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Anisotropic Elliptic $\overrightarrow{p}(\cdot)-$Laplacian Systems

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ABSTRACT: Within the framework of this paper, we aim to prove the existence of distributional solutions in the space $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega,\mathbb{R}^m)$ for a new kind of nonlinear elliptic $\overrightarrow{p}(\cdot)-$anisotropic Laplace systems, such that its right-hand side is a nonlinearity connecting the solution u and given functions $\psi_i \in L^{p_i(\cdot)}(\Omega,\mathbb{R}^m)$, $i = 1,\dots,N$.

Key Words: $\mathbb{R}^m-$anisotropic Lebesgue-Sobolev spaces, elliptic $\overrightarrow{p}(\cdot)-$Laplace systems, variable exponents, distributional solution, existence.

Contents

1. Introduction

Throughout our paper, we will work to demonstrate the existence of distributional solutions for a class of Dirichlet boundary value problems represented by a nonlinear elliptic $\overrightarrow{p}(\cdot)-$ anisotropic Laplace systems, of the form

$$\begin{cases} -\Delta_{\overrightarrow{p}(x)}u + \sum_{i=1}^{N}\Theta_i(x,u,\partial_i u) = \alpha u \sum_{i=1}^{N}(|\psi_i| + \beta \mid u \mid)^{p_i(x)-2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded Lipschitz domain in $\mathbb{R}^N$ $(N \geq 2)$, α, β are strictly positive constants, $\psi_i \in L^{p_i(\cdot)}(\Omega,\mathbb{R}^m)$, $i = 1,\dots,N$, $(m \geq 1)$ independent of u, $-\Delta_{\overrightarrow{p}(x)}$ is the $\overrightarrow{p}(x)-$anisotropic Laplace differential operator defined as follows

$$-\Delta_{\overrightarrow{p}(x)}u := -\sum_{i=1}^{N}\partial_i\big(\mid \partial_i u \mid^{p_i(x)-2} \partial_i u\big),$$

$\Theta_i : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$, $i = 1,\dots,N$, are a Carathéodory functions such that, for almost everywhere $x \in \Omega$ and every s, $\xi \in \mathbb{R}^m$, there exists $c > 0$

$$|\Theta_i(x,s,\xi)| \leq \phi_i(x) + c(|s| + |\xi|)^{p_i(x)-1}, \quad \text{such that } \phi_i \in L^{p_i'(\cdot)}(\Omega). \tag{1.2}$$

System (1.1) is $\overrightarrow{p}(x)$-anisotropic Laplacian operator system type, and that's to involve it the variable exponents anisotropic differential operator (i.e. $-\Delta_{\overrightarrow{p}(x)}$) defined from the space $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega,\mathbb{R}^m)$ to its dual. Here we must recall the importance of this kinds of problems, this appears in the treatment of many scientific phenomena and robotics, including describing models in image processing, electrorheological and thermorheological fluids, as seen in references [22,23,24,25,26,27]. The existence results of systems with this type of differential operators and others with various conditions and data were presented (but not limited to) in the papers [1,2,3,4,5,6,7,8]. Here we tried to depart from the usual (the classical

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case) by considering the right-hand side as a nonlinearity linking the solution u to given functions $\psi_i \in L^{p_i(\cdot)}(\Omega,\mathbb{R}^m)$, $i=1,\dots,N$, without the possibility of formulating to the usual case (i.e., the right-hand side is a datum belongs to certain Lebesgue or Sobolev spaces).

Our proof relied on the sequence of suitable approximate solutions (u_n), thanks to the Theorem of existence of Leray-Schauder's fixed point that ensured its existence, and then we moved to provide prior estimates on the solution and its partial derivatives, where we proved the boundedness of u_n and both strong convergence in $L^{p_i(\cdot)}(\Omega,\mathbb{R}^m)$ and the almost everywhere convergence in Ω for $\partial_i u_n$, $i=1,\dots,N$. Passing to the limit by L^1-strongly sense in both $\mid \partial_i u_n \mid^{p_i(x)-2} \partial_i u_n$, $u_n(\mid \psi_i \mid +\beta \mid u_n \mid)^{p_i(x)-2}$, and $\Theta_i(x,u_n,\partial_i u_n)$. Then we deduce the convergence of u_n to the desired solution u of (1.1).

Basic concepts and definitions with the most important properties of variable exponents anisotropic Lebesgue-Sobolev spaces and their $\mathbb{R}^m$-valued versions are discussed in Section 2. The main result with proof is in Section 3.

2. Preliminaries

In this section we will address the $p(\cdot)-$anisotropic Lebesgue-Sobolev spaces and their $\mathbb{R}^m$-valued versions. For more about these spaces see [10,11,12].

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded open subset. Set

$$\mathcal{C}_+(\overline{\Omega}) = \{p(\cdot) \in C(\overline{\Omega}) : \, p^- = \min_{x\in\overline{\Omega}} p(x) > 1\},$$

where $C(\overline{\Omega})$ is the set of continuous real functions on $\overline{\Omega}$
Let $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$, for every $\alpha,\beta \in \mathbb{R}$ and every $\theta > 0$, the following inequality (it's called Young's inequality)

$$|\alpha\beta| \leq \theta|\alpha|^{p(x)} + c(\theta)|\beta|^{p'(x)},$$

holds true, where $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$ in $\overline{\Omega}$ (the Hölder conjugate of $p(\cdot)$). If $((\alpha,\beta) \neq (0,0))$, the following inequality is true

$$(|\alpha|^{p(x)-2}\alpha - |\beta|^{p(x)-2}\beta)(\alpha-\beta) \geq \begin{cases} 2^{2-p^+}|\alpha-\beta|^{p(x)}, & \text{if } p(x) \geq 2, \\ (p^- - 1)\frac{|\alpha-\beta|^2}{(|\alpha|+|\beta|)^{2-p(x)}}, & \text{if } p(x) \in (1,2). \end{cases} \tag{2.1}$$

Also, for all $p = \text{(constant)} > 0$ and all $\alpha_i \geq 0$, $i=1,\dots,r$, we have

$$(\alpha_1 + \ldots + \alpha_r)^p \leq \max\{1, r^{p-1}\}(\alpha_1^p + \ldots + \alpha_r^p). \tag{2.2}$$

The reflexive Banach $p(\cdot)-$Lebesgue space $L^{p(\cdot)}(\Omega)$ defined by

$$L^{p(\cdot)}(\Omega) := \{\text{measurable functions } u : \Omega \mapsto \mathbb{R}; \int_\Omega |u(x)|^{p(x)} dx < \infty\},$$

under the norm

$$\|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\gamma > 0 :\mid \rho_{p(\cdot)}(u/\gamma) \leq 1\right\},$$

where $u \mapsto \rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)}dx$ is called the convex modular of u.
For every $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the Hölder inequality is defined as

$$\mid \int_\Omega uv\,dx \mid \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$

The Banach $p(\cdot)-$Sobolev space $W^{1,p(\cdot)}(\Omega)$ defined as fellows

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega) : |\partial u| \in L^{p(\cdot)}(\Omega)\right\},$$

when equipped with the norm

$$u \mapsto \|u\|_{W^{1,p(\cdot)}(\Omega)} := \|\partial u\|_{p(\cdot)}. \tag{2.3}$$

We define also the reflexive separable Banach space $\left(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{W^{1,p(\cdot)}(\Omega)}\right)$ by

$$W_0^{1,p(\cdot)}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}.$$

We have the following key results [11,12]. If $u \in L^{p(\cdot)}(\Omega)$, then

$$\min\left(\rho_{p(\cdot)}^{\frac{1}{p^+}}(u), \rho_{p(\cdot)}^{\frac{1}{p^-}}(u)\right) \leq \|u\|_{p(\cdot)} \leq \max\left(\rho_{p(\cdot)}^{\frac{1}{p^+}}(u), \rho_{p(\cdot)}^{\frac{1}{p^-}}(u)\right), \tag{2.4}$$

$$\min\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right) \leq \rho_{p(\cdot)}(u) \leq \max\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right). \tag{2.5}$$

Now we will talk about variable $\overrightarrow{p}(\cdot)-$anisotropic Sobolev spaces $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.
Let $p_i(\cdot) \in C(\overline{\Omega}, [1, +\infty)),\ i = 1, \ldots, N$, and we set for every x in $\overline{\Omega}$

$$\overrightarrow{p}(x) = (p_1(x), \ldots, p_N(x)), \quad p_+(x) = \max_{i\in\{1,\ldots,N\}} p_i(x), \quad p_-(x) = \min_{i\in\{1,\ldots,N\}} p_i(x),$$

$$\overline{p}(x) = N\left(\sum_{i=1}^N \frac{1}{p_i(x)}\right)^{-1}, \quad \overline{p}^\star(x) = \frac{N\overline{p}(x)}{N - \overline{p}(x)} \ \text{ if } \overline{p}(x) < N.$$

The Banach space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined by

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{u \in L^{p_+(\cdot)}(\Omega),\ \partial_i u \in L^{p_i(\cdot)}(\Omega),\ i = 1, \ldots, N\right\},$$

under the norm

$$\|u\|_{\overrightarrow{p}(\cdot)} := \|u\|_{p_+(\cdot)} + \sum_{i=1}^N \|\partial_i u\|_{p_i(\cdot)}. \tag{2.6}$$

The reflexive separable Banach space $\left(\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \|\cdot\|_{\overrightarrow{p}(\cdot)}\right)$ is defined as follows

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega).$$

We have the following embedding [9,10]. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$.

Lemma 2.1 *If $s(\cdot) \in \mathcal{C}_+(\overline{\Omega})$ and $s(\cdot) < \max\{p_+(\cdot),\ \overline{p}^\star(\cdot)\}$ in $\overline{\Omega}$. Then the embedding*

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega) \text{ is compact.} \tag{2.7}$$

Lemma 2.2 *If*

$$p_+(x) < \overline{p}^\star(x),\ \forall x \in \overline{\Omega}. \tag{2.8}$$

Then

$$\|u\|_{p_+(\cdot)} \leq C\sum_{i=1}^N \|\partial_i u\|_{p_i(\cdot)},\ \forall u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{2.9}$$

where $C > 0$ independent of u. Thus,

$$u \mapsto \sum_{i=1}^N \|\partial_i u\|_{p_i(\cdot)} \text{ is equivalent to (2.6) on } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega).$$

Since our paper is devoted to dealing with a system of the form (1.1), we need the spaces $X = L^{p(\cdot)}(\Omega, \mathbb{R}^m)$, $Y = \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m)$, which represent the $\mathbb{R}^m$-valued version for the spaces $L^{p(\cdot)}(\Omega)$, $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ respectively, and becomes a reflexive separable Banach spaces under the norms

$$u \mapsto \|u\|_X := \||u|\|_{p(\cdot)},\ u \mapsto \|u\|_Y := \||u|\|_{\overrightarrow{p}(\cdot)}. \tag{2.10}$$

3. Statement of Results and Proof

Definition 3.1 The vector-valued function $u=(u_1,\dots,u_m)^\top:\Omega\longrightarrow\mathbb{R}^m$ is a distributions solution of the system (1.1) if and only if $u\in W_0^{1,1}(\Omega,\mathbb{R}^m)$, and for every $\varphi\in C_c^\infty(\Omega,\mathbb{R}^m)$,

$$\sum_{i=1}^N\int_\Omega |\partial_i u|^{p_i(x)-2}\,\partial_i u\cdot\partial_i\varphi\,dx+\sum_{i=1}^N\int_\Omega\Theta_i(x,u,\partial_i u)\cdot\varphi\,dx =\alpha\sum_{i=1}^N\int_\Omega u(|\psi_i|+\beta|u|)^{p_i(x)-2}\cdot\varphi\,dx.$$

The main result of our work is represented by the following theorem.

Theorem 3.1 *Let $\overrightarrow{p}(\cdot)\in(\mathcal{C}_+(\overline{\Omega}))^N$ such that $\overline{p}(\cdot)<N$ in $\overline{\Omega}$, and (2.8) is satisfied. Assume that $\psi_i\in L^{p_i(\cdot)}(\Omega,\mathbb{R}^m)$, $i=1,\dots,N$, and Θ_i, $i=1,\dots,N$ be Carathéodory functions are satisfies (1.2). Then the system (1.1) accepts at least one distributional solution u in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega,\mathbb{R}^m)$.*

3.1. Existence of approximate solutions

For every $(\upsilon,\delta)\in L^{p_+(\cdot)}(\Omega,\mathbb{R}^m)\times[0,1]$ given, let us consider the following system related to binary (υ,δ) with the unknown u

$$\begin{cases}-\Delta_{\overrightarrow{p}(x)}u=\delta\left(\alpha\displaystyle\sum_{i=1}^N\upsilon(|\psi_i|+\beta|\upsilon|)^{p_i(x)-2}-\sum_{i=1}^N\Theta_i(x,\upsilon,\partial_i\upsilon)\right) & \text{in }\ \Omega,\\ u=0 & \text{on }\ \partial\Omega.\end{cases}\tag{3.1}$$

Lemma 3.1 *Let $\overrightarrow{p}(\cdot)$, Θ_i, ψ_i, $i=1,\dots,N$ be restricted as in Theorem 3.1. Then, for every $(\upsilon,\delta)\in L^{p_+(\cdot)}(\Omega,\mathbb{R}^m)\times[0,1]$ given, the system (3.1) has only one solution in the weak sense.*

Proof: Let $(\upsilon,\delta)\in L^{p_+(\cdot)}(\Omega,\mathbb{R}^m)\times[0,1]$ given. Through the use of (2.2) and the fact that $\beta|\upsilon|\le|\psi_i|+\beta|\upsilon|$, and that $\psi_i,\ \upsilon\in L^{p_i(\cdot)}(\Omega,\mathbb{R}^m)$ we get for every $i=1,\dots,N$

$$\int_\Omega\left|\upsilon(|\psi_i|+\beta|\upsilon|)^{p_i(x)-2}\right|^{p_i'(x)}dx\le\max\{\beta^{-(p')_-^-},\beta^{-(p')_+^+}\}\int_\Omega\left|(|\psi_i|+\beta|\upsilon|)^{p_i(x)-1}\right|^{p_i'(x)}$$
$$\le c\int_\Omega(|\psi_i|^{p_i(x)}+\max\{\beta^{(p')_-^-},\beta^{(p')_+^+}\}|\upsilon|^{p_i(x)})\,dx\le c'.\tag{3.2}$$

Then, (3.2) and (2.4) means for every $i=1,\dots,N$ that

$$\left\||\upsilon(|\psi_i|+\beta|\upsilon|)^{p_i(x)-2}|\right\|_{p_i'(\cdot)}\le C.\tag{3.3}$$

Also by (1.2), (2.2), and that $\upsilon\in L^{p_i(\cdot)}(\Omega,\mathbb{R}^m)$, we obtain that

$$\int_\Omega|\Theta_i(x,\upsilon,\partial_i\upsilon)|^{p_i'(x)}\ dx\le\int_\Omega\left(|\phi_i|+c(|\upsilon|+|\partial_i\upsilon|)^{p_i(x)-1}\right)^{p_i'(x)}\ dx$$
$$\le c'\int_\Omega(|\phi_i|^{p_i'(x)}+|\upsilon|^{p_i(x)}+|\partial_i\upsilon|^{p_i(x)})\,dx\le c''.\tag{3.4}$$

Then, (3.4) and (2.4) implies for every $i=1,\dots,N$ that

$$\||\Theta_i(x,\upsilon,\partial_i\upsilon)|\|_{p_i'(\cdot)}\le C'.\tag{3.5}$$

Thus, we have proven the boundedness of the right-hand side of (3.1) in $L^{p_i'(\cdot)}(\Omega,\mathbb{R}^m)$. So the existence of a weak solution is a direct result of the main Theorem on monotone operators. Now let's move on to

prove the uniqueness of this solution.
Let u_1, u_2 be two weak solutions of (3.1). So, we have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} \mid \partial_i u_1 \mid^{p_i(x)-2} \partial_i u_1 \cdot \partial_i \varphi \, dx \\
&= \delta\left(\alpha \sum_{i=1}^{N}\int_{\Omega} \upsilon(\mid \psi_i \mid + \beta \mid \upsilon \mid)^{p_i(x)-2} \cdot \varphi \, dx - \sum_{i=1}^{N}\int_{\Omega} \Theta_i(x,\upsilon,\partial_i \upsilon) \cdot \varphi \, dx\right), \qquad (3.6)\\
&\sum_{i=1}^{N}\int_{\Omega} \mid \partial_i u_2 \mid^{p_i(x)-2} \partial_i u_2 \cdot \partial_i \varphi \, dx \\
&= \delta\left(\alpha \sum_{i=1}^{N}\int_{\Omega} \upsilon(\mid \psi_i \mid + \beta \mid \upsilon \mid)^{p_i(x)-2} \cdot \varphi \, dx - \sum_{i=1}^{N}\int_{\Omega} \Theta_i(x,\upsilon,\partial_i \upsilon) \cdot \varphi \, dx\right). \qquad (3.7)
\end{aligned}$$

By taking $\varphi = u_1 - u_2$ as a test function in (3.6) and in (3.7), then subtracting the results side by side, we can deduce that

$$\sum_{i=1}^{N}\int_{\Omega} \left(\mid \partial_i u_1 \mid^{p_i(x)-2} \partial_i u_1 - \mid \partial_i u_2 \mid^{p_i(x)-2} \partial_i u_2\right) \cdot (\partial_i u_1 - \partial_i u_2) \, dx = 0. \qquad (3.8)$$

Since (2.1), we conclude for every $i = 1, \ldots, N$ that

$$\left(\mid \partial_i u_1 \mid^{p_i(x)-2} \partial_i u_1 - \mid \partial_i u_2 \mid^{p_i(x)-2} \partial_i u_2\right) \cdot (\partial_i u_1 - \partial_i u_2) \geq 0.$$

From this and (3.8), we deduce for every $i = 1, \ldots, N$ that

$$\int_{\Omega} \left(\mid \partial_i u_1 \mid^{p_i(x)-2} \partial_i u_1 - \mid \partial_i u_2 \mid^{p_i(x)-2} \partial_i u_2\right) \cdot (\partial_i u_1 - \partial_i u_2) \, dx = 0. \qquad (3.9)$$

Now, after putting, for every $i = 1, \ldots, N$,

$$\Lambda_i = \int_{\Omega} \left(\mid \partial_i u_1 \mid^{p_i(x)-2} \partial_i u_1 - \mid \partial_i u_2 \mid^{p_i(x)-2} \partial_i u_2\right) \cdot (\partial_i u_1 - \partial_i u_2) \, dx,$$

$$\Omega_{i,1} = \{x \in \Omega,\, p_i(x) \geq 2\}, \text{ and } \Omega_{i,2} = \{x \in \Omega,\, p_i(x) \in (1,2)\}, \qquad (3.10)$$

and like the proof steps followed in [13,14,15,16,17,18,19,20,21], we can obtain, for all $i = 1, \ldots, N$

$$\int_{\Omega_{i,1}} |\partial_i(u_1 - u_2)|^{p_i(x)} \, dx \leq c\Lambda_i, \qquad (3.11)$$

$$\text{and },\int_{\Omega_{i,2}} |\partial_i(u_1 - u_2)|^{p_i(x)} \, dx \leq c' \max\left\{\Lambda_i^{\frac{p_i^-}{2}}, \Lambda_i^{\frac{p_i^+}{2}}\right\}. \qquad (3.12)$$

By combining (3.11), (3.12), and (3.9), we get that

$$\int_{\Omega} |\partial_i(u_1 - u_2)|^{p_i(x)} \, dx = 0, \quad i = 1, \ldots, N. \qquad (3.13)$$

Then, from (3.13) and (2.5) we conclude that

$$\left\| |\partial_i(u_1 - u_2)| \right\|_{p_i(\cdot)} = 0, \quad i = 1, \ldots, N. \qquad (3.14)$$

By using the following fact, $\forall u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m)$

$$|\partial_i |u|| \leq |\partial_i u|, \qquad (3.15)$$

and using (2.8), (3.14), we deduce that

$$\left\| |u_1 - u_2| \right\|_{\overrightarrow{p}(\cdot)} = 0, \quad i = 1, \ldots, N. \tag{3.16}$$

Then, (3.16) implies that $u_1 = u_2$. □

Lemma 3.2 *The operator $T : L^{p_+(\cdot)}(\Omega, \mathbb{R}^m) \times [0,1] \longrightarrow L^{p_+(\cdot)}(\Omega, \mathbb{R}^m)$ defined as follows*

$$T(\upsilon, \delta) = u \Leftrightarrow (u \text{ is the only weak solution of the problem } (3.1)),$$

is continuous and compact. Moreover, there exists $C > 0$, such that for every $(\upsilon, \delta) \in L^{p_+(\cdot)}(\Omega, \mathbb{R}^m) \times [0,1]$,

$$\upsilon = T(\upsilon, \delta) \Rightarrow \left\| |\upsilon| \right\|_{p_+(\cdot)} \leq C. \tag{3.17}$$

In addition,

$$\forall \upsilon \in L^{p_+(\cdot)}(\Omega, \mathbb{R}^m) : T(\upsilon, 0) = 0. \tag{3.18}$$

Proof: Choosing u as a test function in the weak formulation of (3.1), and through the use of (2.8), (1.2), (2.2), (2.4), (2.5), Lemma 2.1, and Hölder inequality, we can deduce that

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega |\ \partial_i u\ |^{p_i(x)}\ dx \leq \alpha \sum_{i=1}^{N} \int_\Omega (|\ \psi_i\ | + \beta\ |\ \upsilon\ |)^{p_i(x)-1}\ |\ u\ |\ dx + \sum_{i=1}^{N} \int_\Omega |\ \Theta_i(x, \upsilon, \partial_i \upsilon)\ ||\ u\ |\ dx \\
&\leq 2 \left\| \sum_{i=1}^{N} (|\ \psi_i\ | + |\ \upsilon\ |)^{p_i(x)-1} \right\|_{p_i'(\cdot)} \| |u| \|_{p_i(\cdot)} + 2 \left\| \sum_{i=1}^{N} \left| \phi_i(x) + c(|\upsilon| + |\partial_i \upsilon|)^{p_i(x)-1} \right| \right\|_{p_i'(\cdot)} \| |u| \|_{p_i(\cdot)} \\
&\leq c \left(\sum_{i=1}^{N} \left\| |\ \psi_i\ |^{p_i(x)-1} \right\|_{p_i'(\cdot)} + \sum_{i=1}^{N} \left\| |\ \upsilon\ |^{p_i(x)-1} \right\|_{p_i'(\cdot)} \right) \| |u| \|_{\overrightarrow{p}(\cdot)} \\
&+ c' \left(\sum_{i=1}^{N} \left\| |\ \phi_i\ |^{p_i(x)-1} \right\|_{p_i'(\cdot)} + \sum_{i=1}^{N} \left\| |\ \upsilon\ |^{p_i(x)-1} \right\|_{p_i'(\cdot)} + \sum_{i=1}^{N} \left\| |\ \partial_i \upsilon\ |^{p_i(x)-1} \right\|_{p_i'(\cdot)} \right) \| |u| \|_{\overrightarrow{p}(\cdot)} \\
&\leq c'' \left(1 + \sum_{i=1}^{N} \left(\int_\Omega |\ \upsilon\ |^{p_i(x)}\ dx \right)^{\frac{1}{p_i^-}} + \sum_{i=1}^{N} \left(\int_\Omega |\ \partial_i \upsilon\ |^{p_i(x)}\ dx \right)^{\frac{1}{p_i^-}} \right) \| |u| \|_{\overrightarrow{p}(\cdot)} \\
&\leq C \left(1 + \sum_{i=1}^{N} \| |\upsilon| \|_{p_i(\cdot)}^{\frac{p_i^+}{p_i^-}} + \sum_{i=1}^{N} \| |\partial_i \upsilon| \|_{p_i(\cdot)}^{\frac{p_i^+}{p_i^-}} \right) \| |u| \|_{\overrightarrow{p}(\cdot)} \leq C' \left(1 + \sum_{i=1}^{N} \| |\upsilon| \|_{p_i(\cdot)}^{\frac{p_+^+}{p_-^-}} + \sum_{i=1}^{N} \| |\partial_i \upsilon| \|_{p_i(\cdot)}^{\frac{p_+^+}{p_-^-}} \right) \| |u| \|_{\overrightarrow{p}(\cdot)} \\
&\leq C'' \left(1 + \| |\upsilon| \|_{p_+(\cdot)}^{\frac{p_+^+}{p_-^-}} + \| |\upsilon| \|_{\overrightarrow{p}(\cdot)}^{\frac{p_+^+}{p_-^-}} \right) \| |u| \|_{\overrightarrow{p}(\cdot)} \leq C''' \left(1 + \| |\upsilon| \|_{p_+(\cdot)}^{\frac{p_+^+}{p_-^-}} \right) \| |u| \|_{\overrightarrow{p}(\cdot)}.
\end{aligned} \tag{3.19}$$

Now, we also have through (2.5) that, for all $i = 1, \ldots, N$

$$1 + \int_\Omega |\ \partial_i u\ |^{p_i(x)}\ dx \geq \left\| \partial_i u \right\|_{p_i(\cdot)}^{p_i^-}, \quad \text{and} \quad 1 + \left\| |\partial_i u| \right\|_{p_i(\cdot)}^{p_i^-} \geq \left\| |\partial_i u| \right\|_{p_i(\cdot)}^{p_-^-}.$$

Consequently, we deduce that

$$2 + \int_\Omega |\ \partial_i u\ |^{p_i(x)}\ dx \geq \left\| |\partial_i u| \right\|_{p_i(\cdot)}^{p_-^-},\ i = 1, \ldots, N.$$

So, we obtain that

$$\sum_{i=1}^{N} \int_\Omega |\ \partial_i u\ |^{p_i(x)}\ dx \geq \left(\frac{1}{N} \sum_{i=1}^{N} \left\| |\partial_i u| \right\|_{p_i(\cdot)} \right)^{p_-^-} - 2N\ |\ \Omega\ |. \tag{3.20}$$

Combine (3.19) and (3.20), we conclude that

$$\||u|\|_{\overrightarrow{p}(\cdot)}^{p_-^-} \leq c\left(1+\||v|\|_{p_+(\cdot)}^{\frac{p_+^+}{p_-^-}}\right)\||u|\|_{\overrightarrow{p}(\cdot)}. \tag{3.21}$$

Then, we obtain

$$\||u|\|_{\overrightarrow{p}(\cdot)} \leq C\left(1+\||v|\|_{p_+(\cdot)}^{\frac{p_+^+}{p_-^-(p_-^- -1)}}\right). \tag{3.22}$$

After arriving at this initial estimate of the solution u, let us move on to proving the continuity of T. Let $(v_k) \subset L^{p_+(\cdot)}(\Omega, \mathbb{R}^m)$, $(\delta_k) \subset [0,1]$, $(k \in \mathbb{N},\, k \geq 1)$ be two sequences, such that

$$v_k \longrightarrow v, \quad \text{strongly in } L^{p_+(\cdot)}(\Omega, \mathbb{R}^m), \tag{3.23}$$

$$\delta_k \longrightarrow \delta, \quad \text{strongly in } \mathbb{R}. \tag{3.24}$$

For the previous two limits v and δ, we put $u = T(v, \delta)$, and this equivalent to that, for every $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m)$

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i(x)-2} \partial_i u \cdot \partial_i \varphi \, dx \\ &\quad = \delta\left(\alpha \sum_{i=1}^N \int_\Omega v(|\ \psi_i\ | + \beta |v|)^{p_i(x)-2} \cdot \varphi \, dx - \sum_{i=1}^N \int_\Omega \Theta_i(x, v, \partial_i v) \cdot \varphi \, dx\right).\end{aligned} \tag{3.25}$$

For $n \geq 1$ fixed in $\mathbb{N}$, let us consider the sequence (u_k), such that

$$u_k = T(v_k, \delta_k),\ (k \in \mathbb{N},\, k \geq 1).$$

So, we obtain, for every $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m)$

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega |\partial_i u_k|^{p_i(x)-2} \partial_i u_k \cdot \partial_i \varphi \, dx \\ &\quad = \delta_k\left(\alpha \sum_{i=1}^N \int_\Omega v_k(|\ \psi_i\ | + \beta |v_k|)^{p_i(x)-2} \cdot \varphi \, dx - \sum_{i=1}^N \int_\Omega \Theta_i(x, v_k, \partial_i v_k) \cdot \varphi \, dx\right).\end{aligned} \tag{3.26}$$

Since the sequence (v_k) is bounded in $L^{p_+(\cdot)}(\Omega, \mathbb{R}^m)$ (thanks to (3.23)). Therefore through this and (3.22), we can infer the boundedness of $(u_k)(= T(v_k, \delta_k))$ in $L^{p_+(\cdot)}(\Omega, \mathbb{R}^m)$.
We can then, thanks to the reflexivity of $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m)$ and the compactness of its embedding into $L^{p_+(\cdot)}(\Omega, \mathbb{R}^m)$ (due (2.8) and Lemma 2.1), extract a subsequence (still denoted by (u_k)) and there exists $w \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m)$ such that

$$u_k \rightharpoonup w \quad \text{weakly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m), \tag{3.27}$$

$$\text{and } u_k \longrightarrow w \quad \text{strongly in } L^{p_+(\cdot)}(\Omega, \mathbb{R}^m). \tag{3.28}$$

Through the continuity of $v \mapsto \alpha \sum\limits_{i=1}^N v(|\ \psi_i\ | + \beta |v|)^{p_i(x)-2} - \sum\limits_{i=1}^N \Theta_i(x, v, \partial_i v)$ on $L^{p_+(\cdot)}(\Omega, \mathbb{R}^m)$ towards itself, (3.23), and (3.24), we can pass to the limit in (3.26) as $k \longrightarrow +\infty$, then we obtain for every $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m)$,

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega |\partial_i w|^{p_i(x)-2} \partial_i w \cdot \partial_i \varphi \, dx \\ &\quad = \delta\left(\alpha \sum_{i=1}^N \int_\Omega v(|\ \psi_i\ | + \beta |v|)^{p_i(x)-2} \cdot \varphi \, dx - \sum_{i=1}^N \int_\Omega \Theta_i(x, v, \partial_i v) \cdot \varphi \, dx\right).\end{aligned} \tag{3.29}$$

So, (3.29) means that $w = T(\upsilon, \delta)$.
Then, thanks to Lemma 3.1, we deduce that, $w = u = T(\upsilon, \delta)$ (where u defined in (3.25)). So, we conclude that $u_k \longrightarrow u$ strongly in $L^{p_+(\cdot)}(\Omega, \mathbb{R}^m)$,, and this means the continuity of T.
Let $\widehat{\mathcal{B}} \subset L^{p_+(\cdot)}(\Omega, \mathbb{R}^m)$ be a bounded. Then,
there exists a ball $\mathcal{B}(0, r)$ (i.e. of center 0 and of radius $r > 0$), such that

$$\widehat{\mathcal{B}} \subset \mathcal{B} \subset L^{p_+(\cdot)}(\Omega, \mathbb{R}^m),$$

and this equivalent to that

$$\tilde{\mathfrak{B}} = \widehat{\mathcal{B}} \times [0,1] \subset \mathcal{B} \times [0,1] \subset L^{p_+(\cdot)}(\Omega, \mathbb{R}^m) \times [0,1]. \tag{3.30}$$

Relationship (3.30) implies that, $\tilde{\mathfrak{B}}$ is a bounded of the space $L^{p_+(\cdot)}(\Omega, \mathbb{R}^m) \times [0,1]$.
Let $u \in T(\tilde{\mathfrak{B}})$, then there exists $(\upsilon, \delta) \in \mathcal{B} \times [0,1]$ (i.e. $\||\upsilon|\|_{p_+(\cdot)} \le r$), such that $u = T(\upsilon, \delta)$.
By using (3.22), we deduce that $\||u|\|_{\overrightarrow{p}(\cdot)} \le \varrho$ $(\varrho > 0)$.
This implies that $T(\tilde{\mathfrak{B}}) \subset \mathcal{B}(0, \varrho) \subset \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m)$,
where, $\mathcal{B}(0, \varrho)$ a closed ball (of center 0 and of radius $\varrho > 0$) in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m) \subset L^{p_+(\cdot)}(\Omega, \mathbb{R}^m)$.
Let $(u_k) \subset T(\tilde{\mathfrak{B}})$ be a sequence, then there exists $(\upsilon_k, \delta_k) \in \mathcal{B} \times [0,1]$ (i.e. $\||\upsilon_k|\|_{p_+(\cdot)} \le r$), such that $u_k = T(\upsilon_k, \delta_k)$.
Since $\||u|\|_{\overrightarrow{p}(\cdot)} \le \varrho$, then there exists a subsequence (still denoted by (u_k)) and $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m)$, such that $u_k \rightharpoonup u$ weakly in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m)$.
Thanks to (2.8) and Lemma 2.1, we deduce that $u_k \longrightarrow u$ strongly in $L^{p_+(\cdot)}(\Omega, \mathbb{R}^m)$.
This means that T is compact.
Now, for every $\upsilon \in L^{p_+(\cdot)}(\Omega, \mathbb{R}^m)$ such that $\upsilon = T(\upsilon, \delta)$, we have that

$$\begin{aligned} &\forall \varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m), \sum_{i=1}^{N} \int_\Omega |\partial_i \upsilon|^{p_i(x)-2} \partial_i \upsilon \cdot \partial_i \varphi \, dx = \\ &\delta \left(\alpha \sum_{i=1}^{N} \int_\Omega \upsilon(|\ \psi_i\ | + \beta |\upsilon|)^{p_i(x)-2} \cdot \varphi \, dx - \sum_{i=1}^{N} \int_\Omega \Theta_i(x, \upsilon, \partial_i \upsilon) \cdot \varphi \, dx \right). \end{aligned} \tag{3.31}$$

Taking $\varphi = \upsilon$ in (3.31), and using (2.2), (1.2), the fact that $\beta|\upsilon| \le |\psi_i| + \beta|\upsilon|$, (3.5), (2.4) and that $\psi_i \in L^{p_i(\cdot)}(\Omega, \mathbb{R}^m)$, $\upsilon \in L^{p_+(\cdot)}(\Omega, \mathbb{R}^m)$, we deduce that

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega |\ \partial_i \upsilon\ |^{p_i(x)}\ dx &\le c \sum_{i=1}^{N} \int_\Omega (|\ \psi_i\ | + \beta\ |\ \upsilon\ |)^{p_i(x)} \, dx + \sum_{i=1}^{N} \int_\Omega |\Theta_i(x, \upsilon, \partial_i \upsilon)||\upsilon| \, dx \\ &\le c' \sum_{i=1}^{N} \int_\Omega (|\ \psi_i\ |^{p_i(x)} + c''\ |\ \upsilon\ |^{p_i(x)}) \, dx + 2 \sum_{i=1}^{N} \||\Theta_i(x, \upsilon, \partial_i \upsilon)|\|_{p_i'(\cdot)} \||\upsilon|\|_{p_i(\cdot)} \\ &\le c' \sum_{i=1}^{N} \int_\Omega (|\ \psi_i\ |^{p_i(x)} + c''\ |\ \upsilon\ |^{p_i(x)}) \, dx + c''' \sum_{i=1}^{N} \||\Theta_i(x, \upsilon, \partial_i \upsilon)|\|_{p_i'(\cdot)} \||\upsilon|\|_{p_+(\cdot)} \le C. \end{aligned} \tag{3.32}$$

On the other hand, like proof (3.20), we can deduce that

$$\sum_{i=1}^{N} \int_\Omega |\ \partial_i \upsilon\ |^{p_i(x)}\ dx \ge \left(\frac{1}{N} \sum_{i=1}^{N} \||\partial_i \upsilon|\|_{p_i(\cdot)} \right)^{p_-^-} - 2N\ |\ \Omega\ |\ . \tag{3.33}$$

The combine of (3.32) and (3.33), gives us

$$\sum_{i=1}^{N} \||\partial_i \upsilon|\|_{p_i(\cdot)} \le C'''. \tag{3.34}$$

From (3.34) with using (2.8), (2.9), and (3.15), we obtain (3.17).
Obviously, (3.18) is valid, Because we simply find that, $u = 0 \in L^{p_+(\cdot)}(\Omega, \mathbb{R}^m)$ the only weak solution of (3.1) when $\delta = 0$. Thus, the proof of Lemma 3.2 is completed. □

Lemma 3.3 *Let $\overrightarrow{p}(\cdot)$, Θ_i, ψ_i, $i = 1, \ldots, N$ be restricted as in Theorem 3.1. Then, there exists at least one weak solution $u_n \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m)$ to the approximated problems*

$$\begin{cases} -\Delta_{\overrightarrow{p}(x)} u_n + \sum_{i=1}^{N} \Theta_i(x, u_n, \partial_i u_n) = \alpha u_n \sum_{i=1}^{N} (|\psi_i| + \beta \mid u_n \mid)^{p_i(x)-2} & in \ \ \Omega, \\ u_n = 0 & on \ \ \partial\Omega, \end{cases} \tag{3.35}$$

in this sense

$$\sum_{i=1}^{N} \int_{\Omega} \mid \partial_i u_n \mid^{p_i(x)-2} \partial_i u_n \cdot \partial_i \varphi \, dx + \sum_{i=1}^{N} \int_{\Omega} \Theta_i(x, u_n, \partial_i u_n) \cdot \varphi \, dx$$
$$= \alpha \sum_{i=1}^{N} \int_{\Omega} u_n(\mid \psi_i \mid + \beta \mid u_n \mid)^{p_i(x)-2} \cdot \varphi \, dx, \tag{3.36}$$

for every $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m) \cap L^{\infty}(\Omega, \mathbb{R}^m)$.

Proof: The results of Lemma 3.2 are a direct fulfillment of all the conditions of the Leray-Schauder fixed point theorem, which in turn guarantees us the existence of at least u_n in $L^{p_+(\cdot)}(\Omega, \mathbb{R}^m)$, such that $\Phi(u_n) = u_n$ with $\Phi : u \longmapsto T(u, 1)$. Thus, we have guaranteed the existence of a weak solution for the approximated problems (3.35) in sense of (3.36). Therefore, Lemma 3.3 was proven. □

3.1.1. A priori estimates.

Lemma 3.4 *Let $\overrightarrow{p}(\cdot)$, Θ_i, ψ_i, $i = 1, \ldots, N$ be restricted as in Theorem 3.1. Then there exists $C > 0$, such that*

$$|||u_n|||_{\overrightarrow{p}(\cdot)} \leq C. \tag{3.37}$$

Proof: After taking $\varphi = u_n$ in (3.36), and like the proof of (3.34) we can simply get

$$\sum_{i=1}^{N} \left\| |\partial_i u_n| \right\|_{p_i(\cdot)} \leq c. \tag{3.38}$$

Through (3.38) with using (2.8), and (3.15), we obtain (3.37). □

Lemma 3.5 *There exists a subsequence (still denoted (u_n)) that satisfies the following*

$$\partial_i u_n \longrightarrow \partial_i u \quad \textit{strongly in } L^{p_i(\cdot)}(\Omega, \mathbb{R}^m), \ \textit{and almost everywhere in } \Omega, \ i = 1, \ldots, N. \tag{3.39}$$

Proof: The boundedness of (u_n) in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m)$ due (3.37) allows us to extract a subsequence (still denoted by (u_n)) from (u_n), and guarantees the existence of a function $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m)$, such that

$$u_n \rightharpoonup u \quad \text{weakly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^m) \text{ and almost everywhere in } \Omega. \tag{3.40}$$

By taking $\varphi = u_n - u$ in (3.36), we obtain that

$$\sum_{i=1}^{N} \int_{\Omega} \mid \partial_i u_n \mid^{p_i(x)-2} \partial_i u_n \cdot \partial_i (u_n - u) \, dx$$
$$= \alpha \sum_{i=1}^{N} \int_{\Omega} u_n(\mid \psi_i \mid + \beta \mid u_n \mid)^{p_i(x)-2} \cdot (u_n - u) \, dx - \sum_{i=1}^{N} \int_{\Omega} \Theta_i(x, u_n, \partial_i u_n) \cdot (u_n - u) \, dx. \tag{3.41}$$

Now we have the following

$$\sum_{i=1}^{N}\int_{\Omega}\Big(\mid \partial_i u_n \mid^{p_i(x)-2}\partial_i u_n - \mid \partial_i u \mid^{p_i(x)-2}\partial_i u\Big)\cdot \partial_i(u_n-u)\,dx$$
$$=\sum_{i=1}^{N}\int_{\Omega}\mid \partial_i u_n \mid^{p_i(x)-2}\partial_i u_n\cdot \partial_i(u_n-u)\,dx-\sum_{i=1}^{N}\int_{\Omega}\mid \partial_i u \mid^{p_i(x)-2}\partial_i u\cdot \partial_i(u_n-u). \tag{3.42}$$

By combining (3.42) and (3.41), we deduce that

$$\sum_{i=1}^{N}\int_{\Omega}\Big(\mid \partial_i u_n \mid^{p_i(x)-2}\partial_i u_n - \mid \partial_i u \mid^{p_i(x)-2}\partial_i u\Big)\cdot \partial_i(u_n-u)\,dx$$
$$=\alpha\sum_{i=1}^{N}\int_{\Omega}u_n(\mid \psi_i\mid+\beta\mid u_n\mid)^{p_i(x)-2}\cdot(u_n-u)\,dx-\sum_{i=1}^{N}\int_{\Omega}\Theta_i(x,u_n,\partial_i u_n)\cdot(u_n-u)\,dx$$
$$-\sum_{i=1}^{N}\int_{\Omega}\mid \partial_i u \mid^{p_i(x)-2}\partial_i u\cdot \partial_i(u_n-u). \tag{3.43}$$

Now, since $\partial_i u\in L^{p_i(\cdot)}(\Omega,\mathbb{R}^m)$, then

$$\int_{\Omega}\mid\mid \partial_i u \mid^{p_i(x)-2}\partial_i u\mid^{p_i'(x)}\,dx=\int_{\Omega}\mid \partial_i u\mid^{p_i(x)}\,dx\leq c,\ i=1,\dots,N, \tag{3.44}$$

and this implies that

$$\Big(\mid \partial_i u \mid^{p_i(x)-2}\partial_i u\Big)\text{ uniformly bounded in }L^{p_i'(\cdot)}(\Omega,\mathbb{R}^m),\ i=1,\dots,N. \tag{3.45}$$

Like the proof (3.3) and (3.5), we can obtain for all $i=1,\dots,N$

$$u_n(\mid \psi_i\mid+\beta\mid u_n\mid)^{p_i(x)-2}\text{ uniformly bounded in }L^{p_i'(\cdot)}(\Omega,\mathbb{R}^m), \tag{3.46}$$
$$\text{and },\Theta_i(x,u_n,\partial_i u_n)\text{ uniformly bounded in }L^{p_i'(\cdot)}(\Omega,\mathbb{R}^m). \tag{3.47}$$

By (3.40), (2.4), (3.45), (3.46), and (3.47), we find that, the right-hand side of (3.43) goes to zero when $n\longrightarrow+\infty$. So through this, we conclude that

$$\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\Omega}\Big(\mid \partial_i u_n \mid^{p_i(x)-2}\partial_i u_n - \mid \partial_i u \mid^{p_i(x)-2}\partial_i u\Big)\cdot \partial_i(u_n-u)\,dx=0. \tag{3.48}$$

From (2.1) (i.e. $\big(\mid \partial_i u_1 \mid^{p_i(x)-2}\partial_i u_n - \mid \partial_i u \mid^{p_i(x)-2}\partial_i u\big)\cdot \partial_i(u_n-u)\geq 0$), and (3.48), we find that

$$\lim_{n\to+\infty}\mathsf{A}_{i,n}=0, \tag{3.49}$$

where, $\mathsf{A}_{i,n}=\int_{\Omega}\big(\mid \partial_i u_n \mid^{p_i(x)-2}\partial_i u_n - \mid \partial_i u \mid^{p_i(x)-2}\partial_i u\big)\cdot \partial_i(u_n-u)\,dx$, $i=1,\dots,N$.
Now, considering $\Omega_{i,1}$, $\Omega_{i,2}$ defined in (3.10), and like the proof steps followed in [13,14,15,16,17], we can get, for all $i=1,\dots,N$

$$\int_{\Omega_{i,1}}|\partial_i(u_n-u)|^{p_i(x)}\,dx\leq c\mathsf{A}_{i,n}, \tag{3.50}$$
$$\text{and },\int_{\Omega_{i,2}}|\partial_i(u_n-u)|^{p_i(x)}\,dx\leq c'\max\Big\{\mathsf{A}_{i,n}^{\frac{p_i^-}{2}},\mathsf{A}_{i,n}^{\frac{p_i^+}{2}}\Big\}. \tag{3.51}$$

By passing to the limit in (3.50) and in (3.51) when $n \longrightarrow +\infty$, with the use of (3.49), we obtain that

$$\lim_{n\to+\infty} \int_\Omega |\partial_i(u_n - u)|^{p_i(x)}\, dx = 0. \tag{3.52}$$

From (2.4), we get that

$$|||\partial_i(u_n - u)|||_{p_i(\cdot)} \le \max\left\{ \left(\int_\Omega |\partial_i(u_n - u)|^{p_i(x)}\, dx \right)^{\frac{1}{p^+}}, \left(\int_\Omega |\partial_i(u_n - u)|^{p_i(x)}\, dx \right)^{\frac{1}{p^-}} \right\}. \tag{3.53}$$

Passing to the limit in (3.53) when $n \longrightarrow +\infty$ with the usie of (3.52), we find that

$$\lim_{n\to+\infty} |||\partial_i(u_n - u)|||_{p_i(\cdot)}\, dx = 0. \tag{3.54}$$

Then, (3.54) implies (3.39). Therefore, Lemma 3.5 has been proven. □

3.2. Proof of the Theorem 3.1 :

From (3.39), we deduce that

$$|\, \partial_i u_n\, |^{p_i(x)-2}\, \partial_i u_n \longrightarrow |\, \partial_i u\, |^{p_i(x)-2}\, \partial_i u \;\text{ almost everywhere in}\, \Omega. \tag{3.55}$$

By the use of Young's inequality and that $\partial_i u_n \in L^{p_i(\cdot)}(\Omega, \mathbb{R}^m)$, we find for all $\theta > 0$ that

$$\begin{aligned} \int_\Omega ||\, \partial_i u_n\, |^{p_i(x)-2}\, \partial_i u_n\, |\; dx &= \int_\Omega |\, \partial_i u_n\, |^{p_i(x)-1}\; dx \le C(\theta) + \theta \int_\Omega |\, \partial_i u_n\, |^{p_i(x)}\; dx \\ &\le C(\theta) + c\theta = C'(\theta). \end{aligned} \tag{3.56}$$

Then, for any $\theta > 0$ fixed, we conclude that

$$\left(|\, \partial_i u_n\, |^{p_i(x)-2}\, \partial_i u_n \right) \in L^1(\Omega, \mathbb{R}^m),\; i = 1, \dots, N. \tag{3.57}$$

So, from (3.45), (3.55), (3.57), and Vitali's Theorem, we deduce for every $i = 1, \dots, N$ that

$$|\, \partial_i u_n\, |^{p_i(x)-2}\, \partial_i u_n \longrightarrow |\, \partial_i u\, |^{p_i(x)-2}\, \partial_i u \;\text{ strongly in}\, L^1(\Omega, \mathbb{R}^m). \tag{3.58}$$

From (3.40), we obtain that

$$u_n(|\, \psi_i\, | + \beta\, |\, u_n\, |)^{p_i(x)-2} \longrightarrow u(|\, \psi_i\, | + \beta\, |\, u\, |)^{p_i(x)-2} \quad \text{almost everywhere in } \Omega. \tag{3.59}$$

Like the proof of (3.57) with the use of (2.2) and that ψ_i, $u_n \in L^{p_i(\cdot)}(\Omega, \mathbb{R}^m)$, we can obtain for every $i = 1, \dots, N$ that

$$\left(u_n(|\, \psi_i\, | + \beta\, |\, u_n\, |)^{p_i(x)-2} \right) \in L^1(\Omega, \mathbb{R}^m). \tag{3.60}$$

So, by (3.46), (3.59),(3.60), thanks to Vitali's Theorem, we deduce for every $i = 1, \dots, N$

$$u_n(|\, \psi_i\, | + \beta\, |\, u_n\, |)^{p_i(x)-2} \longrightarrow u(|\, \psi_i\, | + \beta\, |\, u\, |)^{p_i(x)-2} \;\text{ strongly in}\, L^1(\Omega, \mathbb{R}^m). \tag{3.61}$$

In a similar way with the use of (1.2), (3.39), and (3.40), we can obtain that

$$\Theta_i(x, u_n, \partial_i u_n) \longrightarrow \Theta_i(x, u, \partial_i u) \quad \text{almost everywhere in } \Omega, \tag{3.62}$$

$$\text{and }, \Theta_i(x, u_n, \partial_i u_n) \in L^1(\Omega, \mathbb{R}^m). \tag{3.63}$$

Then, by (3.47), (3.62), (3.63), and Vitali's Theorem, we find for every $i = 1, \dots, N$ that

$$\Theta_i(x, u_n, \partial_i u_n) \longrightarrow \Theta_i(x, u, \partial_i u) \;\text{ strongly in}\, L^1(\Omega, \mathbb{R}^m). \tag{3.64}$$

So, we can pass to the limit in (3.36). Thus, Theorem 3.1 was proven.

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