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On the Uniqueness of Fixed Points for Nonlinear-Linear Operator Sums of Krasnosel'skii Type

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ABSTRACT: In this work, we derive sufficient conditions on a linear operator L and a nonlinear mapping K that ensure the existence of a unique fixed point of the sum $L+K$ within the framework of Krasnosel'skii's fixed point theorem. As a special case, when L is the zero operator, our result reduces to the well-known classical Kellogg uniqueness theorem. Moreover, we extend Talman's uniqueness theorem by employing the concept of the measure of noncompactness. In addition, we investigate the asymptotic behavior of the unique fixed point in connection with the Belitskiĭ–Lyubich conjecture. Finally, we present an illustrative application that demonstrates the applicability and effectiveness of the main theoretical result.

Key Words: Fixed point theorems, Fréchet differentiability, compact map, uniqueness.

Contents

1. Introduction

In fixed point theory, one of the more interesting theorems is the Krasnosel'skii's FPT (cf. [1–3]). Inspired by the insight that the inverse of a perturbed differential operator can often be expressed as the combination of a compact and a contraction map, he established a general FPT to address this setting. More precisely, the theorem states that if $\mathsf{N} \subset \mathsf{X}$ is a nonempty, open, bounded, and convex set (X is a Banach space), and if K and C are two maps from $\overline{\mathsf{N}}$ into X satisfying the following conditions:

a. $K\overline{\mathsf{N}} + C\overline{\mathsf{N}} \subset \overline{\mathsf{N}}$,

b. K is continuous on $\overline{\mathsf{N}}$, its image is relatively compact,

c. C is a contraction mapping,

thus, the sum $K+C$ possess a fixed point in $\overline{\mathsf{N}}$ (not necessary unique).

Following Krasnosel'skii's original result, a substantial body of literature has developed, presenting numerous generalizations and refinements of his theorem, along with a wide range of applications in nonlinear integro-differential and nonlinear PDE (see, for instance, [4–10] together with the references cited in those papers).

All of these contributions focus on establishing only the existence of fixed points under different sets of assumptions. The natural extension of this line of inquiry concerns the following question: under what additional conditions can we guarantee that the fixed point is unique?

With this in mind, Kellogg [11] proved that a compact map F from $\overline{\mathsf{N}}$ into $\overline{\mathsf{N}}$ possess a unique fixed point if,

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d. The Fréchet derivative of F is continuous in N, and for every $x \in \mathsf{N}$, the number 1 is in the resolvent set of the derivative $F'(x)$;

e. Given any $x \in \partial\mathsf{N}$, we have $x \neq F(x)$.

The central step in the proof of Kellogg's theorem consists in demonstrating that $\deg(0, \mathbb{I} - F, \mathsf{N}) = 1$. By employing the consideration of Kuratowski's measure of noncompactness, Talman [12] later, adapted this result to a more general class of Darbo contractions (see the Definition below). Further refinements were proposed by Smith and Stuart [13], who proved that the statements of the results in [11, 12] still hold as long as the set

$$\{x \in \mathsf{N}\,;\, 1 \in \sigma(F'(x))\},$$

contains no limit points in N; the rest of hypotheses are unchanged.

Moreover, this class of uniqueness criterion has found applications beyond fixed point theory. Notably, Shih and Wu [14] applied it to establish asymptotic behaviour (see the definition below) within the framework of Schauder's fixed point result, which gives certain answers to Belitskiĭ-Lyubich conjecture (see [16, p. 41]).

Our objective of the present work is to extend the uniqueness of Kellogg's and talman's theorems to the framework of Krasnosel'skii's FPT, particularly in the case where the operator $C = L \in \mathcal{L}(\mathsf{X})$ has a strict contractive iterate, and where K is nonlinear and compact. It should be noted that C.R. Barroso [15] established existence results, in the framework of weak topology, for this class of operators under the assumptions that K is w-continuous and that $K(\overline{\mathsf{N}})$ is w-precompact. When the operator C is zero we obtain as results the theorems of Kellogg, Talman, making our results especially significant in light of Krasnosel'skii's remark made at the beginning of the introduction (see, for instance, [1–8]).

Moreover, inspired by [14], we investigate the asymptotic behaviour of the unique fixed point in the sense of the Belitskiĭ-Lyubich conjecture. This result is obtained under the spectral hypothesis

$$\sup_{x \in \overline{N}} r\left((\mathbb{I} - L)^{-1} K'(x)\right) < 1.$$

An application illustrating the theoretical main result is presented at the end of the paper.

2. Notations and Preliminaries

Throughout this paper, $\mathsf{N} \subset \mathsf{X}$ denotes a nonempty open, bounded, and convex set, where X is a Banach space. We recall that if $F : \mathsf{N} \to \mathsf{X}$ is continuously Fréchet differentiable, the Taylor expansion around $x^* \in \mathsf{N}$ takes the form

$$F(x) = F(x^*) + F'(x^*)(x - x^*) + o(\|x - x^*\|), \quad x \to x^*,$$

where the notation $o(\|x - x^*\|)$ means that

$$\lim_{x \to x^*} \frac{\| F(x) - F(x^*) - F'(x^*)(x - x^*) \|}{\|x - x^*\|} = 0.$$

In the sequel, we will need the following result of Holmes (see [20, Corollary, p. 165]).

Theorem 2.1 *Let $T \in \mathcal{L}(\mathsf{X})$. Then X can be equipped (with an equivalent norm) such that T is a contraction if and only if $r(T) < 1$, where $r(T)$ denotes the spectral radius of T.*

The Kuratowski's measure of noncompactness (see, for example [17, p. 113]) is a mapping defined on the collection of bounded sets of a Banach space X with values in $\mathbb{R}^+$. For a nonempty and bounded subset of $\mathsf{S} \subset \mathsf{X}$, it is defined by

$$\begin{aligned} \alpha(\mathsf{S}) &:= \inf\{\eta > 0\,;\, \mathsf{S} \text{ admits a finite cover by sets of diameter } \leq \eta\} \\ &:= \inf\left\{\eta > 0\,;\, \mathsf{S} \subset \bigcup_{i=1}^{n} S_i,\ S_i \subset X,\ \operatorname{diam}(S_i) \leq \eta,\ n \in \mathbb{N}\right\}. \end{aligned}$$

A mapping $K : \overline{\mathsf{N}} \to \overline{\mathsf{N}}$ is said to verify the χ-Darbo property with a constant $\chi > 0$, if for every $\mathsf{S} \subset \mathsf{N}$, we have

$$\alpha(K(\mathsf{S})) \leq \chi \alpha(\mathsf{S}).$$

If, in addition, $\chi \in (0, 1)$, the map K is referred an α-set contraction.

Moreover, for a map $F : \overline{\mathsf{N}} \to \overline{\mathsf{N}}$, we define its fixed-point set by

$$\mathrm{Fix}(F) = \{x \in \overline{\mathsf{N}}\,,\, F(x) = x\}.$$

In this paper "deg" denotes the topological degree of Leray-Schauder. In order to keep this section concise, we refer the reader to [17, 24] for further details.

3. The Main Results

We begin this section by establishing the following result.

Theorem 3.1 *Let $K : \overline{\mathsf{N}} \to \mathsf{X}$ and $L \in \mathcal{L}(\mathsf{X})$ satisfy the following assumptions:*

i. some iterate of L is a contraction;

ii. $K : \overline{\mathsf{N}} \to \mathsf{X}$ is a compact map, with continuous Fréchet derivative on N;

iii. the operator $\mathbb{I} - L - K'(x)$ is injective, and $K + L$ has no fixed point on $\partial\mathsf{N}$;

iv. $K\overline{\mathsf{N}} + L\overline{\mathsf{N}} \subset \overline{\mathsf{N}}$.

Then, the map $K + L$ possess a unique fixed point $x^* \in \mathsf{N}$.

Proof. By assumption (i), it is clear that that the linear operator $\mathbb{I} - L$ is invertible and

$$(\mathbb{I} - L)^{-1} = (\mathbb{I} - L^k)^{-1} \sum_{p=0}^{k-1} L^p \in \mathcal{L}(\mathsf{X}).$$

Therefore, for any $y \in \overline{\mathsf{N}}$ the equation $x = Lx + Ky$ possess a unique solution $x = (\mathbb{I} - L)^{-1}Ky \in \overline{\mathsf{N}}$. So, according to assumption (iv), $(\mathbb{I} - L)^{-1}K(\overline{\mathsf{N}}) \subset \overline{\mathsf{N}}$.

It is obvious that the map $F = (\mathbb{I} - L)^{-1}K : \overline{\mathsf{N}} \to \overline{\mathsf{N}}$ is compact and continuous.

Furthermore, F is continuously Fréchet differentiable on N and $F'(x) = (\mathbb{I} - L)^{-1}K'(x)$, for all $x \in \mathsf{N}$. Since, $L + K$ has no fixed point on $\partial\mathsf{N}$, then for any $x \in \partial\mathsf{N}$, $F(x) \neq x$.

Now making use of assumption (iii), we infer that

$$\ker(\mathbb{I} - F'(x)) = \ker(\mathbb{I} - L - K'(x)) = \{0\},$$

where $\ker(\mathbb{I} - F'(x))$ denotes the null space of $\mathbb{I} - F'(x)$. Accordingly, $\mathbb{I} - F'(x)$ is injective for each $x \in \mathsf{N}$. Since, the map F is compact and Fréchet differentiable, it follows from [17, Proposition 8.2] that $F'(x)$ is a linear compact operator for each $x \in \mathsf{N}$. By Fredholm alternative (see, for example, [18, Theorem 6.6]) $\mathbb{I} - F'(x)$ is onto. Therefore, $1 \notin \sigma(F'(x))$ for each $x \in \mathsf{N}$.

Consequently, above steps verify that the hypotheses of Kellogg's theorem hold for the mapping F, which therefore possess a unique fixed point $x^* \in \mathsf{N}$, and consequently it is a unique element of N satisfying

$$K(x^*) + L(x^*) = x^*.$$ □

Next, we establish an analogue of Talman's theorem [12, p. 249].

Corollary 3.1 *Assume that $K : \overline{\mathsf{N}} \to \mathsf{X}$ and $L \in \mathcal{L}(\bullet)$ are such that* *i.* *some iterate of L is a contraction;*

ii. *The Fréchet derivative of K is continuous on N, and K satisfies the χ-Darbo property, where $\chi\|(\mathbb{I}-L)^{-1}\|_{\mathcal{L}(\mathsf{X})} < 1$;*

iii. *the operator $\mathbb{I}-L-K'(x)$ is injective, and $K+L$ has no fixed point on $\partial\mathsf{N}$;*

iv. $K\overline{\mathsf{N}} + L\overline{\mathsf{N}} \subset \overline{\mathsf{N}}$.

Then, the map $K+L$ possess a unique fixed point $x^* \in \mathsf{N}$.

Proof. Following Theorem 3.1, we define $F = (\mathbb{I}-L)^{-1}A$. Then, for every bounded subset S of $\overline{\mathsf{N}}$,

$$\begin{aligned}\alpha(F(\mathsf{S})) &\leq \|(\mathbb{I}-L)^{-1}\|_{\mathcal{L}(\mathsf{X})}\alpha(K(\mathsf{S})),\\ &\leq \|(\mathbb{I}-L)^{-1}\|_{\mathcal{L}(X)}\chi\alpha(\mathsf{S}).\end{aligned}$$

Therefore, F is an α-set contraction, because $\chi\|(\mathbb{I}-L)^{-1}\|_{\mathcal{L}(\mathsf{X})} < 1$.

We have seen in the proof of Theorem 3.1 that F is continuously Fréchet differentiable on N, and

$$F'(x) = (\mathbb{I}-L)^{-1}K'(x), \quad \text{for any } x \in \mathsf{N}.$$

Since $K+L$ has no fixed point on $\partial\mathsf{N}$, then for each $x \in \partial\mathsf{N}$, $F(x) \neq x$. By the proof of Theorem 3.1, we known that $F(\overline{\mathsf{N}}) \subset \overline{\mathsf{N}}$, and for each $x \in \mathsf{N}$, $1 \notin \sigma(F^{'}(x))$. Thus, the hypotheses of Talman's theorem (see, [12, p. 249]), and so F possess a unique fixed point $x^* \in \mathsf{N}$. Therefore, x^* is the unique element of N satisfying

$$K(x^*) + L(x^*) = x^*.$$ □

Proceeding as in [14] together with our Theorem 3.1, we are able to establish the following result.

Theorem 3.2 *Let M be an open subset of $(\mathsf{X}, \|.\|)$, and let N be a nonempty open, bounded and convex with $\overline{\mathsf{N}} \subset \mathsf{M}$. Assume that $K : \mathsf{M} \to \mathsf{X}$ and $L \in \mathcal{L}(\mathsf{X})$ are such that*

i. *some iterate of L is a contraction;*

ii. *$K : \mathsf{M} \to \mathsf{X}$ is compact and continuously Fréchet differentiable on M;*

iii. $r\left((\mathbb{I}-L)^{-1}K'(x)\right) < 1$, $\forall x \in \overline{\mathsf{N}}$;

iv. $K\overline{\mathsf{N}} + L\overline{\mathsf{N}} \subset \overline{\mathsf{N}}$.

Then, next statements are valid.

a. *There is a unique $x^* \in \overline{\mathsf{N}}$ such that $Kx^* + Lx^* = x^*$.*

b. *If further X is a complex Banach space, then for every initial point $x_0 \in \overline{\mathsf{N}}$, the sequence of iterates $(F^n(x_0))_{n\in\mathbb{N}}$, where $F = (\mathbb{I}-L)^{-1}K$ converges to unique fixed point x^*.*

Proof. a. As in the previous theorem, we define $F = (\mathbb{I}-L)^{-1}K$. Therefore, by Schauder's FPT, we have $\mathrm{Fix}(F) \neq \emptyset$. To establish uniqueness, we consider two separate cases.

Case 1: $\mathrm{Fix}(F) \subset \mathsf{N}$, that is $\forall x \in \partial\mathsf{N}$, $F(x) \neq x$.
In this case the operator $K+L$ has no fixed point on $\partial\mathsf{N}$. Moreover, for every $x \in \mathsf{N}$, we have $r(F'(x)) < 1$, which implies that $1 \notin \sigma(F'(x))$. Therefore,

$$\ker(\mathbb{I} - F'(x)) = \{0\}.$$

This yields,

$$\ker(\mathbb{I} - L - K'(x)) = \ker(\mathbb{I} - (\mathbb{I}-L)^{-1}K'(x)) = \ker(\mathbb{I} - F'(x)) = \{0\}.$$

Hence, the operator $\mathbb{I} - L - K'(x)$ is injective for each $x \in \mathsf{N}$. Consequently, by theorem 3.1, there exists a unique point $x^* \in \mathsf{N}$ verifying

$$Kx^* + Lx^* = x^*.$$

Case 2: $\mathrm{Fix}(F) \cap \partial\mathsf{N} \neq \emptyset$. The proof of this case is established in essentially the same way as the proof of [14, Theorem 1]. For the sake of completeness, we reproduce the argument here, including some corrections.

Suppose first that $\mathrm{Fix}(F) \cap \partial\mathsf{N} = \{x^*\}$. Continuous Fréchet differentiability of F at x^* leads to

$$F(x) = x^* + F'(x^*)(x - x^*) + \varpi(x - x^*), \varpi(x - x^*) = o(\|x - x^*\|).$$

Because the spectral radius of $F'(x^*)$ is less than 1, Theorem 2.1 guarantees the existence of a norm $|\!|\!|\cdot|\!|\!|$ on X, equivalent to $\|.\|$, $\epsilon \in (0, 1)$ for which

$$|\!|\!|F'(x^*)|\!|\!| < \epsilon.$$

Moreover, since $\varpi(x - x^*) = o(\|x - x^*\|)$, there exists $\delta > 0$ such that

$$|\!|\!|\varpi(x - x^*)|\!|\!| \leq (1 - \epsilon)\ |\!|\!|x - x^*|\!|\!|\,, \qquad x \in \mathsf{U} \subset \mathsf{M},$$

where

$$\mathsf{U} := \{x \in X\,;\ |\!|\!|x - x^*|\!|\!| < \delta\}.$$

Hence, for each $x \in \mathsf{U}$,

$$|\!|\!|F(x) - x^*|\!|\!| \leq \alpha\ |\!|\!|x - x^*|\!|\!|\,, \qquad \alpha = |\!|\!|F'(x^*)|\!|\!| + 1 - \epsilon < 1.$$

It follows that U is invariant under F, and consequently,

$$F(\overline{\mathsf{N}} \cup \overline{\mathsf{U}}) \subset \overline{\mathsf{N}} \cup \overline{\mathsf{U}}.$$

Since $r(F'(x)) < 1$ for all $x \in \overline{\mathsf{N}}$ and F' is continuous, U can be chosen so that it contains no other fixed point of F and such that $r(F'(x)) < 1$ for all $x \in \overline{\mathsf{U}}$. Therefore,

$$1 \notin \sigma(F'(x)), \qquad \forall\, x \in \overline{\mathsf{N}} \cup \overline{\mathsf{U}}.$$

This implies that

$$\mathfrak{R} := \{x \in \overline{\mathsf{N}} \cup \overline{\mathsf{U}}\,;\ 1 \notin \sigma(F'(x))\} = \overline{\mathsf{N}} \cup \overline{\mathsf{U}},$$

which is connected and dense. Define

$$S := \{x \in \overline{\mathsf{N}} \cup \overline{\mathsf{U}}\,;\ F(x) = x\}.$$

Then

$$S \cap \partial(\overline{\mathsf{N}} \cup \overline{\mathsf{U}}) = \emptyset, \quad \deg(F, \mathsf{N} \cup \mathsf{U}, 0) = \pm 1, \quad \text{and } S \cap \mathfrak{R} \neq \emptyset.$$

This concludes that F possess a unique fixed point in $\overline{\mathsf{N}} \cup \overline{\mathsf{U}}$, and hence in $\overline{\mathsf{N}}$, by applying the theorem in [13, p. 238].

From the spectral condition in the hypotheses and the inverse function theorem, it follows that each fixed point on $\partial\mathsf{N}$ is isolated. Repeating the argument for the case of a single boundary fixed point shows that if more than one such point were present, the same reasoning would apply to each separately, which contradicts the construction. Therefore, the case of multiple boundary fixed points cannot arise.

b. According to the first statement, $F = (\mathbb{I} - L)^{-1}K$ admits a unique fixed point x^* in $\overline{\mathsf{N}}$. Since $r(F'(x^*)) < 1$, the same argument as in the proof of **(a)** yields a norm $|\!|\!|\cdot|\!|\!|$ on X, equivalent to $\|\cdot\|$, and an open ball

$$\mathsf{U} = \{x \in \mathsf{X}\,;\ |\!|\!|x - x^*|\!|\!| < \delta\} \subset \overline{\mathsf{N}},$$

such that F is a strict contraction on U with respect to $\|\!\cdot\!\|$. Hence there exists $\alpha \in (0,1)$ such that

$$\|F(x) - x^*\| \leq \alpha \, \|x - x^*\|, \qquad (x \in \mathsf{U}).$$

Therefore,

$$F^k(x) \to x^* \quad \text{as } k \to \infty \quad \text{for any } x \in \mathsf{U}. \tag{3.1}$$

Since $\overline{\mathsf{N}}$ is bounded, the sequence $(F^n)_{n\in\mathbb{N}}$ is uniformly bounded on $\overline{\mathsf{N}}$. By Theorem 6.1 in [21, p. 98], it follows that $(F^n)_{n\in\mathbb{N}}$ is equicontinuous on $\overline{\mathsf{N}}$. Moreover, as F is compact, Arzelà–Ascoli's theorem for compact maps [22, p. 267] ensures the existence of a subsequence $(F^{n_p})_{p\in\mathbb{N}}$ converging uniformly on each compact subset of $\overline{\mathsf{N}}$. Define

$$h(x) := \lim_{p\to\infty} F^{n_p}(x), \qquad (x \in \overline{\mathsf{N}}).$$

By Proposition 3.1 in [23, p. 99], h is holomorphic on $\overline{\mathsf{N}}$. From (3.1), we deduce that $h(x) = x^*$ for all $x \in \mathsf{U}$. Since U is open, the identity theorem implies

$$h(x) = x^* \quad \text{for all } x \in \overline{\mathsf{N}}.$$

Thus, for each $x \in \overline{\mathsf{N}}$, there exists $p(x) \in \mathbb{N}^*$ such that $F^{p(x)}(x) \in \mathsf{U}$. Consequently, for every $n \in \mathbb{N}^*$, using (3.1) we obtain

$$F^{n+p(x)}(x) = F^n(F^{p(x)}(x)) \text{ converges to } x^* \quad \text{as } n \to \infty, \qquad (x \in \overline{\mathsf{N}}).$$

This can be written as,

$$\big((\mathbb{I} - L)^{-1}K\big)^n(x) \text{ converges to } x^* \quad \text{as } n \to \infty, \qquad \forall x \in \overline{\mathsf{N}},$$

which proves item b) of the theorem. □

Remarks and open problems. 1. The second part of Theorem 3.2 does not remain valid in the context of real Banach spaces. We provide here a counterexample to illustrate this fact.

Let $\mathsf{X} = (\mathbb{R}^2, \|.\|_1)$, where $\|(h,k)\|_1 = |h| + |k|$ and let N denote the open unit disk of X. Define the linear operator $L = \kappa \mathbb{I}_{\mathbb{R}^2} \in \mathcal{L}(\mathbb{R}^2)$, where $\varsigma \in (0,1)$ and the nonlinear mapping $K : \overline{\mathsf{N}} \to \mathbb{R}^2$ by

$$A(\mu,\nu) = ((1-\varsigma)f(\mu), (1-\varsigma)f(\nu)),$$

where

$$f(\mu) = \begin{cases} 16\,(\mu+0.5)^3 + 12\,(\mu+0.5)^2\,, & \text{if } \mu \in [-1,-0.5], \\ 0, & \text{if } \mu \in [-0.5, 0.5], \\ -16\,(\mu-0.5)^3 + 12\,(\mu-0.5)^2\,, & \text{if } \mu \in [0.5, 1]. \end{cases}$$

We verify that f is continuously differentiable on $[-1,1]$ and that $|f(\mu)| = 1$ if and only if $|\mu| = 1$. Therefore, assumptions (i), (ii), and (iv) of Theorem 3.2 are readily satisfied.

Let us now define the map $F = (\mathbb{I} - L)^{-1}K$. Then, for every $(\mu,\nu) \in \overline{\mathsf{N}}$, the Fréchet derivative of F is expressed by

$$\mathsf{d}F(\mu,\nu) = (\mathbb{I} - L)^{-1}\mathsf{d}K(\mu,\nu) = \begin{pmatrix} 0 & f'(\nu) \\ f'(\mu) & 0 \end{pmatrix}.$$

Consequently, the spectral radius of the derivative is

$$r(\mathsf{d}F(\mu,\nu)) = |f'(\mu)f'(\nu)|^{1/2} = 0 < 1,$$

for all $(\mu,\nu) \in \overline{\mathsf{N}}$. Hence, by item (a) of Theorem 3.2, the origin $(0,0)$ is the unique fixed point of F in $\overline{\mathsf{N}}$. However, the sequence of iterates at $(1,0)$ does not converge to the unique fixed point $(0,0)$, because $F^n(1,0) = \frac{1-(-1)^n}{2}, \frac{1+(-1)^n}{2}$.

This shows that, although uniqueness conditions hold, the global convergence conclusion of Theorem 3.2 fails in this real Banach space setting.

2. In this example, it can be shown that there exists an infinite set of points $(\mu, \nu) \in \overline{\mathsf{N}}$ for which

$$\lim_{n \to +\infty} F^n(\mu, \nu) = (0, 0).$$

Motivated by this observation, we are naturally led to the following conjecture. Under the assumptions of Theorem 3.2, and in the setting where X is a general real Banach space, one may ask whether there exists a nonempty set $\mathsf{M} \subset \overline{\mathsf{N}} \setminus \{x^*\}$ such that, for every $x_0 \in \mathsf{M}$, the sequence $(F^n(x_0))_{n\in\mathbb{N}}$ converges to the unique fixed point x^*. Moreover, the framework where operator L is nonlinear remains an open problem. □

4. Application

Let $\mathfrak{a}$ be a continuous function defined on $[0,1]\times[0,1]$, and let $s > 1$ be a real number such that $1 < s < m$ with $m \geq 2$. We consider the nonlinear operator $K : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ defined by

$$K(\psi)(\mu) := \int_0^1 \mathfrak{a}(\mu, \nu)\,(\psi(\nu))^s\, d\,\nu,$$

and the linear operator $L : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ defined by

$$L\psi(\mu) := \int_0^\mu \psi(\nu)\, d\nu.$$

Let

$$\mathsf{N} := \{\psi \in \mathcal{C}_+([0,1])\,;\, \|\psi\|_\infty < \mathsf{R}\},$$

where $\mathcal{C}_+([0,1])$ is the space of non-negative continuous functions on $[0,1]$, and where

$$\mathsf{R} := \left(\frac{1}{me^2\Gamma}\right)^{\frac{1}{s-1}}, \quad \text{and} \quad \Gamma := \|\mathfrak{a}\|_\infty. \tag{4.1}$$

Proposition 4.1 *There exists a unique function $\psi^* \in \mathsf{N}$ satisfying*

$$K\psi^* + L\psi^* = \psi^*. \tag{4.2}$$

Proof. We proceed by verifying that the assumptions of Theorem 3.1 are met.

i. We first show that some iterate of L is a contraction, i.e.

$$\|L^k\| < 1, \text{ for some } k \in \mathbb{N}^*.$$

By definition,

$$L\psi(\mu) = \int_0^\mu f(\nu)\, d\nu.$$

Applying Fubini's theorem, the second iterate reads

$$L^2\psi(\mu) = \int_0^\mu \left(\int_0^\tau \psi(\nu)\, d\nu\right) d\tau = \int_0^\mu \left(\int_\nu^\mu d\tau\right) \psi(\nu)\, d\nu = \int_0^\mu (\mu - \nu)\psi(\nu)\, d\nu.$$

More generally, one shows by induction for every $p \in \mathbb{N}^*$,

$$L^p\psi(\mu) = \frac{1}{(p-1)!}\int_0^\mu (\mu - \nu)^{p-1}\psi(\nu)\, d\nu, \quad \forall \mu \in [0,1].$$

Thus, for any $\psi \in \mathcal{C}([0,1])$,

$$\begin{aligned}|L^p\psi(\mu)| &\leq \frac{1}{(p-1)!}\|\psi\|_\infty \int_0^\mu (\mu-\nu)^{p-1}d\nu = \frac{1}{(p-1)!}\|\psi\|_\infty \cdot \frac{\mu^p}{p} \\ &\leq \frac{1}{p!}\|\psi\|_\infty, \quad \forall \mu \in [0,1].\end{aligned}$$

Consequently, for any $p \geq 1$,

$$\|L^p\| \leq \frac{1}{p!},$$

therefore,

$$\|L^p\| \to 0 \text{ as } p \to \infty.$$

In particular, one can find $k \in \mathbb{N}^*$ such that

$$\|L^k\| < 1.$$

ii. The Fréchet derivative of K at ψ is

$$(K'(\psi))(\varphi)(\mu) = \int_0^1 s\mathfrak{a}(\mu,\nu)\,(\psi(\nu))^{s-1}\varphi(\nu)\,d\nu. \tag{4.3}$$

By Taylor's expansion, for $\psi \in \mathsf{N}$ there exists a constant $c > 0$ such that

$$\left|(\psi+\varphi)^s - \psi^s - \psi^{s-1}\varphi\right| \leq c\,|\varphi|^2,$$

whence

$$|K(\psi+\varphi)(\mu) - K(\psi)(\mu) - K'(\psi)(\varphi)(\mu)| \leq c\Gamma\|\varphi\|_\infty^2.$$

Therefore,

$$\lim_{\|\varphi\|_\infty \to 0} \frac{\|K(\psi+\varphi) - K(\psi) - K'(\psi)(\varphi)\|_\infty}{\|\varphi\|_\infty} = 0,$$

and K is continuously Fréchet differentiable on N.

To show compactness of $K : \overline{\mathsf{N}} \to \mathcal{C}([0,1])$, note that for $u \in \overline{\mathsf{N}}$ and $\mu_1, \mu_2 \in [0,1]$,

$$\begin{aligned}|K(\psi)(\mu_1) - K(\psi)(\mu_2)| &= \left|\int_0^1 [\mathfrak{a}(\mu_1,\nu) - \mathfrak{a}(\mu_2,\nu)]\,(\psi(\nu))^s\,d\nu\right| \\ &\leq \|\psi\|_\infty^s \int_0^1 |\mathfrak{a}(\mu_1,\nu) - \mathfrak{a}(\mu_2,\nu)|\,d\nu \\ &\leq \mathsf{R}^s \int_0^1 |\mathfrak{a}(\mu_1,\nu) - \mathfrak{a}(\mu_2,\nu)|\,d\nu.\end{aligned}$$

The uniform continuity of $\mathfrak{a}$ ensures the equicontinuity of $\{K(\psi) : \psi \in \overline{\mathsf{N}}\}$. Since $K(\overline{\mathsf{N}})$ is bounded, the Arzelà–Ascoli theorem yields relative compactness, and hence K is compact.

iii. Since $\mathbb{I} - L - K'(\psi)$ is linear, we verify that

$$\ker(\mathbb{I} - L - K'(\psi)) = \{0\}.$$

Let $\varphi \in \ker(\mathbb{I} - L - K'(\psi))$. Then

$$\varphi = ((\mathbb{I} - L)^{-1}K'(\psi))(\varphi). \tag{4.4}$$

From (4.3), we readily deduce that

$$\|K'(\psi)\|_{\mathcal{L}(\mathcal{C}[0,1])} \leq s\Gamma\mathsf{R}^{s-1}. \tag{4.5}$$

Combining this with the estimate $\|(\mathbb{I}-L)^{-1}\|_{\mathcal{L}(\mathcal{C}[0,1])} \leq e^2$ (see the proof below) together with (4.1), we obtain

$$\|(\mathbb{I}-L)^{-1}K'(\psi)\|_{\mathcal{L}(\mathcal{C}[0,1])} < 1.$$

Hence, (4.4) implies $\varphi = 0$, which proves that $\mathbb{I} - L - K'(\psi)$ is injective.

iv. It remains to establish that

$$(\mathbb{I}-L)^{-1}K(\overline{\mathsf{N}}) \subset \overline{\mathsf{N}}.$$

Since $(\mathbb{I}-L)^{-1} \in \mathcal{L}(\mathcal{C}([0,1]))$, we have

$$\|(\mathbb{I}-L)^{-1}K(\psi)\|_\infty \leq \|(\mathbb{I}-L)^{-1}\|_{\mathcal{L}(\mathcal{C}([0,1]))}\|K(\psi)\|_\infty.$$

From point *(i)*, we know that $\|L^k\|_{\mathcal{L}(\mathcal{C}([0,1]))} \leq \frac{1}{k!} < 1$ for all integers $k > 1$. Thus,

$$(\mathbb{I}-L)^{-1} = (\mathbb{I}-L^k)^{-1}\sum_{p=0}^{k-1} L^p.$$

Hence,

$$\begin{aligned}
\|(\mathbb{I}-L)^{-1}\| &\leq \|(\mathbb{I}-L^k)^{-1}\| \sum_{p=0}^{k-1} \|L^p\| \\
&\leq \left\| \sum_{p\geq 0} L^{kp} \right\| \left(\sum_{p\geq 0} \|L^p\| \right) \\
&\leq \left(\sum_{p\geq 0} \frac{1}{(kp)!} \right) \cdot \left(\sum_{p\geq 0} \frac{1}{p!} \right) \leq e^2.
\end{aligned}$$

Let $\psi \in \overline{\mathsf{N}}$. Then,

$$|K(\psi)(\mu)| \leq \int_0^1 |\mathfrak{a}(\mu,\nu)|(\psi(\nu))^s \, d\nu \leq \mathsf{R}^s \sup_{\nu\in[0,1]} |\mathfrak{a}(\mu,\nu)|,$$

so that

$$\|K(\psi)\|_\infty \leq \mathsf{R}^s\Gamma.$$

Therefore,

$$\|(\mathbb{I}-L)^{-1}K(\psi)\|_\infty \leq e^2\mathsf{R}^s\Gamma.$$

According to (4.1), it follows that

$$\|(\mathbb{I}-L)^{-1}K(\psi)\|_\infty \leq \frac{1}{m}\mathsf{R} < \mathsf{R}.$$

Consequently,

$$(\mathbb{I}-L)^{-1}K(\overline{\mathsf{N}}) \subset \overline{\mathsf{N}},, \quad (\mathbb{I}-L)^{-1}K(\psi) \neq \psi, \quad \forall\, \psi \in \partial\mathsf{N}.$$

That is, the operator $K+L$ has no fixed point on $\partial\mathsf{N}$. By Theorem 3.1, there is a unique $\psi^* \in \mathsf{N}$ satisfying

$$K(\psi^*) + L(\psi^*) = \psi^*. \qquad \square$$

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