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Curve Theory on an Extended Manifold

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ABSTRACT: A structure is an almost complex structure if $J^2 = -I, I$ in a complex manifold M, where J is a tensor field of type (1,1) and I is the identity tensor field. Consider that ${}^kM$ is the extended complex manifold of the manifold M. The order of an extended complex manifold is k. In the extended complex manifold ${}^kM$, the extended almost complex structure satisfies the condition $(J_k)^2 = -I$. In this paper, we study some properties of various lifts in an extended complex structure in an extended complex manifold ${}^kM$. We define and study various properties of the submanifold ${}^kV$ of the extended complex manifold ${}^kM$. In addition, we elaborated the conditions of the existing distributions of real dimensions of the extended complex manifold ${}^kM$. Finally, we define Haantje's tensor on the extended complex manifold ${}^kM$.

Keywords: Complex manifold, extended manifold, Frenet curve.

Contents

1. Introduction

The significance of lifting theory in differential geometry lies in its capacity to generalize geometric structures on any manifold through the use of lift functions. The geometric structures, for example, an almost complex structure, an almost product structure, on the base manifold admit lifts, namely, the complete and vertical lifts, to the canonical extended manifold. Tekkoyun et. al. [1] studied the geometric structures of an extended complex manifold ${}^kM$ of k-th order of the complex manifold M and established higher-order vertical and complete lifts of functions, vector fields, and 1-forms on M to ${}^kM$. Tekkoyun and Civelek [2] investigated higher-order complete, vertical and horizontal lifts of the complex structures on the complex manifold M to the extended complex manifold ${}^kM$. Das and the author [3,4] have studied almost r-contact structures on the tangent bundle using the complete and vertical lifts. The geometric structures such as almost complex, metallic structure, almost Hermitian structure, etc., on the manifold to the tangent bundle are studied by earlier investigators [5-11], [14,15,33] and [18-22].

Properties of the slant curve [24,29] were studied by J. E Lee and others, curvature and torsion on Trans Sasakii manifold were well defined by Acet et. al. [13,16] and [25-28]. Geometry of null curves were elaborately explained by Duggal et. al [30], mean curvature and slant curve was defined by J. E. Lee et. al [31] .

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2. Preliminaries

Let M be a $2m$-real dimensional manifold and ${}^kM$ its k-th order extended manifold. A tensor field J_k of type (1,1) on ${}^kM$ is called an extended almost complex structure [23,32] on ${}^kM$ if J_k is an endomorphism of the tangent space $Tp({}^kM)$ such that $(J_k)^2 = -I$ at every point p of ${}^kM$. An extended manifold ${}^kM$ with an extended almost complex structure J_k is called an extended almost complex manifold [1,16].

Suppose $(\aleph^{ri}, \Re^{ri})$ is a system of real coordinates defined at any point p of ${}^kM$ covered by neighborhood ${}^kU$ and $\{\frac{\partial}{\partial\aleph^{ri}}, \frac{\partial}{\partial\Re^{ri}}\}$ is the natural base over the real field R of the tangent space $Tp({}^kM)$ of ${}^kM$. The manifold ${}^kM$ is called an extended complex manifold if ${}^kM$ is covered by neighborhood ${}^kU$ and a local coordinate system $(\aleph^{ri}, \Re^{ri})$ defined in the neighborhood ${}^kU$ such that

$$J_k(\frac{\partial}{\partial\aleph^{ri}}) = \frac{\partial}{\partial\Re^{ri}}, \tag{2.1}$$

$$J_k(\frac{\partial}{\partial\Re^{ri}}) = -\frac{\partial}{\partial\aleph^{ri}}. \tag{2.2}$$

If $k = 0, J_0$ is called an almost complex structure and a manifold ${}^0M = M$ with an almost complex structure J_0 is said to be the complex manifold.

Let $z^{ri} = \aleph^{ri} + i\Re^{ri}, i = \sqrt{-1}$ be an extended complex local coordinate system on a neighborhood ${}^kU$ of any point p of ${}^kM$. Then $\frac{\partial}{\partial z^{ri}} = \frac{1}{2}\{\frac{\partial}{\partial\aleph^{ri}} - i\frac{\partial}{\partial\Re^{ri}}\}, \frac{\partial}{\partial\bar{z}^{ri}} = \frac{1}{2}\{\frac{\partial}{\partial\aleph^{ri}} + i\frac{\partial}{\partial\Re^{ri}}\}$ and the endomorphism J_k is given by

$$J_k(\frac{\partial}{\partial z^{ri}}) = i\frac{\partial}{\partial z^{ri}}, J_k(\frac{\partial}{\partial\bar{z}^{ri}}) = -i\frac{\partial}{\partial\bar{z}^{ri}}. \tag{2.3}$$

Let M be any complex manifold and ${}^kM$ its k-th order extension. If f is a function on M, then the functions f^{v^k} and f^{c^k} denote the vertical and complete lifts of the function f on ${}^kM$, respectively and are given by [1,12,17]

$$f^{v^k} = f \circ \tau M \circ \tau^2 M \circ ... \circ \tau^{k-1}M, \tag{2.4}$$

and

$$f^{c^k} = \dot{z}^{ri}(\frac{\partial f^{c^k}}{\partial z^{ri}})^v + \dot{\bar{z}}^{ri}(\frac{\partial f^{c^k}}{\partial\bar{z}^{ri}})^v, \tag{2.5}$$

where $\tau^{k-1}M \rightarrow^k M^{k-1}M$ is a canonical projection.

The extended properties for the vertical and complete lifts of the complex functions are given as follows:

$$\begin{aligned} i)\ \ (f+g)^{v^r} &= f^{v^r} + g^{v^r}, (f.g)^{v^r} = f^{v^r}.g^{v^r} \\ ii)\ \ (f+g)^{c^r} &= f^{c^r} + g^{c^r}, (f.g)^{c^r} = \sum_{j=0}^{r} C^r_j f^{c^{r-j}v^j}.g^{c^j v^{r-j}}, \end{aligned} \tag{2.6}$$

where f and g are complex functions and C^r_j represents the combination.

Let $\aleph$ be a complex vector field with the local expression $\aleph = Z^{0i}\frac{\partial}{\partial z^{ki}} + \bar{Z}^{0i}\frac{\partial}{\partial\bar{z}^{ki}}$. Then the local expressions of the vertical and complete lifts of $\aleph$ to ${}^kM$ are respectively given by

$$\aleph^{v^k} = (Z^{0i})^{v^k}\frac{\partial}{\partial z^{ki}} + (\bar{Z}^{0i})^{v^k}\frac{\partial}{\partial\bar{z}^{ki}}. \tag{2.7}$$

and

$$\aleph^{c^k} = C^r_j(Z^{0i})^{v^{k-r}c^r}\frac{\partial}{\partial z^{ri}} + C^r_j(\bar{Z}^{0i})^{v^{k-r}c^r}\frac{\partial}{\partial\bar{z}^{ri}}. \tag{2.8}$$

The extended properties for the vertical and complete lifts of the complex vector fields are given as follows:

$$\begin{aligned} i)\ (\aleph+\Re)^{v^r} &= \aleph^{v^r}+\Re^{v^r}, (\aleph+\Re)^{c^r}=\aleph^{c^r}+\Re^{c^r}, \\ ii)\ \aleph^{v^k}(f^{c^k}) &= (\aleph f)^{v^k}, \quad \aleph^{c^k}(f^{c^k})=(\aleph f)^{c^k}, \\ iii)\quad (f\aleph)^{v^r} &= f^{v^r}.\aleph^{v^r}, (f\aleph)^{c^r}=\sum_{j=0}^{r} C_j^r f^{c^{r-j}v^j}.\aleph^{c^j v^{r-j}}, \\ iv)\ \aleph^{v^r}(f^{v^r}) &= 0, \quad \aleph^{c^r}(f^{c^r})=(\aleph f)^{c^r},\ \aleph^{c^r}(f^{v^r})=\aleph^{v^r}(f^{c^r})=(\aleph f)^{v^r}, \\ v)\ [\aleph^{v^r},\Re^{v^r}] &= 0, [\aleph^{c^r},\Re^{c^r}]=[\aleph,\Re]^{c^r}, [\aleph^{v^r},\Re^{c^r}]=[\aleph^{c^r},\Re^{v^r}]=[\aleph,\Re]^{v^r}, \end{aligned} \tag{2.9}$$

where $\aleph, \Re$ are the complex vector fields and f is the complex function.

Let α be a complex 1-form with the local expression $\alpha=\alpha_{0i}dz^{0i}+\bar{\alpha}_{0i}d\bar{z}^{0i}$. Then the local expression of the vertical and complete lifts of α to ${}^kM$ are, respectively, given by

$$\alpha^{v^k}=(\alpha_{0i})^{v^k}dz^{0i}+(\bar{\alpha}_{0i})^{v^k}d\bar{z}^{0i}, \tag{2.10}$$

and

$$\alpha^{c^k}=(\alpha_{0i})^{c^{k-r}v^r}dz^{ri}+(\bar{\alpha}_{0i})^{c^{k-r}v^r}d\bar{z}^{ri}. \tag{2.11}$$

The extended properties for the vertical and the complete lifts of complex 1-forms are given as follows [1]:

$$\begin{aligned} i)(\alpha+\lambda)^{v^r} &= \alpha^{v^r}+\lambda^{v^r}, (\alpha+\lambda)^{c^r}=\alpha^{c^r}+\lambda^{c^r}, \\ ii)\quad (f\alpha)^{v^r} &= f^{v^r}\alpha^{v^r}, (f\alpha)^{c^r}=\sum_{j=0}^{r} C_j^r f^{c^{r-j}v^j}\alpha^{c^r v^{r-j}}. \end{aligned} \tag{2.12}$$

Let M be any complex manifold and ${}^kM$ its k-th order extension. If F be a tensor field of type (1,1). Then

$$\begin{aligned} \alpha^{c^k}(F^{c^k}) &= (\alpha F)^{c^k}, \quad \alpha^{v^k}(F^{v^k})=(\alpha F)^{v^k}, \\ F^{c^k}(\aleph^{c^k}) &= (F\aleph)^{c^k}, \quad F^{v^k}(\aleph^{c^k})=(F\aleph)^{v^k}, \end{aligned} \tag{2.13}$$

where $\aleph$ and α are a vector field and a 1-form respectively.

3. Frenet Curve in Extended Manifold

Let M be any complex manifold and ${}^kM$ its k-th order extension. If the tensor field F of type (1,1) on M satisfies the equation

$$F^2+I=0, \tag{3.1}$$

where I is an identity tensor field. Then F is called an almost complex structure on M [32].

Let $\Gamma : I \to M = ({}^kM, g)$ be a Frenet curve in extended manifold ${}^kM$ with Frenet frame field (T, N, B). Here defined T, N, B are tangent vector field, normal vector field and binormal vector field respectively. Let us defined Levi Civita connection as $\triangledown$ of $({}^kM, g)$. Frenet triples must satisfy the following Frenet-Serret conditions

$$\triangledown_T T=\kappa N, \triangledown_T N=-\kappa T+\tau B, \triangledown_T B=-\tau N, \tag{3.2}$$

where τ and $\kappa=|\triangledown_T T|$ are the geodesic torsion and geodesic curvature of the Γ respectively.

Shape of Frenet curve Γ is helix if both κ and τ are constant.

Now we recall the fundamental concept of contact metric geometry Let ${}^kM$ ba an extended manifold. A contact form over extended manifold is a one form η such that $d\eta\wedge\eta \neq 1$ on ${}^kM$. An extended complex manifold along with contact form η is called contact extended manifold. The Reeb vector field ξ is a unique vector field satisfies the equations

$$\eta(\xi) = 1, d\eta(\xi.\star) = 0.$$

On the contact extended manifold $({}^kM, \eta)$,there exist two types of Frenet equation.

3.1. Frenet Equation of type 1

Γ be a null curve on extended manifold and η be the vector field, Frenet frame is formed by two null vectors ξ and η and so type 1 Frenet equation is defined subsequently. From $g(\xi,\xi) = 0$, $g(\xi,\eta) = 1$ so $g(\nabla_\xi\xi,\xi) = 0$ and $g(\nabla_\xi\xi,\eta) = g(\xi,\nabla_\xi\eta) = h$, where h is a smooth function on ${}^kM$ leads to the equation

$$\nabla_\xi\xi = h\xi + T_1, \tag{3.3}$$

where T_1 is a non null tangent vector.

Now first curvature function defined by κ_1 and defined as

$$\rho_1 = \|T_1\|$$

and $\kappa_1 = \epsilon_1\rho_1$ where $\epsilon_1 = \pm1$. If ω_1 is unit vector field along Γ then we have

$$\omega_1 = \rho_1^{-1}T_1.$$

Hence equation (3.3) become

$$\nabla_\xi\xi = h\xi + \epsilon_1\kappa_1\omega_1. \tag{3.4}$$

On generalising (3.4) we get Frenet equation of type 1.

3.2. Frenet equation of type 2

For null curve Γ the pseudo orthonormal basis consist of ξ and set of vectors L_t, L_{t+1} defined as

$$L_t = \frac{\omega_t + \omega_{t+1}}{\sqrt{2}}, \quad L_{t+1} = \frac{\omega_{t+1} + \omega_t}{\sqrt{2}}, \tag{3.5}$$

where ω_1 and ω_2 are vector fields from Frenet frame of type 1 for particular index t defined over ${}^kM$. There are $k-1$choice for L_t. For specific value of L_t the vector field will be null and curvature function κ_1 is defined as

$$\nabla_\xi\xi - h\xi = \kappa_1 L_1 \tag{3.6}$$

where L_1 is a null vector field along Γ and L_1 is perpendicular to ξ. If L_1 exists, then there must exist L_2 along Γ such that $g(L_1, L_2) = 1$. Equation (3.5) holds for $t = 1$, so $\omega_1 + \omega_2$ is perpendicular to ξ and we have

$$\nabla_\xi\xi = h\xi + k_1(\omega_1 + \omega_2),$$

where $k_1 = \frac{\kappa_1}{\sqrt{2}}$, implies that

$$\nabla_\xi N = -hN + k_2\omega_1 + k_3\omega_2 + S_1$$

where S_1 is a vector field perpendicular to ξ, N, ω_1 and ω_2. let us have function β_2 derived as

$$\beta_2 = \|S_1\|,$$

which is zero for any t and we have

$$\omega_3 = \beta_2^{-1}S_1$$

on extended manifold ${}^kM$ along Γ we have a Frenet frame of type 2.

A contact extended manifold ${}^kM$ together with contact metric structure is called *extended contact metric manifold* satisfying the following equation

$$(\nabla_{\aleph\phi})\Re = g(\aleph + h\aleph, \Re)\xi - \eta(Y)(\aleph + h\aleph), \tag{3.7}$$

for all $\aleph, Y \in {}^kM$ and $h = \text{Ł}_{\zeta\phi/2}$.

4. Slant Curve in the Extended Manifold

Let ${}^kM$ be an extended contact metric manifold. On ${}^kM$, their exist a Frenet curve $\Gamma(\sigma)$, where σ is the arc length that parameterises the frenet curve. The Frenet curve is slant curve if it imparts a contact angle $\phi(\sigma)$ and this angle defined as

$$cos\ \phi(\sigma) = g(T(\sigma), \xi). \tag{4.1}$$

For the Frenet curve as a slant of the fixed curvature the contact angle must be constant. If this angle is 90^o, then curve is known as Legendre curve which is valid iff $g(T(\sigma), \xi) = 0$.

Now let us take a locally orthonormal field on the extended manifold ${}^kM$ as $(\aleph, \theta\aleph, \xi)$ satisfying the condition $\eta(\aleph) = 0$.

Let Frenet curve Γ on extended manifold ${}^kM$. considering the differentiation of equation (4.1) we have

$$-\phi' sin\phi = g(\kappa N, \xi) + g(T, -\theta T) = \kappa\eta(N). \tag{4.2}$$

where θ is a tensor such that $g(\theta T, \xi) = \eta(N)$ and so by virtue of equation 4.2 we have the following theorem:

Theorem 4.1 *The Frenet curve Γ in the extended manifold ${}^kM$ is a slant curve iff $\eta(N) = 0$.*

Theorem 4.2 *The Frenet curve Γ on the extended manifold ${}^kM$ is a slant curve iff the ratio of torsion and curvature is constant.*

Proof: T, N and ξ of a slant curve $\Gamma(\sigma)$ has the form

$$\begin{aligned} T &= sin\phi[cos\delta(\sigma)\aleph + sin\delta(\sigma)\theta\aleph] + cos\phi\xi, && (4.3)\\ N &= -sin\delta(\sigma)\aleph + cos\delta(\sigma)\theta\aleph, && (4.4)\\ \xi &= cos\phi T \pm sin\phi B. && (4.5) \end{aligned}$$

for some $\delta(\sigma)$.Differentiating $g(N, \xi) = 0$ along Γ and using Frenet-Serret formula, we have

$$\kappa cos\phi + (-1 \pm \tau)sin\phi = 0. \tag{4.6}$$

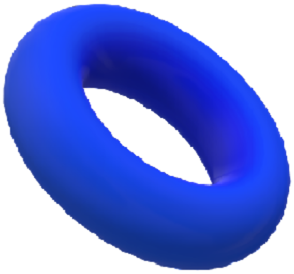
The ratio of torsion and curvature is constant and if Γ is a slant curve.

Proposition 4.1 *If Γ is a slant curve, then its curvature $\tau = \pm 1$.*

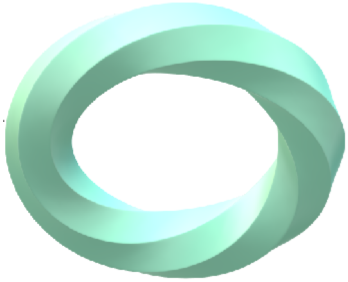
4.1. Hopf Cylinder:

The Hopf cylinder over Γ is the flat surface whose mean curvature is half of the geodesic curvature of $\bar{\Gamma}$, where $\bar{\Gamma}$is the Frenet curve whose inverse image is the flat surface over the extended manifold ${}^kM$.

If the extended manifold is the unit sphere S^3 then $S_{\bar{\Gamma}} = \pi^{-1}(\bar{\Gamma})$, where $S_{\bar{\Gamma}}$ is the inverse image and Π coincides with the Hopf fibering $S^3(1) \to S^2(4)$. In this case $S_{\bar{\Gamma}}$ is a small circle and its Hopf cylinder is the constant mean curvature torus. On the contrary, if $S_{\bar{\Gamma}}$ is big circle, then Hopf cylinder is a Clifford minimal torus.



Mean Curvature Torus



Clifford Minimal Torus

Now let us consider a slant curve Γ with the contact angle ϕ in an extended manifold. As

$$\bar{\Gamma} = \pi \ o \ \Gamma$$

is the projection of Γ onto ${}^k\bar{M}$, then the length parameter $\bar{\sigma}$ of $\bar{\Gamma}$ is

$$\bar{\sigma} = \frac{\sigma}{sin\phi} \tag{4.7}$$

. In the defined Frenet formula $(\bar{T}(\bar{\sigma}), \bar{N}(\bar{\sigma})$ of $\bar{\Gamma}$ is given by

$$\bar{T}(\bar{\sigma}) = \frac{1}{sin\phi}\pi * T(\sigma), \quad \bar{N}(\bar{\sigma}) = \pm\pi * N(\sigma)$$

For the Frenet curve over extended manifold, the signed curvature $\bar{\kappa}$ is as

$$\bar{\kappa}(\bar{\sigma}) = \frac{\pm 1}{sin^2\Phi}\kappa(\sigma)$$

From equation (4.7) it is evident that projection $\Gamma(\sigma) = \bar{\pi}(\Gamma(\sigma))$ is the curve, where σ is the arc length and Γ is a horizontal lift of $\bar{\Gamma}$, also the curvature $\bar{\kappa}(\sigma) = \pm\kappa(\sigma)$ So for the Hopf cylinder $S = \pi^{-1}(\bar{\Gamma})$ the Reeb vector field is tangent to S and S contains Γ.

4.2. Biharmonic curve:

Over extended manifold ${}^kM$ Frenet curve Γ is biharmonic if it satisfies

$$\nabla_T^3 T + R(\kappa N, T)T = 0 \tag{4.8}$$

Theorem 4.3 *In any extended manifold Frenet biharmonic curve with constant holomorphic directional curvature is a slant helix satisfying*

$$\kappa^2 + \tau^2 = 1 + (H-1)sin^2\phi.$$

Any proper biharmonic curve in extended manifold satisfies the equation given in theorem 4.3 which specify

$$\kappa cos\phi + (-1 \pm r)sin\phi = 0.$$

5. Mean Curvature of Slant Curve

Let ${}^kM$ be an extended manifold and the frenet curve $\Gamma(\sigma)$ is defined over it such that

$$\Gamma = \Gamma(\sigma) : I \rightarrow {}^kM$$

Let us defined pullback vector bundle $\Gamma^{*}TM$ as

$$\Gamma^{*}TM = \bigcup_{\sigma \in I} T_{\Gamma(\sigma)}\ {}^{k}M. \tag{5.1}$$

The Levi Civita connection ∇ on ${}^{k}M$ has further deduced the new connection ∇^{Γ} on $\Gamma^{*}T\ {}^{k}M$ as

$$\nabla^{\Gamma}_{\sigma'} V = \nabla_{\dot{\Gamma}} V \tag{5.2}$$

The mean curvature vector field H of a curve Γ in the extended manifold is

$$H = \nabla_{\dot{\Gamma}}\dot{\Gamma} = \kappa N \tag{5.3}$$

Differentiating $N = \theta\dot{\Gamma}$ along with the Frenet curve then

$$\nabla_{\aleph\theta} Y = g(\aleph, \Re)\xi - \eta(\Re)\aleph. \tag{5.4}$$

Using equation (5.4) we have torsion $\tau = 1$.

For a slant curve in extended manifold ${}^{k}M$ along with equations (4.2), (4.3), (4.4) and (5.3), we get

$$\nabla_{\dot{\Gamma}} H = \nu\xi, \tag{5.5}$$

existence of equation (5.5) reveals that

$$\begin{aligned} \kappa^{2} &= -\nu cos\phi_{o} \\ \kappa' &= 0 \\ \kappa\tau &= \nu sin\phi_{o}, \end{aligned} \tag{5.6}$$

hence we have the theorem

Theorem 5.1 *If Γ be the slant curve in the extended manifold, then tangent , normal and binormal satisfy the equation*

$$\nabla_{\dot{\Gamma}}\nabla_{\dot{\Gamma}}\dot{\Gamma} = -\kappa^{2}T + \kappa' N + \kappa\tau B \tag{5.7}$$

Theorem 5.2 *Let Γ be a slant curve on the extended manifold, then*
i. Γ is geodesic, then it has a mean curvature vector field iff $\nu = 0$.
ii. If $\nu =$ constant then Γ is helix.
iii. If $\nabla_{\dot{\Gamma}} H = \nu\xi$, then Γ is legendre curve iff $\nabla_{\dot{\Gamma}} H = 0$

Theorem 5.3 *Let Γ be a slant curve in the extended manifold. Then*

$$\nabla_{\dot{\Gamma}}\nabla_{\dot{\Gamma}}\nabla_{\dot{\Gamma}}\dot{\Gamma} = 3\kappa\kappa' T + (\kappa'' - \kappa^{2} - \kappa\tau^{2})N + (2\kappa'\tau + \kappa\tau')B \tag{5.8}$$

In extended manifold ${}^{k}M$ slant curve Γ satisfies $\nabla_{\Gamma} = \nu\xi$ iff

$$\begin{aligned} 3\kappa\kappa' &= \nu cos\phi_{o} \\ -\kappa'' + \kappa^{2} + \kappa\tau^{2} &= 0 \\ -(2\kappa'\tau + \kappa\tau') &= \nu sin\phi_{o}. \end{aligned} \tag{5.9}$$

So we have the following theorem

Theorem 5.4 *If Γ is a slant curve in the extended manifold ${}^{k}M$, then Γ has no proper mean curvature field.*

Proof: If $\nu = \nu_{o} \neq 0$ but is a real constant. Considering equation (5.9)

$$\kappa^{2} = \frac{2}{3}(\nu_{o} cos\phi_{o})\sigma + a \tag{5.10}$$

where a is constant. Considering equations (5.9) and (5.10) it reflects contradiction in the assumption, hence Γ has no proper mean curvature field.

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