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Particle Swarm Optimization Based-Algorithm for an Optimal Completion of Incomplete Pairwise Comparison Matrices

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ABSTRACT: This study introduces a robust method based on Particle Swarm Optimization (PSO) to estimate the missing entries in incomplete Pairwise Comparison Matrices (PCMs), commonly utilized in multi-criteria decision-making processes. Incompleteness, due to missing data or subjective biases, often affects the consistency and reliability of decisions. The proposed method refines the estimates iteratively to minimize the consistency ratio and improve matrix coherence. Numerical experiments show that PSO offers an efficient solution. A comparison with the Geometric Mean Method (GMM) across various matrix sizes demonstrates that PSO achieves higher accuracy, particularly in terms of consistency and estimation quality.

Key Words: Pairwise comparison matrix, analytic hierarchy process, decision-making theory, particle swarm optimization, geometric mean method.

Contents

1. Introduction

The Pairwise Comparison Matrices (PCMs) are an integral part of the Analytic Hierarchy Process (AHP) a widely used decision support method outlined by Saaty [1], because they are used to compare the competing alternatives systematically. To this day, pairwise comparison method is not only embedded in AHP, but also in newer decision-making frameworks like the Best-Worst Method (BWM), and Heuristic Rating Estimation (HRE) [2,3] or older frameworks like MACBETH, PROMETHEE, and ELECTRE [4,5,6].

Pairwise comparisons continue to present an active area of research owing to their conceptual simplicity and intuitive nature.This concept has been investigated in various settings in the literature, including dominance-based rough set theory [7], fuzzy pairwise comparisons (FPC) [8], more general formulations based on group theory [9], numerical ranking methods avoiding conventional scales [10], consistency inspection on the matrices [11], ordinal pairwise comparisons [12], and the structural analysis of pairwise ranking systems [13].

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When conducted properly, pairwise comparisons can be used to establish a final ranking of the alternatives. For a set of n alternatives, the number of required comparisons is $\frac{n(n-1)}{2}$, a number that grows very rapidly with n. Since these judgments are typically provided by experts, the process often becomes demanding in terms of time and cost.

This limitation has spurred the development of ranking methods that require fewer comparisons. Recent research highlights the effectiveness of metaheuristic algorithms for handling pairwise comparison matrices, particularly for estimating missing values in incomplete matrices [14,15]. A multitude of studies therefore propose alternative methodologies to address the problem of incompleteness in PCMs and estimate the missing judgments [16,17].

Metaheuristic algorithms represent optimization techniques inspired by processes observed in natural systems. A prominent example is Particle Swarm Optimization (PSO), which is modeled on the collective behaviors exhibited by animal groups such as flocks or swarms. Originally introduced by Kennedy and Eberhart [18], PSO simulates social interactions among individuals to iteratively search for optimal solutions across diverse problem domains. In this framework, particles navigate the search space by continuously updating their positions based on both their previous performance and the best solutions identified by other particles. This cooperative mechanism enables PSO to efficiently explore complex search landscapes and converge toward near-optimal or optimal solutions.

This paper aims to develop a mechanism for approximation missing elements in incomplete pairwise comparison matrices while maintaining their underlying consistency features. To do this, we introduce an upgraded Particle Swarm Optimization (PSO) strategy, a well-known metaheuristic technique that estimates missing judgments with appropriate values and repeatedly improves matrix consistency until a preset acceptable threshold is met.

The rest of this paper is organized as follows: Section 2 gives a brief explanation of the Analytic Hierarchy Process (AHP) and discusses the problems caused by incomplete pairwise comparison matrices. Section 3 describes the suggested PSO-based methodology for estimating missing entries in matrices. Section 4 evaluates the method's effectiveness using numerical experiments on matrices of various dimensions, confirming its accuracy and robustness in recovering missing values while maintaining consistency.

2. Pairwise Comparison Matrix (PCM)

2.1. Overview of the AHP method

In multi-criteria decision analysis, AHP serves as a powerful tool that decomposes problems into hierarchical levels for comparative evaluation of competing factors.

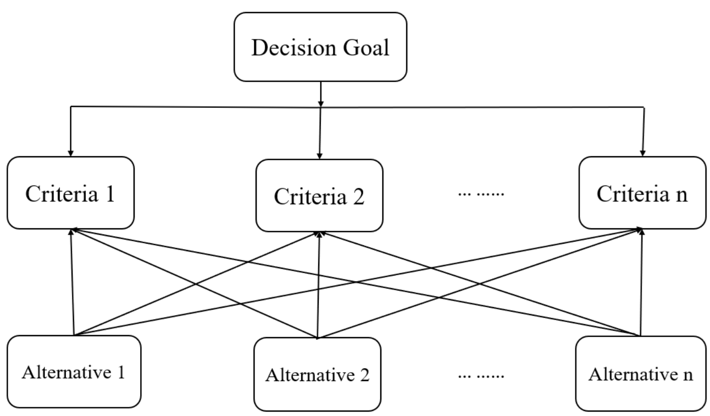


Figure 1: Process of the AHP method

Introduced by Saaty [1], the Analytic Hierarchy Process (AHP) decomposes complex decisions into hierarchical structures, assigns weights to criteria through pairwise comparisons, and evaluates alternatives systematically. The process consists of five key steps :

1. **Structure the Hierarchy** : organize the decision components into a hierarchy including the goal, criteria, and alternatives.

2. **Pairwise Comparison of Criteria** : assess the relative importance of criteria through pairwise comparisons using a nine-point scale.

3. **Pairwise Comparison of Alternatives** : evaluate alternatives against criteria based on pairwise comparisons.

4. **Calculate Priorities** : numerical priorities are calculated for each alternative based on the weighted criteria.

5. **Final Decision** : the alternative with the highest overall priority score is selected as the most desirable solution.

The Analytic Hierarchy Process (AHP), introduced by T.L.Saaty [19], is a structured technique designed to assist with complex decision-making. It decomposes a problem into a multi-level hierarchical structure of goals, criteria, and alternatives. Through a systematic pairwise comparison of elements at each level, AHP helps decision-makers evaluate and prioritize factors by assigning them relative weights. This approach has been widely applied in domains such as business, public policy, engineering, healthcare, and education, where structured decision frameworks are essential.

2.2. Consistency in Pairwise Comparison Matrices

In the context of multi-criteria decision analysis (MCDA) [20], pairwise comparison is a foundational tool. The quantitative use of this technique dates back to the work of Thurstone in 1927. Later, Saaty formalized and extended the concept to develop AHP, a widely used methodology for handling decisions involving multiple criteria [21,22]. Ensuring consistency in these pairwise comparisons is vital, as it directly impacts the reliability and validity of the resulting prioritization.

The Pairwise Comparison (PC) technique is widely used to establish a ranking among a set of alternatives, represented as $A = \{a_1, a_2, a_3, \dots, a_n\}$. In this framework, each alternative is assigned a positive real value, $w(a_i)$, known as its weight or priority. The ranking is determined by evaluating the alternatives in pairs, allowing for a relative judgment. These judgments are compiled into a square matrix called the pairwise comparison matrix (PCM), which provides a structured way to represent the preferences.

Definition 2.1 *Let $A = (a_{ij}) \in \mathbb{R}_+^{n\times n}$. Matrix A is called a pairwise comparison matrix if it satisfies the following two properties :*

1. ***Unit elements on the main diagonal:*** $a_{ii} = 1$ *for all* $i = 1, \dots, n$.

2. ***Positive reciprocity:*** $a_{ji} = \dfrac{1}{a_{ij}}$ *for all* $i \neq j$.

Therefore, A has the following generic form :

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ \frac{1}{a_{12}} & 1 & a_{23} & \cdots & a_{2n} \\ \frac{1}{a_{13}} & \frac{1}{a_{23}} & 1 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \frac{1}{a_{3n}} & \cdots & 1 \end{pmatrix} \tag{2.1}$$

Each entry a_{ij} in the matrix indicates the preference intensity of alternative a_i over a_j under a specific criterion. These comparisons are typically made using a numerical scale that reflects how strongly one element is preferred over another.

Table 1 illustrates the fundamental 1-9 scale proposed by Saaty for conducting pairwise judgments:

Table 1: Saaty's 1-9 scale for pairwise comparisons [1]

Linguistic Term	Numerical Value
Equal importance	1
Slightly more important	3 or $\frac{1}{3}$
Clearly more important	5 or $\frac{1}{5}$
Strongly more important	7 or $\frac{1}{7}$
Extremely more important	9 or $\frac{1}{9}$
Intermediate levels	2, 4, 6, 8 or their reciprocals

Definition 2.2 *A pairwise comparison matrix* $\mathbf{A} = (a_{ij}) \in \mathcal{A}^{n\times n}$ *is said to be* consistent *if :*

$$a_{ih} = a_{ij} \cdot a_{jh}, \quad \forall\, 1 \le i, j, h \le n$$

If this condition does not hold for some i, j, or h, the matrix is considered inconsistent.

Definition 2.3 *Let* $\mathbf{A} \in \mathcal{A}^{n\times n}$ *be a pairwise comparison (PC) matrix of order n. The corresponding Consistency Index (CI) of* $\mathbf{A}$ *is defined as follows :*

$$CI = \frac{\lambda_{\max} - n}{n-1} \tag{2.2}$$

where $\lambda_{\max}$ *is the largest eigenvalue of the matrix* $\mathbf{A}$.

When the consistency index (CI) equals zero, it implies that the largest eigenvalue $\lambda_{\max}$ is equal to the order n of the pairwise comparison matrix. The CI serves as a quantitative measure of how much a given matrix deviates from perfect consistency [21,22]. A method for estimating an upper bound for CI, particularly when the comparison values are limited to a fixed scale, was introduced in [23].

Saaty suggested the use of a discrete scale for pairwise comparisons, specifically:

$$a_{ij} \in \left\{\frac{1}{9}, \frac{1}{8}, \dots, \frac{1}{2}, 1, 2, \dots, 8, 9\right\}, \quad \forall\, 1 \le i, j \le n. \tag{2.3}$$

This scale helps to standardize expert judgments and facilitates consistency evaluation. In addition, Saaty proposed a normalized inconsistency measure by comparing the CI of the matrix to a benchmark known as the *Random Index* (RI).

For a given matrix size (order), the Random Index is calculated by averaging the Consistency Indices from a large sample of randomly generated pairwise comparison matrices. It provides a reference for determining the acceptable level of inconsistency. Values of RI_n for $n = 1$ to 10 have been published based on extensive simulations, as reported by [24] and further validated in the studies of [25] and [26].

Definition 2.4 *Let* $\mathbf{A} \in \mathcal{A}^{n\times n}$ *be any PC matrix of size n. The Consistency Ratio (CR) of* $\mathbf{A}$ *is defined by :*

$$CR = \frac{CI}{RI_n} \tag{2.4}$$

Table 2: Random Consistency Index

n	1	2	3	4	5	6	7	8	9	10
$\mathcal{RI}_n$	0	0	0.58	0.9	1.12	1.24	1.32	1.41	1.45	1.49

A threshold for the acceptability of inconsistency has also been proposed by Saaty.

Definition 2.5 *Let* $\mathbf{A} \in \mathcal{A}^{n\times n}$ *be any PC matrix of size n. The matrix A is consistent enough to be accepted if* $CR \le 0,1$.

Theorem 2.1 *A PC matrix $A = (a_{ij})$ is consistent perfectly, if and only if* $\lambda_{max} = n$

The complete proof of this theorem can be found in reference [27].

Example 1 :

Consider the following pairwise comparison matrix of size 3×3 :

$$B = \begin{bmatrix} 1 & 1/3 & 2 \\ 3 & 1 & 6 \\ 1/2 & 1/6 & 1 \end{bmatrix}$$

and $a_{12} = \frac{1}{3} = 2 \times \frac{1}{6} = a_{13}a_{32}$, Consequently, A is consistently

$$\begin{vmatrix} 1-\lambda & \frac{1}{3} & 2 \\ 3 & 1-\lambda & 6 \\ \frac{1}{2} & \frac{1}{6} & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - 1 + 3(\lambda) = \lambda^2(3-\lambda)$$

So, $(\lambda)^2(3-\lambda) = 0 \Longleftrightarrow \lambda_1 = \lambda_2 = 0 \;\; and \;\; \lambda_3 = \lambda_{max} = 3$
According to Theorem 2.1, matrix B is perfectly consistent.

2.3. Incomplete Pairwise Comparison Matrix (IPCM)

As previously noted, the pairwise comparison (PC) matrix captures the judgments made between all pairs of alternatives. However, in practical applications, completing all required comparisons can become increasingly difficult as the number of alternatives grows. The number of necessary pairwise evaluations increases quadratically, specifically requiring $\frac{n(n-1)}{2}$ comparisons for n alternatives in order to preserve reciprocity.

For instance, 8 alternatives demand 28 comparisons, whereas 11 alternatives require as many as 55. When dealing with a large number of alternatives, this process can become burdensome—particularly because it depends on expert input, and experts are often limited by time constraints.

To cope with these challenges, the concept of incomplete pairwise comparison matrices has been introduced. In these matrices, some of the comparison values are not specified. These undefined elements are typically denoted with a question mark (?), indicating a missing judgment. An illustration of such a matrix is provided below. We now proceed to define the incomplete pairwise comparison matrix formally.

Definition 2.6 *Matrice $A = (a_{ij})$ is said an Incomplete Pairwise Comparison Matrix (IPCM) for n alternative such that $a_{ij} \in \mathbb{R}_+ \cup \{?\}$,$\forall 1 \le i, j \le n$ $a_{ij} \in \mathbb{R}_+$ implies $a_{ij} = \frac{1}{a_{ji}}$ and $a_{ij} =?$ implies $a_{ji} =?$ is said to be missing.*

Example 2 :

Consider the following IPCM matrix of size 4.

$$A = \begin{bmatrix} 1 & ? & 3 & ? \\ ? & 1 & 2 & 6 \\ \frac{1}{3} & \frac{1}{2} & 2 & 5 \\ ? & \frac{1}{6} & \frac{1}{5} & 1 \end{bmatrix}$$

The use of graph representations for incomplete pairwise comparison matrices is a useful method for visualising the structure of known elements.

Definition 2.7 *An IPCM $A = (a_{ij})$ can be modeled using an undirected graph $G_A = (V_A, E_A)$. In this representation, the vertices $V_A = \{1, 2, \ldots, n\}$ denote the alternatives under consideration. An edge e_{ij} is included in the set E_A if and only if the judgment comparing alternative i to alternative j is available, meaning a_{ij} is defined and $i \neq j$.*

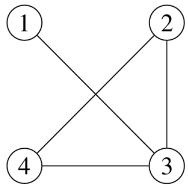


Figure 2: Graph structure associated with the pairwise comparison matrix A in Example 2.

2.4. Geometric mean approach for IPCM

In the literature, several approaches [28,29,30] have been proposed to estimate the missing elements in pairwise comparison matrices (PCMs). Among these, the Geometric Mean Method (GMM) has gained significant attention due to its simplicity and computational efficiency. This method provides a direct way to approximate the unknown judgments while maintaining logical consistency within the PCM. As discussed in [31], the GMM was later adapted by replacing each missing element $a_{ij} =?$ with the ratio $\frac{w_i}{w_j}$, and computing the geometric mean of each row to derive consistent and computationally tractable weights.

$$w_i = \left(\prod_{j=1}^{n} a_{ij} \right)^{\frac{1}{n}}, \quad i = 1, \dots, n$$

The preceding idea results in the following ranking algorithm:

- **Step 1** Constract an auxiliary matrix $B = (b_{ij})_{n\times n}$ from the given incomplete PC matrix $A = (a_{ij})_{n\times n}$, such that

$$b_{ij} = \begin{cases} 1, & \text{if } i \neq j \;\; and \;\; a_{ij} =? \\ 0, & \text{if } i \neq j \;\; and \;\; a_{ij} \neq ? \\ n - k_i, & \text{if } i = j \end{cases}$$

  where k_i represents the number of missing comparisons in the ith row of A. Then, a vector of constant terms

$$\mathbf{v} = \begin{pmatrix} \sum_{\substack{j=1 \\ a_{1j}\neq ?}}^{n} \log a_{1j} \\ \vdots \\ \vdots \\ \sum_{\substack{j=1 \\ a_{nj}\neq ?}}^{n} \log a_{nj} \end{pmatrix}$$

- **Step 2** Solve the next system of linear equations for $\tilde{\mathbf{w}} = (\tilde{w}_1, \dots, \tilde{w}_n)^T$:

$$B\tilde{\mathbf{w}} = \mathbf{v}$$

  and generate a weight vector $W = (w_1, w_2, \dots, w_n)^T$ of the form:

$$W = (\exp(\tilde{w_1}), \exp(\tilde{w_2}), \dots, \dots, \exp(\tilde{w_n}))$$

- **Step 3** Normalise W and get the resultant vector W_{GM}:

$$W_{GM} = \left(\frac{w_1}{\sum_{j=1}^{n} w_j}, \frac{w_2}{\sum_{j=1}^{n} w_j}, \dots, \frac{w_n}{\sum_{j=1}^{n} w_j} \right)^T$$

Example 3 :

Consider the incomplete pairwise comparison (PC) matrix A^* of size 4, containing three missing elements represented as $\frac{w_i}{w_j}$.

$$A^* = \begin{bmatrix} 1 & \frac{w_1}{w_2} & \frac{w_1}{w_3} & 2 \\ \frac{w_2}{w_1} & 1 & 3 & \frac{w_2}{w_4} \\ \frac{w_3}{w_1} & \frac{1}{3} & 1 & 2 \\ \frac{1}{2} & \frac{w_4}{w_2} & \frac{1}{2} & 1 \end{bmatrix}$$

3. Bio-inspired Technique for Improving Incomplete PCM

3.1. Particle Swarm Optimization: A Review

Particle Swarm Optimization (PSO) is a stochastic optimization technique designed for nonlinear functions, inspired by the replication of social behavior and introduced by Dr. Eberhart and Dr. Kennedy [32] in 1995. The method's origin traces back to observations from computer simulations of flocking birds and schooling fish conducted by Reynolds, Heppner, and Grenander [33,34]. These studies demonstrated that individuals in a moving group maintain an optimal distance from each other while aligning their movements relative to the local motions of their neighbors. Extensive research has investigated the application of Particle Swarm Optimization (PSO) across a wide range of engineering domains [35,36]. In addition, PSO has been combined with fuzzy logic to address bilateral free-boundary shape optimization problems, as demonstrated in [37]. The fundamental concept of PSO is inspired by the collective foraging behavior observed in biological systems, where imitation plays a key role in competitive food searching. In such systems, individuals explore an environment with randomly distributed food sources; once one individual discovers a source, others tend to mimic its behavior. This socially guided mechanism, shaped by environmental cues and neighborhood interactions, forms the basis of an optimization strategy aimed at locating the global optimum by leveraging the tendencies of neighboring individuals. Each particle updates its trajectory by balancing two influences: the overall trends observed within its neighborhood and its own historical best experience. Formally, in the PSO algorithm, particles exist within a D-dimensional solution space, where each particle (Fig. 3) represents a candidate solution to the optimization problem. Particles have access only to limited local information, specifically details about their neighbors as defined by the population topology, which can be either static [38] or dynamic [39]. Each particle is described by three key attributes: its position in the solution space, its velocity, and the objective function value corresponding to its current location.

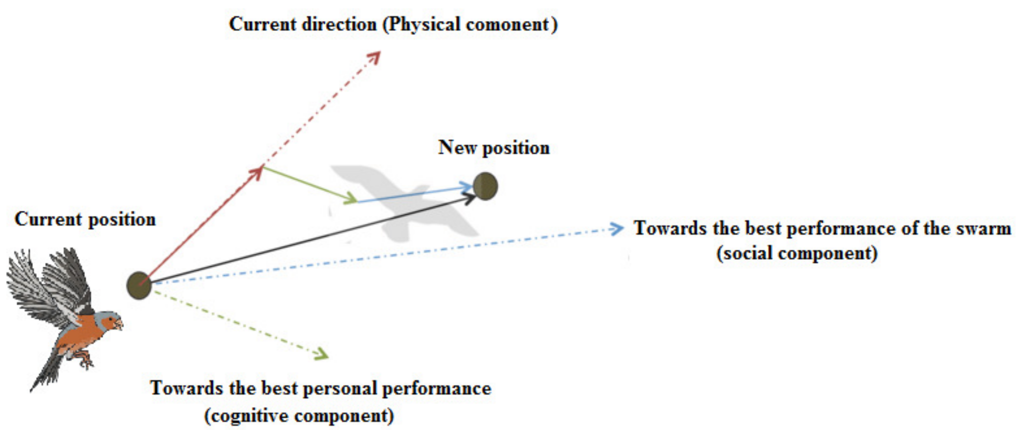


Figure 3: Architecture of the Particle Swarm Optimization (PSO) Algorithm

Particle movement is governed by the following update rules :

$$v_i^{k+1} = \omega v_i^k + c_1 r_1 (pbest_i^k - x_i^k) + c_2 r_2 (gbest_i^k - x_i^k) \tag{3.1}$$

$$x_i^{k+1} = x_i^k + v_i^{k+1} \tag{3.2}$$

Here, c_1 and c_2 denote acceleration coefficients, ω is the inertia weight that balances the impact of the previous velocity of the particle, and r_1, r_2 are independent random variables uniformly distributed in $[0, 1]$ [40]. The term v_i^k represents the velocity of the particle i in iteration k, $pbest_i^k$ is the best personal position found by the particle i up to iteration k, and $gbest_i^k$ is the best position found by the neighborhood of particle i at iteration k. The objective function evaluates each particle's position, guiding the swarm toward convergence on the optimal solution. The flowchart [4] illustrates the working process of the PSO algorithm

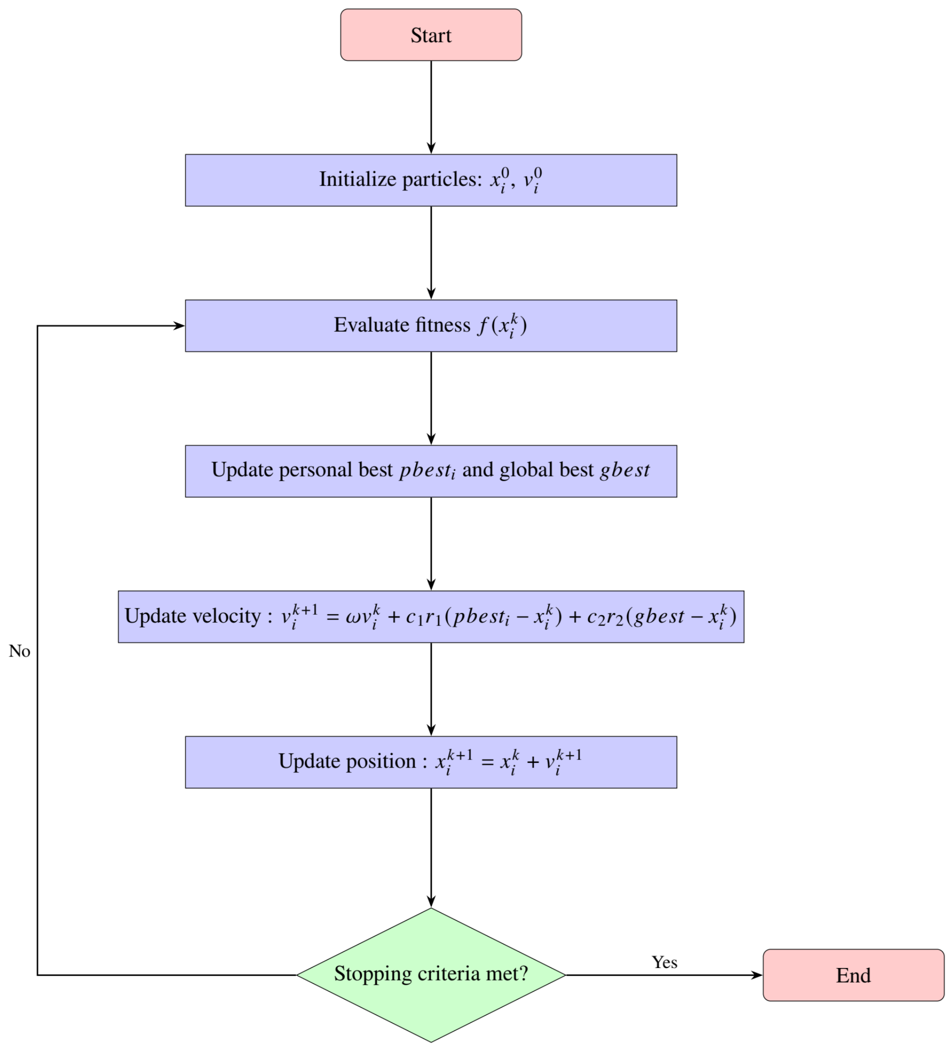


Figure 4: Flowchart of the Particle Swarm Optimization (PSO) Process

3.2. PSO Approach for Incomplete Pairwise Comparison Matrices

Particle Swarm Optimization (PSO) is an effective technique for addressing the challenge of incomplete pairwise comparison matrices. It aids in completing matrices of varying sizes, thereby helping experts save time and streamline the selection of pairwise comparison matrices within the AHP method for decision-making.

Initialization

The initialization phase involves randomly generating a set of particles, each representing a potential solution. The objective is to identify the incomplete pairwise comparison matrices using $\frac{n(n-1)}{2}$ entries from the range $[\frac{1}{9}, 9]$. To achieve this, we generate k random matrices, each represented by a vector in the following format:

$$(a_{1,2}, \ldots, ?, \ldots, a_{1,n}, a_{2,3}, ?, \ldots, a_{2,n}, \ldots, ?, a_{n-1,n})$$

In this representation (?) denotes the missing elements.

It is important to highlight that we need to create random vectors with m missing entries, each indicating their position within the matrix. Subsequently, we can generate the corresponding complete matrices to facilitate the continuation of the AHP process.

Example : Encoding IPC Matrices of Size 5

Consider the following incomplete pairwise comparison (IPC) matrix of size five :

$$\left[2, \frac{1}{3}, ?, 5, ?, 4, 6, 5, ?, 3\right] \Longrightarrow \begin{pmatrix} 1 & 2 & \frac{1}{3} & ? & 5 \\ \frac{1}{2} & 1 & ? & 4 & 6 \\ 3 & ? & 1 & 5 & ? \\ ? & \frac{1}{4} & \frac{1}{5} & 1 & 3 \\ \frac{1}{5} & \frac{1}{6} & ? & \frac{1}{3} & 1 \end{pmatrix}$$

The proposed method uses particle swarm optimization (PSO) to estimate the missing elements, as shown in Figure 5. The PSO approach allows experts to save time by focusing on refining and selecting appropriate pairwise comparison (PC) matrices within the analytic hierarchy process (AHP) method, ultimately supporting informed decision-making for the application in question.

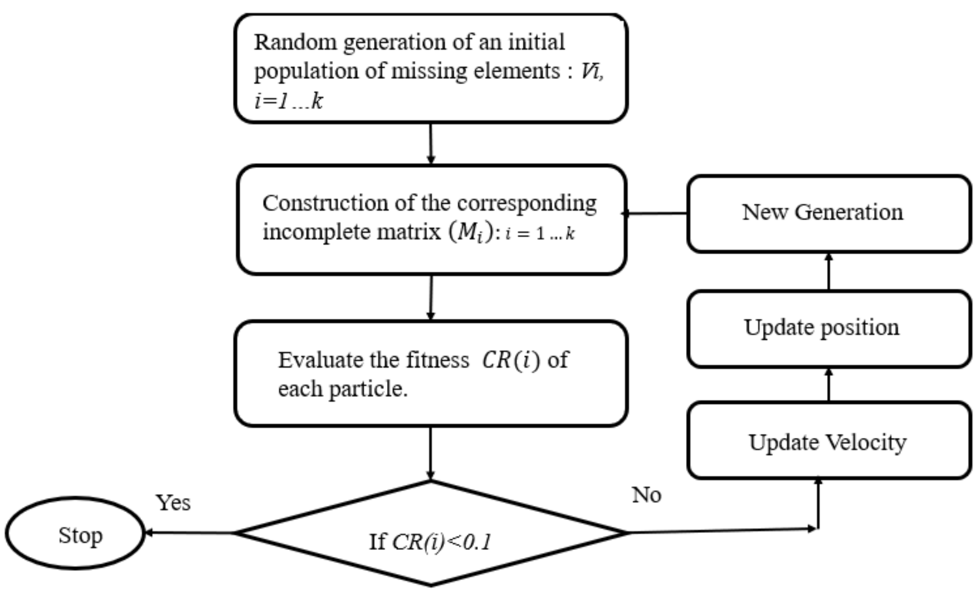


Figure 5: Diagram of the PSO Algorithm Applied to Incomplete Pairwise Comparison (IPC) Matrices

This methodology integrates an enhanced PSO with the AHP method. PSO explores various possibilities for missing data, while the AHP method assesses the acceptability of matrices by calculating the

consistency ratio (CR) until a sufficient level of consistency is reached.
The Algorithm 1 below outline the procedure for reconstructing incomplete PC matrices using the PSO approach.

Algorithm 1 Completion of Incomplete Pairwise Comparison Matrix using PSO

Input : an incomplete pairwise comparison matrix $A = (a_{ij})_{n\times n}$ with $m < n$ missing values.
Output : a completed matrix A with estimated values

Initialization
Determine m and n from A
Initialize a swarm of k particles $X^{(0)} = \{X_1^{(0)}, \dots, X_k^{(0)}\}$, where each $X_i^{(0)} \in \mathbb{R}^m$
Randomly assign each component of $X_i^{(0)}$ from $\left\{\frac{1}{9}, \frac{1}{8}, \dots, 1, \dots, 8, 9\right\}$
Evaluation
for each particle $X_i^{(0)}$ **do**
  fill in missing entries of A to form $M_i^{(0)}$
  compute the Consistency Ratio CR_i
end for
Iterative Optimization
while stopping criterion not met **do**
  for each particle $i = 1, \dots, k$ **do**
    update velocity :

$$v_i \leftarrow \omega v_i + c_1 r_1 (p_i - x_i) + c_2 r_2 (p_g - x_i)$$

    update position :

$$x_i \leftarrow x_i + v_i$$

    fill missing values in A using x_i to obtain M_i
    compute CR_i
    if CR_i improved **then**
      update personal best p_i
      if p_i better than all others **then**
        update global best p_g
      end if
    end if
  end for
end while
Output
Return completed matrix A using global best p_g.

4. Experimental Results and Analysis

4.1. Completion of PCMs and Achievement of the Consistency Threshold

These experiments were designed to assess the performance of the Particle Swarm Optimization (PSO) algorithm. In practice, the algorithm was able to generate pairwise comparison matrices with consistency ratios below the commonly accepted threshold of $CR < 0.1$, and it did so within a reasonable computational time.

To further examine the reliability of PSO, we performed a series of simulations on problems of different sizes. As the matrix dimensions increased, the settings of the algorithm-namely, the population size and the maximum number of iterations were adjusted proportionally. This adaptation was necessary to cope with the additional complexity introduced by larger instances.

The PSO parameters employed in the study are detailed as follows :

- Cognitive coefficient: $c_1 = 1.5$,

- Social coefficient: $c_2 = 1.5$,
- Inertia weight: $w = 0.7$.

Table 3 provides an overview of the problem encoding scheme corresponding to the different matrix sizes investigated.

Table 3: Matching vectors(particles) of random samples of matrix of various sizes

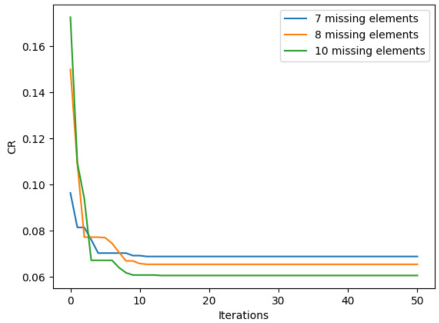
Matrices size	Particle(vector)
4	[?, 2, 4, ?, 2, 9]
6	$[?, \frac{1}{3}, 3, ?, 4, 3, 8, ?, 7, 8, 8,?, \frac{1}{3}, \frac{1}{2}, ?]$
12	$[3, ?,\frac{1}{2}, \frac{1}{4}, 2, ?, \frac{1}{6}, \frac{1}{5}, 2, \frac{1}{3}, 2, \frac{1}{3}, \frac{1}{7}, ?, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6},?, \frac{1}{2}, 3, \frac{1}{3}, ?, 2, 2, \frac{1}{6}, \frac{1}{5}, 4, ?, 5, \frac{1}{3}, 3, 4, 2, \frac{1}{2}, 8, ?, 6, 9, 3, 3, \frac{1}{2}, 6, 2, 2, \frac{1}{2}, \frac{1}{5}, ?, 4, \frac{1}{3}, 4, \frac{1}{5}, \frac{1}{7}, 4, \frac{1}{2}, 5, \frac{1}{2}, ?, 2, 4, 8, 4, 9, ?, 2, 7]$
15	$[3, ?\ 4, \frac{1}{4}, \frac{1}{4}, ?, ?, 2, 3, 3, 4, \frac{1}{2}, 3, 8, ?, 2, \frac{1}{4}, ?, 2, 3, 2, \frac{1}{4}, \frac{1}{2}, 4, \frac{1}{3}, 6, 4, \frac{1}{3}, \frac{1}{9}, \frac{1}{4}, \frac{1}{5}, ?, \frac{1}{5}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{5}, \frac{1}{3}, 2, ?, \frac{1}{4}, 2, \frac{1}{2}, 2, \frac{1}{2}, \frac{1}{4}, ?, \frac{1}{2}, 3, 6, 3, 6, 4, 8, 3, 2, ?, 6, 8, 9, 6, 5, 4, 2, 5, 7, 2, 7, ?, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{4}, ?, \frac{1}{9}, 3, 3, \frac{1}{3}, ?, \frac{1}{4}, 2, \frac{1}{3}, 2, 5, \frac{1}{5}, \frac{1}{4}, 2, \frac{1}{2}, 3, ?, 3, 2, 2, 3, 7, 4, 2, ?, 8, \frac{1}{3}, \frac{1}{2}, 2, ?, 4, 3]$

Table 4: Evaluation of Consistency Ratios, Matrix Sizes, and Iterations for Various Levels of Missing Data

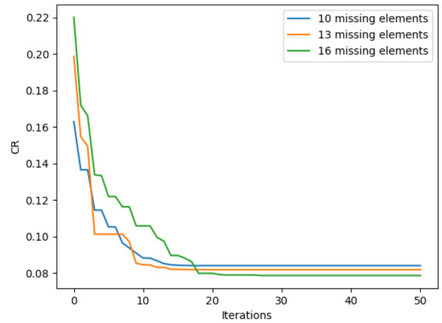
Missing elements m	**Matrix sizes** n	**Consistency ratio** (CR)	**Iterations**
7		$0,069$	15
8	10	$0,065$	20
10		$0,061$	25
10		$0,084$	15
13	12	$0,082$	35
16		$0,079$	40
15		$0,093$	15
18	15	$0,095$	40
20		$0,093$	50
20		$0,082$	40
25	20	$0,091$	45
28		$0,091$	50

During the process of the PSO method, it is observed that the consistency ratio index (CR) decreases progressively, enabling experts to select an appropriate complete matrix for the intended application as the iterations progress.

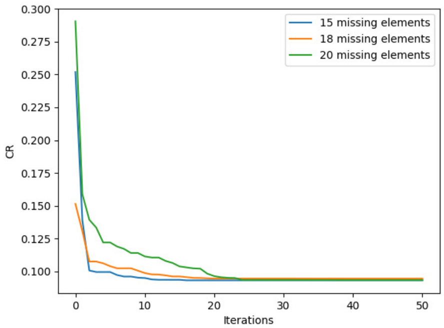
Table 4 reports the achieved consistency ratios, all below 0.1, along with the respective iteration counts. These findings suggest that completing the missing data does not require a substantial increase in iterations, even as the number of absent comparisons grows. To evaluate the effect of varying quantities of missing entries in incomplete pairwise comparison matrices, several test cases were generated featuring different numbers of absent elements. For instance, one case involved a matrix of size 10 with 7, 8, and 10 missing entries, which correspond to 14, 16, and 20 missing comparisons in the matrix, respectively. Figure 6 illustrates the progression of the consistency ratio (CR) throughout the particle swarm optimization process for selected incomplete matrices of sizes 10, 12, 15, and 20, each with varying levels of missing data. The results indicate that the proposed method effectively reconstructs the missing judgments even for larger matrices and when the proportion of missing elements is substantial.



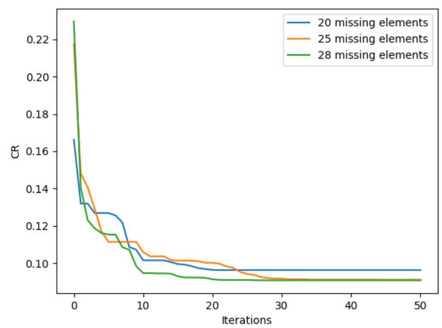
(a) Matrix size $n = 10$



(b) Matrix size $n = 12$



(c) Matrix size $n = 15$



(d) Matrix size $n = 20$

Figure 6: Performance of the PSO algorithm in generating consistent PC matrices for various matrix sizes.

These experiments confirm that PSO is capable of reconstructing incomplete pairwise comparison matrices of diverse sizes while ensuring the consistency of the completed matrices remains within acceptable limits.

4.2. PSO and GMM: A Comparative study

Table 5 presents the consistency ratio (CR) results for pairwise comparison matrices (PCMs) of increasing sizes, from $n = 4$ to $n = 15$, with a corresponding rise in the number of missing entries. Overall, both PSO and GMM produce similar CR values, with absolute differences typically below 0,01. For instance, at $n = 4$ and $n = 8$, the CRs are identical for both methods (0,073 and 0,051, respectively), while at $n = 10$, the difference remains minimal (0,077 for PSO versus 0,079 for GMM). However, more significant discrepancies appear at $n = 6$ and $n = 15$. When $n = 6$ with five missing entries, PSO yields a CR of 0,052 compared to 0,062 for GMM, for $n = 15$, where twenty entries are missing, the CR is 0,082 with PSO and 0,092 with GMM.

Despite these variations, all CR values remain below the standard threshold of 0,1, indicating that both methods maintain acceptable levels of consistency. The lowest CRs are observed at $n = 6$ (PSO) and $n = 8$ (both methods), suggesting better consistency in smaller or moderately incomplete matrices. As matrix size increases beyond $n = 10$, CR values tend to rise, reflecting the increased difficulty of maintaining consistency under more severe incompleteness. PSO produces CR values equal to or lower than GMM in four of six cases, pointing to a slight advantage in consistency optimization, especially under high levels of missing data. However, both approaches demonstrate robustness and practical reliability.

Table 5: Comparison of PSO and GMM Methods Based on CR Values for Matrices of Varying Sizes

Incomplete PCMs		**Calculated Consistency Ratio (CR)**	
Matrix size	**Missing element**	**CR by PSO**	**CR by GMM**
4	2	0,073	0,073
6	5	0,052	0,062
8	8	0,051	0,051
10	9	0,077	0,079
12	10	0,084	0,089
15	20	0,082	0,092

5. Conclusion

Incomplete pairwise comparison matrices represent a recurrent challenge in numerous applications of the AHP, particularly when expert input is limited or partially unavailable. In this work, we propose the use of PSO, a population-based metaheuristic, to estimate the missing judgments and reconstruct complete pairwise matrices. The enhanced PSO algorithm introduced here is designed to iteratively complete the matrix while preserving a satisfactory level of consistency. Extensive numerical experiments on matrices of different sizes confirm the effectiveness of PSO in approximating missing values and helping decision makers refine their judgments. Moreover, a comparative study with the GMM demonstrates that PSO consistently achieves comparable, and in many cases slightly improved, consistency ratios, particularly under conditions of high incompleteness. These findings underscore the robustness and practical utility of the PSO-based approach in addressing incompleteness in AHP decision model.

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