H}$. Then, $\mathbb{T}$ is an extended interpolative modified Kannan-type $\mathbb{F}$-contraction if $\exists$ a positive number τ and $c_1, c_2 \in (0,1)$ with $c_1+c_2 < 1$ such that*

$$\tau + \mathbb{F}(\mathtt{d}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v})) \leq c_1\mathbb{F}(\mathtt{d}(\mathtt{u}, \mathbb{T}\mathtt{u})) + c_2\mathbb{F}(\frac{1}{s}\mathtt{d}(\mathtt{v}, \mathbb{T}\mathtt{v}))$$

$\forall$ $\mathtt{u}, \mathtt{v} \in \mathcal{H} \setminus Fix(\mathbb{T})$ with $\mathtt{d}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}) > 0$ and $\mathbb{F} \in \mathfrak{F}$.

Definition 1.8 *[12] Consider the b-metric space $(\mathcal{H}, \mathtt{d}, s)$ and a self-map $\mathbb{T}$ on $\mathcal{H}$. Then, $\mathbb{T}$ is an extended interpolative modified Ćirić-Reich-Rus-type $\mathbb{F}$-contraction if $\exists$ a positive number τ and $c_1, c_2 \in [0,1)$ with $c_1 + c_2 < 1$ and $\mathbb{F} \in \mathfrak{F}$ such that*

$$\tau + \mathbb{F}(\mathtt{d}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v})) \leq c_1\mathbb{F}(\mathtt{d}(\mathtt{u}, \mathtt{v})) + c_2\mathbb{F}(\mathtt{d}(\mathtt{u}, \mathbb{T}\mathtt{u})) + (1 - c_1 - c_2)\mathbb{F}(\frac{1}{s}\mathtt{d}(\mathtt{v}, \mathbb{T}\mathtt{v}))$$

$\forall$ $\mathtt{u}, \mathtt{v} \in \mathcal{H} \setminus Fix(\mathbb{T})$ and $\mathtt{d}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}) > 0$.

Konwar et al. [12] established the following fixed point theorems.

Theorem 1.1 *[12] Consider the complete b-metric space $(\mathcal{H}, \mathtt{d}, s)$ and $\mathbb{T} : \mathcal{H} \to \mathcal{H}$ be a continuous self-map. If $\mathbb{T}$ is an extended Kannan-type $\mathbb{F}$-contraction, then $\mathbb{T}$ has a fixed point in $\mathcal{H}$.*

Theorem 1.2 *[12] Consider the complete b-metric space $(\mathcal{H}, \mathtt{d}, s)$ and $\mathbb{T} : \mathcal{H} \to \mathcal{H}$ be a continuous self-map. If $\mathbb{T}$ is an extended interpolative Ćirić-Reich-Rus-type $\mathbb{F}$-contraction, then $\mathbb{T}$ has a fixed point in $\mathcal{H}$.*

Motivated by the above developments, this paper is devoted in exploring the existence of fixed point for extended interpolative $\mathbb{F}$-contractions within the framework of $\mathcal{S}$-metric spaces. Specifically, we study two significant forms: the extended interpolative Kannan-type and Reich-Rus-Ćirić-type $\mathbb{F}$-contractions. These mappings blend the ideas of interpolation and auxiliary function-based contraction into the $\mathcal{S}$-metric setting. Our main objective is to establish conditions ensuring the existence of fixed point for such mappings. By doing so, we not only extend existing theories but also contribute a unified treatment that encompasses several known results as special cases, thereby enriching the structure of fixed point theory in $\mathcal{S}$-metric spaces.

2. Main Results

Here, we introduce the concept of extended interpolative Kannan-type $\mathbb{F}$-contraction in $\mathcal{S}$-metric space.

Definition 2.1 *Let $(\mathcal{H}, \mathcal{S})$ be an $\mathcal{S}$-metric space and $\mathbb{T}$ be a self-map on $\mathcal{H}$. Then, $\mathbb{T}$ is called an extended interpolative Kannan-type $\mathbb{F}$-contraction if $\exists$ a positive number τ and $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ such that*

$$\tau + \mathbb{F}(\mathcal{S}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}, \mathbb{T}\mathtt{w})) \leq \alpha\mathbb{F}(\mathcal{S}(\mathtt{u}, \mathtt{u}, \mathbb{T}\mathtt{u})) + \beta\mathbb{F}(\mathcal{S}(\mathtt{v}, \mathtt{v}, \mathbb{T}\mathtt{v})) + (1 - \alpha - \beta)\mathbb{F}(\mathcal{S}(\mathtt{w}, \mathtt{w}, \mathbb{T}\mathtt{w})) \tag{2.1}$$

$\forall$ $\mathtt{u}, \mathtt{v}, \mathtt{w} \in \mathcal{H} \setminus Fix(\mathbb{T})$ with $\mathcal{S}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}, \mathbb{T}\mathtt{w}) > 0$ and $\mathbb{F} \in \mathfrak{F}$.

Theorem 2.1 *Let $(\mathcal{H}, \mathcal{S})$ be a complete $\mathcal{S}$- metric space and let $\mathbb{T}$ be a continuous self-map on $\mathcal{H}$. If $\mathbb{T}$ is an extended interpolative Kannan-type $\mathbb{F}$-contraction, then $\mathbb{T}$ has a fixed point in $\mathcal{H}$.*

Proof: Consider $\mathtt{u}_0 \in \mathcal{H}$ and define a sequence $\mathtt{u}_\mathtt{p} = \mathbb{T}^\mathtt{p}(\mathtt{u}_0)$ $\forall$ natural number $\mathtt{p}$. If $\mathtt{u}_\mathtt{p} = \mathbb{T}\mathtt{u}_\mathtt{p}$ for some $\mathtt{p}$, then this point becomes a fixed point of $\mathbb{T}$. Thus, we assume that $\mathtt{u}_\mathtt{p} \neq \mathtt{u}_{\mathtt{p}+1}$ $\forall$ $\mathtt{p} \in \mathbb{N}$.

Putting $\mathtt{u} = \mathtt{v} = \mathtt{u_p}$ and $\mathtt{w} = \mathtt{u_{p-1}}$ in (2.1), we get

$$\begin{aligned}\tau + \mathbb{F}(\mathcal{S}(\mathtt{u_{p+1}}, \mathtt{u_{p+1}}, \mathtt{u_p})) &= \tau + \mathbb{F}(\mathcal{S}(\mathbb{T}\mathtt{u_p}, \mathbb{T}\mathtt{u_p}, \mathbb{T}\mathtt{u_{p-1}})) \\ &\leq \alpha\mathbb{F}(\mathcal{S}(\mathtt{u_p}, \mathtt{u_p}, \mathbb{T}\mathtt{u_p})) + \beta\mathbb{F}(\mathcal{S}(\mathtt{u_p}, \mathtt{u_p}, \mathbb{T}\mathtt{u_p})) \\ &\quad +(1-\alpha-\beta)\mathbb{F}(\mathcal{S}(\mathtt{u_{p-1}}, \mathtt{u_{p-1}}, \mathbb{T}\mathtt{u_{p-1}})) \\ &= \alpha\mathbb{F}(\mathcal{S}(\mathtt{u_p}, \mathtt{u_p}, \mathtt{u_{p+1}})) + \beta\mathbb{F}(\mathcal{S}(\mathtt{u_p}, \mathtt{u_p}, \mathtt{u_{p+1}})) \\ &\quad +(1-\alpha-\beta)\mathbb{F}(\mathcal{S}(\mathtt{u_{p-1}}, \mathtt{u_{p-1}}, \mathtt{u_p})) \end{aligned} \tag{2.2}$$

Suppose $\mathcal{S}(\mathtt{u_{p-1}}, \mathtt{u_{p-1}}, \mathtt{u_p}) < \mathcal{S}(\mathtt{u_p}, \mathtt{u_p}, \mathtt{u_{p+1}})$ for some $\mathtt{p} \geq 1$; then from (2.2), we have

$$\begin{aligned}\tau + \mathbb{F}(\mathcal{S}(\mathtt{u_{p+1}}, \mathtt{u_{p+1}}, \mathtt{u_p})) &< \alpha\mathbb{F}(\mathcal{S}(\mathtt{u_p}, \mathtt{u_p}, \mathtt{u_{p+1}})) + \beta\mathbb{F}(\mathcal{S}(\mathtt{u_p}, \mathtt{u_p}, \mathtt{u_{p+1}})) \\ &\quad +(1-\alpha-\beta)\mathbb{F}(\mathcal{S}(\mathtt{u_p}, \mathtt{u_p}, \mathtt{u_{p+1}}))\end{aligned}$$

implies

$$\tau + \mathbb{F}(\mathcal{S}(\mathtt{u_{p+1}}, \mathtt{u_{p+1}}, \mathtt{u_p})) < \mathbb{F}(\mathcal{S}(\mathtt{u_{p+1}}, \mathtt{u_{p+1}}, \mathtt{u_p})),$$

which is a contradiction.
Therefore,

$$\mathcal{S}(\mathtt{u_p}, \mathtt{u_p}, \mathtt{u_{p+1}}) \leq \mathcal{S}(\mathtt{u_{p-1}}, \mathtt{u_{p-1}}, \mathtt{u_p}) \ \forall \mathtt{p} \geq 1.$$

From (2.2), we obtain

$$\tau + \mathbb{F}(\mathcal{S}(\mathtt{u_{p+1}}, \mathtt{u_{p+1}}, \mathtt{u_p})) \leq \mathbb{F}(\mathcal{S}(\mathtt{u_p}, \mathtt{u_p}, \mathtt{u_{p-1}})).$$

Thus,

$$\mathbb{F}(\mathcal{S}(\mathtt{u_{p+1}}, \mathtt{u_{p+1}}, \mathtt{u_p})) \ \leq \ \mathbb{F}(\mathcal{S}(\mathtt{u_p}, \mathtt{u_p}, \mathtt{u_{p-1}})) - \tau \ \leq \ \cdots \ \leq \ \mathbb{F}(\mathcal{S}(\mathtt{u_1}, \mathtt{u_1}, \mathtt{u_0})) - \mathtt{p}\tau \ \forall \ \mathtt{p} \geq 1. \tag{2.3}$$

Letting $\mathtt{p} \to +\infty$, we have

$$\lim_{\mathtt{p}\to\infty} \mathbb{F}(\mathcal{S}(\mathtt{u_{p+1}}, \mathtt{u_{p+1}}, \mathtt{u_p})) = -\infty.$$

Thus, from F_2,

$$\lim_{\mathtt{p}\to+\infty} \mathcal{S}(\mathtt{u_{p+1}}, \mathtt{u_{p+1}}, \mathtt{u_p}) = 0.$$

Denote $\theta_{\mathtt{p}} = \mathcal{S}(\mathtt{u_{p+1}}, \mathtt{u_{p+1}}, \mathtt{u_p})$. Hence,

$$\lim_{\mathtt{p}\to+\infty} \theta_{\mathtt{p}} = 0.$$

From F_3, $\exists\lambda \in (0,1)$ such that

$$\lim_{\mathtt{p}\to+\infty} {\theta_{\mathtt{p}}}^{\lambda}\mathbb{F}(\theta_{\mathtt{p}}) = 0.$$

From (2.3),

$${\theta_{\mathtt{p}}}^{\lambda}\mathbb{F}(\theta_{\mathtt{p}}) - {\theta_{\mathtt{p}}}^{\lambda}\mathbb{F}(\theta_0) \leq -{\theta_{\mathtt{p}}}^{\lambda}\mathtt{p}\tau < 0.$$

Letting $\mathtt{p} \to +\infty$ in above inequality,

$$\lim_{\mathtt{p}\to+\infty} \mathtt{p}{\theta_{\mathtt{p}}}^{\lambda} = 0.$$

From above equation, we know that $\exists$ a natural number $\mathtt{p_0}$ such that $\mathtt{p}{\theta_{\mathtt{p}}}^{\lambda} \leq 1, \forall \ \mathtt{p} \geq \mathtt{p_0}$. This implies

$$\theta_{\mathtt{p}} \leq \frac{1}{\mathtt{p}^{\frac{1}{\lambda}}}, \ \ \forall \ \mathtt{p} \ \geq \mathtt{p_0} \tag{2.4}$$

Now, we aim to prove that the $\{\mathtt{u_p}\}$ is a Cauchy sequence. To do this, we take any $\mathtt{m}, \mathtt{p} \in \mathbb{N}$ such that $\mathtt{m} > \mathtt{p} \geq \mathtt{p}_0$. By using (2.4) and the triangle inequality, we get

$$\begin{aligned}
\mathcal{S}(\mathtt{u_p}, \mathtt{u_p}, \mathtt{u_m}) &\leq 2\mathcal{S}(\mathtt{u_p}, \mathtt{u_p}, \mathtt{u_{p+1}}) + 2\mathcal{S}(\mathtt{u_{p+1}}, \mathtt{u_{p+1}}, \mathtt{u_{p+2}}) + \cdots + 2\mathcal{S}(\mathtt{u_{m-2}}, \mathtt{u_{m-2}}, \mathtt{u_{m-1}}) \\
&\quad + \mathcal{S}(\mathtt{u_{m-1}}, \mathtt{u_{m-1}}, \mathtt{u_m}) \\
&\leq 2\mathcal{S}(\mathtt{u_p}, \mathtt{u_p}, \mathtt{u_{p+1}}) + 2\mathcal{S}(\mathtt{u_{p+1}}, \mathtt{u_{p+1}}, \mathtt{u_{p+2}}) + \cdots + 2\mathcal{S}(\mathtt{u_{m-2}}, \mathtt{u_{m-2}}, \mathtt{u_{m-1}}) \\
&\quad + 2\mathcal{S}(\mathtt{u_{m-1}}, \mathtt{u_{m-1}}, \mathtt{u_m}) \\
&= 2(\theta_{\mathtt{p}} + \theta_{\mathtt{p+1}} + \cdots + \theta_{\mathtt{m-1}}) \\
&= 2\sum_{i=\mathtt{p}}^{\mathtt{m-1}} \theta_i \\
&\leq 2\sum_{i=\mathtt{p}}^{\infty} \theta_i \\
&\leq 2\sum_{i=\mathtt{p}}^{\infty} \frac{1}{i^{\frac{1}{\lambda}}}
\end{aligned}$$

As $\mathtt{p} \to +\infty$, this sum goes to zero. Therefore,

$$\lim_{\mathtt{p},\mathtt{m} \to +\infty} \mathcal{S}(\mathtt{u_p}, \mathtt{u_p}, \mathtt{u_m}) \to 0,$$

which means $\{\mathtt{u_p}\}$ is a Cauchy sequence in $\mathcal{H}$. Since $(\mathcal{H}, \mathcal{S})$ is a complete $\mathcal{S}$-metric space, $\{\mathtt{u_p}\}$ converges to some point $\mathtt{u}^\star \in \mathcal{H}$, we have

$$\lim_{\mathtt{p} \to +\infty} \mathtt{u_p} = \mathtt{u}^\star.$$

To prove that $\mathtt{u}^\star$ is a fixed point of $\mathbb{T}$, consider any subsequence $\{\mathtt{u}_{\mathtt{p}_t}\}$ of $\{\mathtt{u_p}\}$. Then,

$$\mathtt{u}^\star = \lim_{\mathtt{p} \to +\infty} \mathtt{u}_{\mathtt{p}_t+1} = \lim_{\mathtt{p} \to +\infty} \mathbb{T}(\mathtt{u}_{\mathtt{p}_t}) = \mathbb{T}(\lim_{\mathtt{p} \to +\infty} \mathtt{u}_{\mathtt{p}_t}) = \mathbb{T}(\mathtt{u}^\star).$$

Therefore, $\mathtt{u}^\star$ is a fixed point of $\mathbb{T}$.

□

The following corollary is derived from Definition 2.1 and Theorem 2.1.

Corollary 2.1 *Let $(\mathcal{H}, \mathcal{S})$ be a complete $\mathcal{S}$- metric space and let $\mathbb{T}$ be a continuous self-map on $\mathcal{H}$ such that $\exists$ a positive number τ and $c_1, c_2 \in [0,1)$ with $c_1 + c_2 < 1$ satisfying*

$$\tau + \mathbb{F}(\mathcal{S}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v})) \leq c_1\mathbb{F}(\mathcal{S}(\mathtt{u}, \mathtt{u}, \mathbb{T}\mathtt{u})) + c_2\mathbb{F}(\mathcal{S}(\mathtt{v}, \mathtt{v}, \mathbb{T}\mathtt{v}))$$

$\forall\ \mathtt{u}, \mathtt{v} \in \mathcal{H} \setminus Fix(\mathbb{T})$ *with* $\mathcal{S}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}) > 0$ *and* $\mathbb{F} \in \mathfrak{F}$. *Then, $\mathbb{T}$ has a fixed point in $\mathcal{H}$.*

We now present the extended interpolative Reich-Rus-Ćirić-type $\mathbb{F}$-contraction and prove a fixed point theorem.

Definition 2.2 *Let $(\mathcal{H}, \mathcal{S})$ be an $\mathcal{S}$-metric space and $\mathbb{T}$ be a self-map on $\mathcal{H}$. Then, $\mathbb{T}$ is called an extended interpolative Reich-Rus-Ćirić-type $\mathbb{F}$-contraction if $\exists$ a positive number τ and $\alpha, \beta, \gamma \in [0,1)$ with $\alpha + \beta + \gamma < 1$ such that*

$$\begin{aligned}
\tau + \mathbb{F}(\mathcal{S}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}, \mathbb{T}\mathtt{w})) &\leq \alpha\mathbb{F}(\mathcal{S}(\mathtt{u}, \mathtt{v}, \mathtt{w})) + \beta\mathbb{F}(\mathcal{S}(\mathtt{u}, \mathtt{u}, \mathbb{T}\mathtt{u})) + \gamma\mathbb{F}(\mathcal{S}(\mathtt{v}, \mathtt{v}, \mathbb{T}\mathtt{v})) \\
&\quad + (1 - \alpha - \beta - \gamma)\mathbb{F}(\mathcal{S}(\mathtt{w}, \mathtt{w}, \mathbb{T}\mathtt{w})) \qquad (2.5)
\end{aligned}$$

$\forall\ \mathtt{u}, \mathtt{v}, \mathtt{w} \in \mathcal{H} \setminus Fix(\mathbb{T})$ *with* $\mathcal{S}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}, \mathbb{T}\mathtt{w}) > 0$ *and* $\mathbb{F} \in \mathfrak{F}$.

Theorem 2.2 *Let $(\mathcal{H}, \mathcal{S})$ be a complete $\mathcal{S}$- metric space and let $\mathbb{T}$ be a continuous self-map on $\mathcal{H}$. If $\mathbb{T}$ is an extended interpolative Reich-Rus-Ćirić-type $\mathbb{F}$-contraction, then $\mathbb{T}$ has a fixed point in $\mathcal{H}$.*

Proof: Consider $u_0 \in \mathcal{H}$ and define a sequence $u_p = \mathbb{T}^p(u_0)$ for all natural number p. If $u_p = \mathbb{T}u_p$ for some p, then this point becomes a fixed point of $\mathbb{T}$. Thus, we assume that $u_p \neq u_{p+1}$ $\forall$ $p \in \mathbb{N}$.
Putting $u = v = u_p$ and $w = u_{p-1}$ in (2.5), we get

$$\begin{aligned}
\tau + \mathbb{F}(\mathcal{S}(u_{p+1}, u_{p+1}, u_p)) &= \tau + \mathbb{F}(\mathcal{S}(\mathbb{T}u_p, \mathbb{T}u_p, \mathbb{T}u_{p-1})) \\
&\leq \alpha\mathbb{F}(\mathcal{S}(u_p, u_p, u_{p-1})) + \beta\mathbb{F}(\mathcal{S}(u_p, u_p, \mathbb{T}u_p)) + \gamma\mathbb{F}(\mathcal{S}(u_p, u_p, \mathbb{T}u_p)) \\
&\quad +(1-\alpha-\beta-\gamma)\mathbb{F}(\mathcal{S}(u_{p-1}, u_{p-1}, \mathbb{T}u_{p-1})) \\
&= \alpha\mathbb{F}(\mathcal{S}(u_p, u_p, u_{p-1})) + \beta\mathbb{F}(\mathcal{S}(u_p, u_p, u_{p+1})) + \gamma\mathbb{F}(\mathcal{S}(u_p, u_p, u_{p+1})) \\
&\quad +(1-\alpha-\beta-\gamma)\mathbb{F}(\mathcal{S}(u_{p-1}, u_{p-1}, u_p)) \qquad (2.6)
\end{aligned}$$

Suppose $\mathcal{S}(u_{p-1}, u_{p-1}, u_p) < \mathcal{S}(u_p, u_p, u_{p+1})$ for some $p \geq 1$; then from (2.6), we have

$$\begin{aligned}
\tau + \mathbb{F}(\mathcal{S}(u_{p+1}, u_{p+1}, u_p)) &< \alpha\mathbb{F}(\mathcal{S}(u_p, u_p, u_{p+1})) + \beta\mathbb{F}(\mathcal{S}(u_p, u_p, u_{p+1})) + \gamma\mathbb{F}(\mathcal{S}(u_p, u_p, u_{p+1})) \\
&\quad +(1-\alpha-\beta-\gamma)\mathbb{F}(\mathcal{S}(u_p, u_p, u_{p+1}))
\end{aligned}$$

implies

$$\tau + \mathbb{F}(\mathcal{S}(u_{p+1}, u_{p+1}, u_p)) < \mathbb{F}(\mathcal{S}(u_{p+1}, u_{p+1}, u_p)),$$

which is a contradiction.
Therefore,

$$\mathcal{S}(u_p, u_p, u_{p+1}) \leq \mathcal{S}(u_{p-1}, u_{p-1}, u_p) \ \forall p \geq 1.$$

From (2.6), we obtain

$$\tau + \mathbb{F}(\mathcal{S}(u_{p+1}, u_{p+1}, u_p)) \leq \mathbb{F}(\mathcal{S}(u_p, u_p, u_{p-1})).$$

Thus,

$$\mathbb{F}(\mathcal{S}(u_{p+1}, u_{p+1}, u_p)) \leq \mathbb{F}(\mathcal{S}(u_p, u_p, u_{p-1})) - \tau \leq \cdots \leq \mathbb{F}(\mathcal{S}(u_1, u_1, u_0)) - p\tau, \ \forall \ p \geq 1. \qquad (2.7)$$

Letting $p \to +\infty$, we have

$$\lim_{p\to+\infty} \mathbb{F}(\mathcal{S}(u_{p+1}, u_{p+1}, u_p)) = -\infty.$$

Thus, from F_2,

$$\lim_{p\to+\infty} \mathcal{S}(u_{p+1}, u_{p+1}, u_p) = 0.$$

Denote $\theta_p = \mathcal{S}(u_{p+1}, u_{p+1}, u_p)$. Hence,

$$\lim_{p\to+\infty} \theta_p = 0.$$

From F_3, $\exists\lambda \in (0,1)$ such that

$$\lim_{p\to+\infty} {\theta_p}^{\lambda} F(\theta_p) = 0.$$

From (2.7),

$${\theta_p}^{\lambda} F(\theta_p) - {\theta_p}^{\lambda} F(\theta_0) \leq -{\theta_p}^{\lambda} p\tau < 0.$$

Letting $p \to +\infty$ in above inequality,

$$\lim_{p\to+\infty} p{\theta_p}^{\lambda} = 0.$$

From above equation, we know that $\exists$ a natural number p_0 such that $p{\theta_p}^{\lambda} \leq 1$ $\forall p \geq p_0$. This implies

$$\theta_p \leq \frac{1}{p^{\frac{1}{\lambda}}}, \quad \forall \ p \geq p_0. \qquad (2.8)$$

Now, we aim to prove that the $\{\mathtt{u}_\mathtt{p}\}$ is a Cauchy sequence. To do this, we take any $\mathtt{m}, \mathtt{p} \in \mathbb{N}$ such that $\mathtt{m} > \mathtt{p} \geq \mathtt{p}_0$. By using (2.8) and the triangle inequality, we get

$$\begin{aligned}
\mathcal{S}(\mathtt{u}_\mathtt{p}, \mathtt{u}_\mathtt{p}, \mathtt{u}_\mathtt{m}) &\leq 2\mathcal{S}(\mathtt{u}_\mathtt{p}, \mathtt{u}_\mathtt{p}, \mathtt{u}_{\mathtt{p}+1}) + 2\mathcal{S}(\mathtt{u}_{\mathtt{p}+1}, \mathtt{u}_{\mathtt{p}+1}, \mathtt{u}_{\mathtt{p}+2}) + \cdots + 2\mathcal{S}(\mathtt{u}_{\mathtt{m}-2}, \mathtt{u}_{\mathtt{m}-2}, \mathtt{u}_{\mathtt{m}-1}) \\
&\quad + \mathcal{S}(\mathtt{u}_{\mathtt{m}-1}, \mathtt{u}_{\mathtt{m}-1}, \mathtt{u}_\mathtt{m}) \\
&\leq 2\mathcal{S}(\mathtt{u}_\mathtt{p}, \mathtt{u}_\mathtt{p}, \mathtt{u}_{\mathtt{p}+1}) + 2\mathcal{S}(\mathtt{u}_{\mathtt{p}+1}, \mathtt{u}_{\mathtt{p}+1}, \mathtt{u}_{\mathtt{p}+2}) + \cdots + 2\mathcal{S}(\mathtt{u}_{\mathtt{m}-2}, \mathtt{u}_{\mathtt{m}-2}, \mathtt{u}_{\mathtt{m}-1}) \\
&\quad + 2\mathcal{S}(\mathtt{u}_{\mathtt{m}-1}, \mathtt{u}_{\mathtt{m}-1}, \mathtt{u}_\mathtt{m}) \\
&= 2(\theta_\mathtt{p} + \theta_{\mathtt{p}+1} + \cdots + \theta_{\mathtt{m}-1}) \\
&= 2\sum_{i=\mathtt{p}}^{\mathtt{m}-1} \theta_i \\
&\leq 2\sum_{i=n}^{\infty} \theta_i \\
&\leq 2\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\lambda}}}
\end{aligned}$$

As $\mathtt{p} \to +\infty$, this sum goes to zero. Therefore, $\lim_{\mathtt{p},\mathtt{m}\to+\infty} \mathcal{S}(\mathtt{u}_\mathtt{p}, \mathtt{u}_\mathtt{p}, \mathtt{u}_\mathtt{m}) = 0$, which means $\{\mathtt{u}_\mathtt{p}\}$ is a Cauchy sequence in $\mathcal{H}$. Since $(\mathcal{H}, \mathcal{S})$ is a complete $\mathcal{S}$-metric space, $\{\mathtt{u}_\mathtt{p}\}$ converges to some point $\mathtt{u}^\star \in \mathcal{H}$, we have

$$\lim_{\mathtt{p}\to+\infty} \mathtt{u}_\mathtt{p} = \mathtt{u}^\star.$$

To prove that $\mathtt{u}^\star$ is a fixed point of $\mathbb{T}$, consider any subsequence $\mathtt{u}_{\mathtt{p}_t}$ of $\{\mathtt{u}_\mathtt{p}\}$. Then,

$$\mathtt{u}^\star = \lim_{\mathtt{p}\to+\infty} \mathtt{u}_{\mathtt{p}_t+1} = \lim_{\mathtt{p}\to+\infty} \mathbb{T}(\mathtt{u}_{\mathtt{p}_t}) = \mathbb{T}(\lim_{\mathtt{p}\to+\infty} \mathtt{u}_{\mathtt{p}_t}) = \mathbb{T}(\mathtt{u}^\star).$$

Therefore, $\mathtt{u}^\star$ is a fixed point of $\mathbb{T}$. □

Remark 2.1 *(i) If we take $\alpha = 0$ in Theorem 2.2, we obtain Theorem 2.1. Thus, extended interpolative Kannan-type $\mathbb{F}$-contraction is a particular case of extended interpolative Reich-Rus-Ćirić-type $\mathbb{F}$-contraction.*

(ii) If we take $\mathbb{F}(t) = \ln(t)$ (for $t > 0$) in Theorem 2.1, the inequality (2.1) reduces to

$$\mathcal{S}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}, \mathbb{T}\mathtt{w}) \leq e^{-\tau}[\mathcal{S}(\mathtt{u}, \mathtt{u}, \mathbb{T}\mathtt{u})]^\alpha.[\mathcal{S}(\mathtt{v}, \mathtt{v}, \mathbb{T}\mathtt{v})]^\beta.[\mathcal{S}(\mathtt{w}, \mathtt{w}, \mathbb{T}\mathtt{w})]^{1-\alpha-\beta}. \tag{2.9}$$

So, we can state a theorem as follows,
Let $(\mathcal{H}, \mathcal{S})$ be a complete $\mathcal{S}$- metric space and let $\mathbb{T}$ be a continuous self-map on $\mathcal{H}$ such that $\exists$ a positive number τ and $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ satisfying the condition (2.9) $\forall\, \mathtt{u}, \mathtt{v}, \mathtt{w} \in \mathcal{H} \setminus Fix(\mathbb{T})$ with $\mathcal{S}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}, \mathbb{T}\mathtt{w}) > 0$. Then, $\mathbb{T}$ has a fixed point in $\mathcal{H}$.

(iii) If we take $\mathbb{F}(t) = \ln(t)$ (for $t > 0$) in Theorem 2.2, the inequality (2.5) reduces to

$$\mathcal{S}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}, \mathbb{T}\mathtt{w}) \leq e^{-\tau}[\mathcal{S}(\mathtt{u}, \mathtt{v}, \mathtt{w})]^\alpha.[\mathcal{S}(\mathtt{u}, \mathtt{u}, \mathbb{T}\mathtt{u})]^\beta.[\mathcal{S}(\mathtt{v}, \mathtt{v}, \mathbb{T}\mathtt{v})]^\gamma.[\mathcal{S}(\mathtt{w}, \mathtt{w}, \mathbb{T}\mathtt{w})]^{1-\alpha-\beta-\gamma}. \tag{2.10}$$

So, we can state a theorem as follows,
Let $(\mathcal{H}, \mathcal{S})$ be a complete $\mathcal{S}$- metric space and let $\mathbb{T}$ be a continuous self-map on $\mathcal{H}$ such that $\exists$ a positive number τ and $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$ satisfying the condition (2.10) $\forall\, \mathtt{u}, \mathtt{v}, \mathtt{w} \in \mathcal{H} \setminus Fix(\mathbb{T})$ with $\mathcal{S}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}, \mathbb{T}\mathtt{w}) > 0$. Then, $\mathbb{T}$ has a fixed point in $\mathcal{H}$.

Example 2.1 *Consider $\mathcal{H} = \{0, 1, 2\}$. We define $\mathcal{S} : \mathcal{H}^3 \to [0, +\infty)$ by $\mathcal{S}(\mathtt{u}, \mathtt{v}, \mathtt{w}) = \max\{|\mathtt{u} - \mathtt{v}|, |\mathtt{v} - \mathtt{w}|, |\mathtt{w} - \mathtt{u}|\}$ and a self-map $\mathbb{T}$ on $\mathcal{H}$ as $\mathbb{T}(0) = 1, \mathbb{T}(1) = 1, \mathbb{T}(2) = 0$.*
Obviously, $(\mathcal{H}, \mathcal{S})$ is a complete $\mathcal{S}$- metric space. Now, we consider $\mathtt{u}, \mathtt{v}, \mathtt{w} \in \mathcal{H} \setminus Fix(\mathbb{T})$ with $S(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}, \mathbb{T}\mathtt{w}) > 0$. Here $Fix(\mathbb{T}) = \{1\}$.

Hence, $(\mathtt{u},\mathtt{v},\mathtt{w}) \in \{(0,0,2),(2,0,0),(0,2,0),(2,2,0),(0,2,2),(2,0,2)\}$.
We choose $\tau = 0.2$, $\alpha = 0.3$, $\beta = 0.4$ *and* $\mathbb{F}(t) = \ln(t), t > 0$. *Then* $\mathbb{F} \in \mathfrak{F}$.
For extended interpolative Kannan-type $\mathbb{F}$*-contraction:*
Case(i): When $\mathtt{u} = 0, \mathtt{v} = 0, \mathtt{w} = 2$

$$\begin{aligned}\tau + \mathbb{F}(\mathcal{S}(\mathbb{T}\mathtt{u},\mathbb{T}\mathtt{v},\mathbb{T}\mathtt{w})) &= 0.2 + \mathbb{F}(\mathcal{S}(\mathbb{T}0,\mathbb{T}0,\mathbb{T}2))\\ &= 0.2 + \ln(1)\\ &= 0.2\end{aligned}$$

and

$$\begin{aligned}&\alpha\mathbb{F}(\mathcal{S}(\mathtt{u},\mathtt{u},\mathbb{T}\mathtt{u})) + \beta\mathbb{F}(\mathcal{S}(\mathtt{v},\mathtt{v},\mathbb{T}\mathtt{v})) + (1-\alpha-\beta)(\mathbb{F}(\mathcal{S}(\mathtt{w},\mathtt{w},\mathbb{T}\mathtt{w})))\\ = \;& 0.3\mathbb{F}(\mathcal{S}(0,0,\mathbb{T}0)) + 0.4\mathbb{F}(\mathcal{S}(0,0,\mathbb{T}0)) + 0.3(\mathbb{F}(\mathcal{S}(2,2,\mathbb{T}2)))\\ = \;& 0.3\ln(1) + 0.4\ln(1) + 0.3\ln(2)\\ = \;& 0.3 \times 0.693 = 0.2079\end{aligned}$$

Therefore,

$$\tau + \mathbb{F}(\mathcal{S}(\mathbb{T}\mathtt{u},\mathbb{T}\mathtt{v},\mathbb{T}\mathtt{w})) \leq \alpha\mathbb{F}(S(\mathtt{u},\mathtt{u},\mathbb{T}\mathtt{u})) + \beta\mathbb{F}(S(\mathtt{v},\mathtt{v},\mathbb{T}\mathtt{v})) + (1-\alpha-\beta)(\mathbb{F}(S(\mathtt{w},\mathtt{w},\mathbb{T}\mathtt{w}))), \tag{2.11}$$

$\forall\ (\mathtt{u},\mathtt{v},\mathtt{w}) \in \{(0,0,2),(2,0,0),(0,2,0)\}$.
Case (ii): When $\mathtt{u} = 2, \mathtt{v} = 2, \mathtt{w} = 0$

$$\begin{aligned}\tau + \mathbb{F}(\mathcal{S}(\mathbb{T}\mathtt{u},\mathbb{T}\mathtt{v},\mathbb{T}\mathtt{w})) &= 0.2 + \mathbb{F}(\mathcal{S}(\mathbb{T}2,\mathbb{T}2,\mathbb{T}0))\\ &= 0.2 + \ln(1)\\ &= 0.2\end{aligned}$$

and

$$\begin{aligned}&\alpha\mathbb{F}(\mathcal{S}(\mathtt{u},\mathtt{u},\mathbb{T}\mathtt{u})) + \beta\mathbb{F}(\mathcal{S}(\mathtt{v},\mathtt{v},\mathbb{T}\mathtt{v})) + (1-\alpha-\beta)(\mathbb{F}(\mathcal{S}(\mathtt{w},\mathtt{w},\mathbb{T}\mathtt{w})))\\ = \;& 0.3\mathbb{F}(\mathcal{S}(2,2,\mathbb{T}2)) + 0.4\mathbb{F}(\mathcal{S}(2,2,\mathbb{T}2)) + 0.3(\mathbb{F}(\mathcal{S}(0,0,\mathbb{T}0)))\\ = \;& 0.3\ln(2) + 0.4\ln(2) + 0.3\ln(1)\\ = \;& 0.7 \times 0.693 = 0.4851\end{aligned}$$

Therefore,

$$\tau + \mathbb{F}(\mathcal{S}(\mathbb{T}\mathtt{u},\mathbb{T}\mathtt{v},\mathbb{T}\mathtt{w})) \leq \alpha\mathbb{F}(S(\mathtt{u},\mathtt{u},\mathbb{T}\mathtt{u})) + \beta\mathbb{F}(S(\mathtt{v},\mathtt{v},\mathbb{T}\mathtt{v})) + (1-\alpha-\beta)(\mathbb{F}(S(\mathtt{w},\mathtt{w},\mathbb{T}\mathtt{w})))$$

$\forall\ (\mathtt{u},\mathtt{v},\mathtt{w}) \in \{(2,2,0),(0,2,2),(2,0,2)\}$.
Hence, all the conditions of Theorem 2.1 are satisfied.
Next, for extended interpolative Reich-Rus-Ćirić-type $\mathbb{F}$*-contraction:*
We choose $\tau = 0.2$, $\alpha = 0.3$, $\beta = 0.4$, $\gamma = 0.2$ *and* $F(t) = \ln(t), t > 0$.
Case(i): When $\mathtt{u} = 0, \mathtt{v} = 0, \mathtt{w} = 2$

$$\begin{aligned}\tau + \mathbb{F}(\mathcal{S}(\mathbb{T}\mathtt{u},\mathbb{T}\mathtt{v},\mathbb{T}\mathtt{w})) &= 0.2 + \mathbb{F}(\mathcal{S}(\mathbb{T}0,\mathbb{T}0,\mathbb{T}2))\\ &= 0.2\end{aligned}$$

and

$$\begin{aligned}&\alpha\mathbb{F}(\mathcal{S}(\mathtt{u},\mathtt{v},\mathtt{w})) + \beta\mathbb{F}(\mathcal{S}(\mathtt{u},\mathtt{u},\mathbb{T}\mathtt{u})) + \gamma\mathbb{F}(\mathcal{S}(\mathtt{v},\mathtt{v},\mathbb{T}\mathtt{v})) + (1-\alpha-\beta-\gamma)(F(\mathcal{S}(\mathtt{w},\mathtt{w},\mathbb{T}\mathtt{w})))\\ = \;& 0.3\mathbb{F}(\mathcal{S}(0,0,2) + 0.4\mathbb{F}(\mathcal{S}(0,0,\mathbb{T}0)) + 0.2\mathbb{F}(\mathcal{S}(0,0,\mathbb{T}0) + 0.1(\mathbb{F}(\mathcal{S}(2,2,\mathbb{T}2)))\\ = \;& 0.3\ln(2) + 0.4\ln(1) + 0.2\ln(1) + 0.1\ln(2)\\ = \;& 0.3 \times 0.693 + 0.1 \times 0.693\\ = \;& 0.4 \times 0.693\\ = \;& 0.2772\end{aligned}$$

Therefore,

$$\tau + \mathbb{F}(\mathcal{S}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}, \mathbb{T}\mathtt{w})) \leq \alpha\mathbb{F}(S(\mathtt{u}, \mathtt{u}, \mathbb{T}\mathtt{u})) + \beta\mathbb{F}(S(\mathtt{v}, \mathtt{v}, \mathbb{T}\mathtt{v})) + (1-\alpha-\beta)(\mathbb{F}(S(\mathtt{w}, \mathtt{w}, \mathbb{T}\mathtt{w}))).$$

$\forall\ (\mathtt{u}, \mathtt{v}, \mathtt{w}) \in \{(0,0,2), (2,0,0), (0,2,0)\}$.
Case (ii): When $\mathtt{u} = 2, \mathtt{v} = 2, \mathtt{w} = 0$

$$\begin{aligned} \tau + \mathbb{F}(\mathcal{S}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}, \mathbb{T}\mathtt{w})) &= 0.2 + \mathbb{F}(\mathcal{S}(\mathbb{T}2, \mathbb{T}2, \mathbb{T}0)) \\ &= 0.2 \end{aligned}$$

and

$$\begin{aligned} & \alpha\mathbb{F}(\mathcal{S}(\mathtt{u}, \mathtt{v}, \mathtt{w})) + \beta\mathbb{F}(\mathcal{S}(\mathtt{u}, \mathtt{u}, \mathbb{T}\mathtt{u})) + \gamma\mathbb{F}(\mathcal{S}(\mathtt{v}, \mathtt{v}, \mathbb{T}\mathtt{v})) + (1-\alpha-\beta-\gamma)(\mathbb{F}(\mathcal{S}(\mathtt{w}, \mathtt{w}, \mathbb{T}\mathtt{w}))) \\ = & 0.3\mathbb{F}(\mathcal{S}(2,2,0)) + 0.4\mathbb{F}(\mathcal{S}(2,2,\mathbb{T}2)) + 0.2\mathbb{F}(\mathcal{S}(2,2,\mathbb{T}2)) + 0.1(\mathbb{F}(\mathcal{S}(0,0,\mathbb{T}0))) \\ = & 0.3\ln(2) + 0.4\ln(2) + 0.2\ln(2) + 0.1\ln(1) \\ = & 0.3 \times 0.693 + 0.4 \times 0.693 + 0.2 \times 0.693 + 0.1 \times 0 \\ = & 0.9 \times 0.693 = 0.6237 \end{aligned}$$

Therefore,

$$\begin{aligned} \tau + \mathbb{F}(\mathcal{S}(\mathbb{T}\mathtt{u}, \mathbb{T}\mathtt{v}, \mathbb{T}\mathtt{w})) \leq\ & \alpha\mathbb{F}(\mathcal{S}(\mathtt{u}, \mathtt{v}, \mathtt{w})) + \beta\mathbb{F}(\mathcal{S}(\mathtt{u}, \mathtt{u}, \mathbb{T}\mathtt{u})) + \gamma\mathbb{F}(\mathcal{S}(\mathtt{v}, \mathtt{v}, \mathbb{T}\mathtt{v})) \\ & + (1-\alpha-\beta-\gamma)(\mathbb{F}(\mathcal{S}(\mathtt{w}, \mathtt{w}, \mathbb{T}\mathtt{w}))) \end{aligned}$$

$\forall\ (\mathtt{u}, \mathtt{v}, \mathtt{w}) \in \{(2,2,0), (0,2,2), (2,0,2)\}$.
Hence, all the conditions of Theorem 2.2 are satisfied. Here, $\mathbb{T}$ *has a fixed point and* 1 *is the fixed point.*

3. Application

Fixed point theorems serve as a fundamental tool in proving the existence of solutions to various types of integral equations. Among them, the class of Volterra integral equations, which involve integration over a variable domain, appears frequently in applied mathematics, physics and engineering problems. In this section, we apply Corollary 2.1 to an integral equation.
Let $J = [0, \mathtt{r}], \mathtt{r} > 0$ and $\mathcal{H} = C(J, \mathbb{R})$ be the set of all real-valued continuous functions defined on J. Suppose

$$\begin{aligned} \mathcal{S}(\mathsf{f}, \mathsf{g}, \mathsf{h}) &= \sup_{s\in J} |\mathsf{f}(s) - \mathsf{g}(s)| + \sup_{s\in J} |\mathsf{g}(s) - \mathsf{h}(s)| \\ &= ||\mathsf{f} - \mathsf{g}|| + ||\mathsf{g} - \mathsf{h}||, \end{aligned}$$

where $\mathcal{S}$ is an $\mathcal{S}$-metric on $\mathcal{H}$.
We consider the integral equation

$$\mathsf{f}(t) = \gamma(t) + \int_0^{\mathtt{r}} \mathbb{K}(t,s)\mathtt{q}(s, \mathsf{f}(s))\mathtt{d}s,\ \ t \in [0, \mathtt{r}], \tag{3.1}$$

where

(i) $\gamma : J \to \mathbb{R}$ and $\mathtt{q} : J \times \mathbb{R} \to \mathbb{R}$ are continuous;

(ii) $\forall\ t \in J,\ \mathbb{K} : J \times J \to \mathbb{R}$ is continuous and measurable at $s \in J$;

(iii) $\forall\ t, s \in J,\ \mathbb{K}(t,s) \geq 0$ and $\forall\ t \in J,\ \int_0^r \mathbb{K}(t,s)g(s, \mathsf{f}(s))\mathtt{d}s \leq 1$.

Theorem 3.1 *Assume that conditions (i), (ii) and (iii) are satisfied. Let* τ *be a positive number and* $c_1, c_2 \in (0,1)$ *with* $c_1 + c_2 < 1$ *be such that*

$$\begin{aligned} & |\mathtt{q}(t, \mathsf{f}(t)) - \mathtt{q}(t, \mathsf{h}(t))| \\ \leq\ & e^{-\tau}\left(||\mathsf{f} - \int_0^{\mathtt{r}} \mathbb{K}(t,s)\mathtt{q}(s,\mathsf{f}(s))\mathtt{d}s||\right)^{c_1} . \left(||\mathsf{h} - \int_0^{\mathtt{r}} \mathbb{K}(t,s)\mathtt{q}(s,\mathsf{f}(s))\mathtt{d}s||\right)^{c_2} \end{aligned} \tag{3.2}$$

$\forall t \in J$ and all $\mathsf{f}, \mathsf{h} \in \mathbb{R}$ such that

$$\mathsf{f}(t) \neq \int_0^{\mathtt{r}} \mathbb{K}(t,s)\mathsf{q}(s,\mathsf{f}(s))\mathsf{d}s,$$
$$\mathsf{h}(t) \neq \int_0^{\mathtt{r}} \mathbb{K}(t,s)\mathsf{q}(s,\mathsf{f}(s))\mathsf{d}s$$

while $\mathsf{f}(t) \neq \mathsf{h}(t)$ $\forall t \in J$. Then the integral equation (3.1) has a solution in $\mathcal{H}$.

Proof: We define a self-map $\mathbb{T} : \mathcal{H} \to \mathcal{H}$ as

$$\mathbb{T}\mathsf{f}(t) = \gamma(t) + \int_0^{\mathtt{r}} \mathbb{K}(t,s)\mathsf{q}(s,\mathsf{f}(s))\mathsf{d}s, \;\; t \in [0,\mathtt{r}].$$

For each $t \in [0,\mathtt{r}]$,

$$\begin{aligned}
|\mathbb{T}\mathsf{f}(t) - \mathbb{T}\mathsf{h}(t)| &= \left|\int_0^{\mathtt{r}} \mathbb{K}(t,s)\{\mathsf{q}(t,\mathsf{f}(s)) - \mathsf{q}(t,\mathsf{h}(s))\}\mathsf{d}s\right| \\
&\leq \int_0^{\mathtt{r}} \mathbb{K}(t,s)\,|\mathsf{q}(t,\mathsf{f}(s)) - \mathsf{q}(t,\mathsf{h}(s))|\,\mathsf{d}s \\
&\leq \int_0^{\mathtt{r}} \mathbb{K}(t,s).e^{-\tau}.\,(||\mathsf{f} - \mathbb{T}\mathsf{f}||)^{c_1}.\,(||\mathsf{h} - \mathbb{T}\mathsf{h}||)^{c_2}\,\mathsf{d}s \\
&= e^{-\tau}\,(||\mathsf{f} - \mathbb{T}\mathsf{f}||)^{c_1}.\,(||\mathsf{h} - \mathbb{T}\mathsf{h}||)^{c_2}\int_0^{\mathtt{r}} \mathbb{K}(t,s)\mathsf{d}s \\
&\leq e^{-\tau}\,(||\mathsf{f} - \mathbb{T}\mathsf{f}||)^{c_1}.\,(||\mathsf{h} - \mathbb{T}\mathsf{h}||)^{c_2}
\end{aligned}$$

Taking supremum over $t \in J$ on the above inequality,

$$\begin{aligned}
\mathcal{S}(\mathbb{T}\mathsf{f}, \mathbb{T}\mathsf{f}, \mathbb{T}\mathsf{h}) &= ||\mathbb{T}\mathsf{f} - \mathbb{T}\mathsf{h}|| \\
&\leq e^{-\tau}\,(||\mathsf{f} - \mathbb{T}\mathsf{f}||)^{c_1}.\,(||\mathsf{h} - \mathbb{T}\mathsf{h}||)^{c_2} \\
&= e^{-\tau}\,(\mathcal{S}(\mathsf{f},\mathsf{f},\mathbb{T}\mathsf{f}))^{c_1}.\,(\mathcal{S}(\mathsf{h},\mathsf{h},\mathbb{T}\mathsf{h}))^{c_2}
\end{aligned}$$

implies

$$\begin{aligned}
&\frac{\mathcal{S}(\mathbb{T}\mathsf{f}, \mathbb{T}\mathsf{f}, \mathbb{T}\mathsf{h})}{(\mathcal{S}(\mathsf{f},\mathsf{f},\mathbb{T}\mathsf{f}))^{c_1}.\,(\mathcal{S}(\mathsf{h},\mathsf{h},\mathbb{T}\mathsf{h}))^{c_2}} \leq e^{-\tau} \\
&\Rightarrow \ln\left[\frac{\mathcal{S}(\mathbb{T}\mathsf{f}, \mathbb{T}\mathsf{f}, \mathbb{T}\mathsf{h})}{(\mathcal{S}(\mathsf{f},\mathsf{f},\mathbb{T}\mathsf{f}))^{c_1}.\,(\mathcal{S}(\mathsf{h},\mathsf{h},\mathbb{T}\mathsf{h}))^{c_2}}\right] \leq -\tau \\
&\Rightarrow \ln\left[\mathcal{S}(\mathbb{T}\mathsf{f}, \mathbb{T}\mathsf{f}, \mathbb{T}\mathsf{h})\right] - \ln\left[(\mathcal{S}(\mathsf{f},\mathsf{f},\mathbb{T}\mathsf{f}))^{c_1}.\,(\mathcal{S}(\mathsf{h},\mathsf{h},\mathbb{T}\mathsf{h}))^{c_2}\right] \leq -\tau \\
&\Rightarrow \tau + \ln\left[\mathcal{S}(\mathbb{T}\mathsf{f}, \mathbb{T}\mathsf{f}, \mathbb{T}\mathsf{h})\right] \leq \ln\left[(\mathcal{S}(\mathsf{f},\mathsf{f},\mathbb{T}\mathsf{f}))^{c_1}.\,(\mathcal{S}(\mathsf{h},\mathsf{h},\mathbb{T}\mathsf{h}))^{c_2}\right] \\
&\Rightarrow \tau + \ln\left[\mathcal{S}(\mathbb{T}\mathsf{f}, \mathbb{T}\mathsf{f}, \mathbb{T}\mathsf{h})\right] \leq c_1\ln[\mathcal{S}(\mathsf{f},\mathsf{f},\mathbb{T}\mathsf{f})] + c_2\ln[\mathcal{S}(\mathsf{h},\mathsf{h},\mathbb{T}\mathsf{h})]
\end{aligned}$$

By taking $\mathbb{F}(t) = \ln(t),\; t > 0$, we have

$$\tau + \mathbb{F}[\mathcal{S}(\mathbb{T}\mathsf{f}, \mathbb{T}\mathsf{f}, \mathbb{T}\mathsf{h})] \leq c_1\mathbb{F}[\mathcal{S}(\mathsf{f},\mathsf{f},\mathbb{T}\mathsf{f})] + c_2\mathbb{F}[\mathcal{S}(\mathsf{h},\mathsf{h},\mathbb{T}\mathsf{h})]$$

$\forall\, \mathsf{f}, \mathsf{h} \in \mathcal{H} \setminus Fix(\mathbb{T})$ with $\mathcal{S}(\mathbb{T}\mathsf{f}, \mathbb{T}\mathsf{f}, \mathbb{T}\mathsf{h}) < 0$.
That is, all the conditions of Corollary 2.1 are satisfied and hence $\mathbb{T}$ has a fixed point. Therefore, the integral equation (3.1) has a solution. □

4. Conclusion

In this study, we propose the concepts of extended interpolative Kannan-type $\mathbb{F}$-contraction and extended interpolative Reich-Rus-Ćirić-type F-contraction in $\mathcal{S}$-metric space. We develop a comprehensive theoretical framework for these new interpolative contractions and establish several fixed point results under suitable conditions. Additionally, an application is included to demonstrate the main theoretical findings.

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References

1. S. Sedghi, N. Shobe, A. Aliouche: *A generalization of fixed point theorems in $S-$ metric spaces*, Mat. Vesnik., 64(3), 258-266, (2012)
2. S. Sedghi and N. V. Dung: *Fixed point theorems on S-metric spaces*, Mat. Vesnik., 66(1), 113-124, (2014)
3. K. M. Devi, Y. Rohen, K. A. Singh: *Fixed Points of Modified F-contractions in S-metric spaces*, J. Math. Comput. Sci, 12, 1-11, (2022)
4. T. Thaibema, Y. Rohen, T. Stephen, O. Budhichandra: *Fixed point of rational F-contractions in S-metric spaces*, J. Math. Comput. Sci, 12, 1-11, (2022)
5. B. Khomdram, Y. Rohen, Y. M. Singh and M. S. Khan: *Fixed point theorems of generalized $S-\beta-\psi$ contractive type mappings.*, Mathematica Moravica, 22(1), 81-92, (2018)
6. D. Wardowski: *Fixed points of a new type of contractive mapping in complete metric spaces.*, Fixed Point Theory Appl. 2012, 94, (2012)
7. S. Chaipornjareansri: *Fixed Point Theorems for F_w-contractions in complete S-metric spaces*, Thai J. Math, 98-109, (2016)
8. D. Bajovic, S. Mitrovic, S. Buhmiler, S.Radenovic: *Remarks on some results of Extended Interpolative Hardy-Rogers-Geraghty-Wardowski Contractions and Ćirić-Reich-Rus type F-contractions*, Sahand Commun. Math. Anal., 1-14, (2024)
9. S. Sedghi, M. M. Rezaee, T. Dosenovic, S. Radenovic: *Common fixed point theorems for contractive mappings satisfying ϕ-maps in S-metric spaces*, Acta Univ. Sapientiae Math., 8(2), 298-311, (2016)
10. R. Kannan: *Some results on fixed points*, Bull. Calcutta Math. Soc., 60, 71-76, (1972)
11. E. Karapinar: *Revisiting the Kannan Type Contractions via Interpolation*, Advances in the Theory of Nonlinear Analysis and its Applications, 2(2), 85-87, (2018)
12. N. Konwar, P. Debnath: *Fixed Point Results for a Family of Interpolative F-Contractions in b-Metric Spaces*, Axioms, 11, 6219, (2022)
13. I. Yildirim:*On extended interpolative single and multivalued F-contractions*, Turkish J. of Mathematics, 46(1), 688-698, (2022)
14. P. Debnath, H. M. Srivastava: *New Extensions of Kannan's and Reich's Fixed Point Theorems for Multivalued Maps Using Wardowski's Technique with Application to Integral Equations*, Symmetry, 1-7, (2020)
15. B. Mohammadi, V. Parvaneh and H. Aydi: *On extended interpolative Ćirić-Reich-Rus type F-contractions and an application*, J. of Inequal. Appl., 2019, 290, (2019)
16. E. Karapinar, R. Agarwal and H. Aydi: *Interpolative Reich-Rus-Ćirić type contractions on Partial Metric Spaces*, Mathematics, 6, 1-7, (2018)

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