Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 3** : 1–7.
ISSN-0037-8712
doi:10.5269/bspm.79314

Patterns of Integer Solutions to Non-Homogeneous Ternary Sextic Diophantine Equation $A\left(x^2+y^2\right)-(2A-1)xy=\left(k^2+(4A-1)s^2\right)z^6$

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ABSTRACT: Fascinating patterns of integer solutions to non-homogeneous ternary sextic Diophantine equation given by $\mathrm{A}\left(\mathrm{x}^2+\mathrm{y}^2\right)-(2\ \mathrm{A}-1)\mathrm{xy}=\left(\mathrm{k}^2+(4\ \mathrm{A}-1)\mathrm{s}^2\right)\mathrm{z}^6$ is obtained through employing substitution technique and factorization method.

Key Words: Ternary sextic equation, non-homogeneous sextic equation, substitution technique, factorization method.

Contents

1. Introduction

It is well-known that a diophantine equation is an algebraic equation with integer coefficients involving two or more unknowns such that the only solutions focused are integer solutions. No doubt that diophantine equations are rich in variety [1-4] .There is no universal method available to know whether a diophantine equation has a solution or finding all solutions if it exists. In particular, on solving sextic Diophantine equations with variables at least three, the problems illustrated in [5-24] are observed. This paper focuses on finding integer solutions to the non-homogeneous sextic equation with three unknowns $A\left(x^2+y^2\right)-(2A-1)xy=\left(k^2+(4A-1)s^2\right)z^6$. Varieties of solutions in integers are obtained through methods of substitution and factorization.

2. Method of Analysis

The non-homogeneous polynomial equation of degree six with three unknowns to be solved for the integer solutions is

$$A\left(x^2+y^2\right)-(2A-1)xy=\left(k^2+(4A-1)s^2\right)z^6, k\neq s \tag{2.1}$$

The introduction of the transformations

$$\mathrm{x}=\mathrm{u}+\mathrm{v}, \mathrm{y}=\mathrm{u}-\mathrm{v}, \mathrm{u}\neq\mathrm{v} \tag{2.2}$$

in (1) leads to the sextic equation

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2010 *Mathematics Subject Classification:* 11D99.
Submitted October 02, 2025. Published December 20, 2025

$$u^2+(4A-1)v^2=\left(k^2+(4A-1)s^2\right)z^6 \tag{2.3}$$

The process of obtaining varieties of non-zero integer solutions to (2.1) is illustrated below:

2.1. Procedure

By inspection, it is seen that (2.3) is satisfied by

$$\mathrm{u}=\mathrm{kz}^3, \mathrm{v}=\mathrm{sz}^3$$

In view of (2.2), the values of x, y satisfying (2.1) are obtained as

$$x=(k+s)z^3, y=(k-s)z^3$$

2.2. Procedure

By scrutiny, it is observed that (2.3) is satisfied by

$$u=k\left(k^2+(4A-1)s^2\right)^{3\alpha}\beta^{3t}, v=s\left(k^2+(4A-1)s^2\right)^{3\alpha}\beta^{3t}; \beta>1, \alpha, t>0$$

and

$$z=\left(k^2+(4A-1)s^2\right)^{\alpha}\beta^{t} \tag{2.4}$$

In view of (2.2), we have

$$x=(k+s)\left(k^2+(4A-1)s^2\right)^{3\alpha}\beta^{3t}, y=(k-s)\left(k^2+(4A-1)s^2\right)^{3\alpha}\beta^{3t}. \tag{2.5}$$

Thus, (2.4) & (2.5) satisfy (2.1).

2.3. Procedure

Let

$$z=a^2+(4A-1)b^2 \tag{2.6}$$

Substituting (2.6) in (2.3) and employing factorization , consider

$$\begin{aligned} u+i\sqrt{4A-1}v &= (k+i\sqrt{4A-1}s)(a+i\sqrt{4A-1}b)^6 \\ &=(k+i\sqrt{4A-1}s)[f(a,b)+i\sqrt{4A-1}g(a,b)] \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} f(a,b) &= a^6-105a^4b^2+735a^2b^4-343b^6 \\ g(a,b) &= 6a^5b-140a^3b^3+294ab^5 \end{aligned} \tag{2.8}$$

On equating the coefficients of corresponding terms in (2.7), we have

$$\begin{aligned} u &= kf(a,b)-(4A-1)sg(a,b) \\ v &= sf(a,b)+kg(a,b) \end{aligned}$$

In view of (2.2), one has

$$\begin{aligned} x &= (k+s)f(a,b)+(k-(4A-1)s)g(a,b) \\ y &= (k-s)f(a,b)-(k+(4A-1)s)g(a,b) \end{aligned} \tag{2.9}$$

Thus, (2.6) & (2.9) satisfy (2.1).

2.4. Procedure

Rewrite (2.3) as

$$u^2+(4A-1)v^2=\left(k^2+(4A-1)s^2\right)z^6*1 \tag{2.10}$$

The integer 1 on the R.H.S. of (2.10) is write

$$1=\frac{(F(A,n)+i(2n-1)\sqrt{4A-1})(F(A,n)-i(2n-1)\sqrt{4A-1})}{(G(A,n))^2} \tag{2.11}$$

where

$$F(A,n)=(8A-2)n^2-(8A-2)n+2A-1$$
$$G(A,n)=(8A-2)n^2-(8A-2)n+2A$$

Substituting (2.6) & (2.11) in (2.10) and employing factorization, consider

$$\begin{aligned}
u+i\sqrt{4A-1}v&=(k+i\sqrt{4A-1}s)(a+i\sqrt{4A-1}b)^6\frac{(F(A,n)+i(2n-1)\sqrt{4A-1})}{(G(A,n))}\\
&=(k+i\sqrt{4A-1}s)[f(a,b)+i\sqrt{4A-1}g(a,b)]\frac{(F(A,n)+i(2n-1)\sqrt{4A-1})}{(G(A,n))}\\
&=[f(a,b)+i\sqrt{4A-1}g(a,b)]\frac{(F(A,k,s,n)+i\sqrt{4A-1}G(A,k,s,n))}{(G(A,n))}
\end{aligned} \tag{2.12}$$

where

$$F(A,k,s,n)=k(F(A,n))-(4A-1)s(2n-1)$$
$$G(A,k,s,n)=s(F(A,n))+k(2n-1)$$

Equating the coefficients of corresponding terms in (2.12), one obtains

$$\begin{aligned}
u&=\frac{[f(a,b)F(A,k,s,n)-(4A-1)g(a,b)G(A,k,s,n)]}{(G(A,n))}\\
v&=\frac{[g(a,b)F(A,k,s,n)+f(a,b)G(A,k,s,n)]}{(G(A,n))}
\end{aligned} \tag{2.13}$$

As the main thrust of this paper is to obtain integer solutions, replacing a by $(\mathrm{G}(\mathrm{A},\mathrm{n}))\mathrm{P}$ and b by $(\mathrm{G}(\mathrm{A},\mathrm{n}))\mathrm{Q}$ in (2.6) & (2.13) and employing (2.2), the corresponding integer solutions to (1) are given by

$$\begin{aligned}
\mathrm{x}&=\mathrm{x}(\mathrm{P},\mathrm{Q},\mathrm{k},\mathrm{s},\mathrm{n})\\
&=(\mathrm{G}(\mathrm{A},\mathrm{n}))^5[\mathrm{F}(\mathrm{A},\mathrm{k},\mathrm{s},\mathrm{n})\{\mathrm{f}(\mathrm{P},\mathrm{Q})+\mathrm{g}(\mathrm{P},\mathrm{Q})\}+\mathrm{G}(\mathrm{A},\mathrm{k},\mathrm{s},\mathrm{n})\{\mathrm{f}(\mathrm{P},\mathrm{Q})-(4\mathrm{A}-1)\mathrm{g}(\mathrm{P},\mathrm{Q})\}]\\
\mathrm{y}&=\mathrm{y}(\mathrm{P},\mathrm{Q},\mathrm{k},\mathrm{s},\mathrm{n})\\
&=(\mathrm{G}(\mathrm{A},\mathrm{n}))^5[\mathrm{F}(\mathrm{A},\mathrm{k},\mathrm{s},\mathrm{n})\{\mathrm{f}(\mathrm{P},\mathrm{Q})-\mathrm{g}(\mathrm{P},\mathrm{Q})\}-\mathrm{G}(\mathrm{A},\mathrm{k},\mathrm{s},\mathrm{n})\{\mathrm{f}(\mathrm{P},\mathrm{Q})+(4\mathrm{A}-1)\mathrm{g}(\mathrm{P},\mathrm{Q})\}]\\
\mathrm{z}&=\mathrm{z}(\mathrm{A},\mathrm{P},\mathrm{Q},\mathrm{n})\\
&=(\mathrm{G}(\mathrm{A},\mathrm{n}))^2\left(\mathrm{P}^2+(4\mathrm{A}-1)\mathrm{Q}^2\right)
\end{aligned}$$

Remark 2.1 *It is to be seen that ,in addition to (2.11), one may have*

$$1=\frac{\left((4A-1)r^2-t^2+i\sqrt{4A-1}(2rt)\right)\left((4A-1)r^2-t^2-i\sqrt{4A-1}(2rt)\right)}{\left((4A+1)r^2+t^2\right)^2}$$
$$1=\frac{\left(r^2-(4A-1)t^2+i\sqrt{4A-1}(2rt)\right)\left(r^2-(4A-1)t^2-i\sqrt{4A-1}(2rt)\right)}{\left(r^2+(4A-1)t^2\right)^2}$$

Following the above analysis , two more sets of integer solutions to (2.1) are obtained.

2.5. Procedure

It is worth to mention that the expression $\left(\mathrm{k}^2+(4\ \mathrm{A}-1)\mathrm{s}^2\right)$ is a perfect square when

$$\mathrm{k}=(2\ \mathrm{A}-1)\mathrm{s} \tag{2.14}$$

Note that

$$\left(k^2+(4A-1)s^2\right)=(2As)^2$$

The substitution of the transformations

$$x=2As(u+v), y=2As(u-v), u\neq v \tag{2.15}$$

in (2.1) gives

$$u^2+(4A-1)v^2=z^6 \tag{2.16}$$

Substituting (2.6) in (2.16) and employing factorization , we get after some algebra

$$\mathrm{u}=\mathrm{f}(\mathrm{a},\ \mathrm{b}), \mathrm{v}=\mathrm{g}(\mathrm{a},\ \mathrm{b})$$

given in (2.8). In view of (2.15), we get

$$\begin{aligned} x&=2As[f(a,b)+g(a,b)], \\ y&=2As[f(a,b)-g(a,b)]. \end{aligned} \tag{2.17}$$

Thus ,(2.1) is satisfied by (2.6) & (2.15).

Remark 2.2 *The option*

$$\mathrm{v}=\mathrm{z}^2 \tag{2.18}$$

in (2.16) leads to

$$\mathrm{u}^2=\mathrm{z}^4\left(\mathrm{z}^2-(4\ \mathrm{A}-1)\right)$$

which is satisfied by

$$\mathrm{z}=\pm 2\ \mathrm{A}, \mathrm{u}=\pm 4\ \mathrm{A}^2(2\ \mathrm{A}-1)$$

and from (2.18), we obtain

$$\mathrm{v}=4\ \mathrm{A}^2$$

Thus , from (2.15) ,the following two sets of integer solutions to (2.1) are obtained:

$$\left(16\mathrm{sA}^4, 16\mathrm{sA}^4-16\mathrm{sA}^3, \pm 2\ \mathrm{A}\right), \left(-16\mathrm{sA}^4+16\mathrm{sA}^3, -16\mathrm{sA}^4, \pm 2\ \mathrm{A}\right)$$

Remark 2.3 *The choice*

$$\mathrm{u}=\mathrm{z}^2 \tag{2.19}$$

in (2.16) gives

$$(4A-1)v^2=z^4\left(z^2-1\right) \tag{2.20}$$

Assume

$$\mathrm{z}^2 = (4\ \mathrm{A} - 1)\mathrm{R}^2 + 1 \tag{2.21}$$

The above equation (2.21) is well-known pellian equation whose general solution ($\mathrm{z_n}, \mathrm{R_n}$) is given by

$$\mathrm{z_n} = \frac{\mathrm{f_n}}{2}, \mathrm{R_n} = \frac{\mathrm{g_n}}{2\sqrt{4\ \mathrm{A} - 1}} \tag{2.22}$$

where

$$f_n = \left(z_0 + R_0\sqrt{4A-1}\right)^{n+1} + \left(z_0 - R_0\sqrt{4A-1}\right)^{n+1} \tag{2.23}$$
$$g_n = \left(z_0 + R_0\sqrt{4A-1}\right)^{n+1} + \left(z_0 - R_0\sqrt{4A-1}\right)^{n+1}$$

in which ($\mathrm{z_0}, \mathrm{R_0}$) is the initial solution to (2.21).
From (2.20), we get

$$\mathrm{v_n} = (\mathrm{z_n})^2 * \mathrm{R_n}$$

From (2.19), we have

$$\mathrm{u_n} = \mathrm{z_n^2}$$

In view of (2.15) , we obtain

$$x_n = 2\,\mathrm{As}\,(1 + R_n)\,(z_n)^2 = \frac{As}{4\sqrt{4A-1}}\left[2\sqrt{4A-1} + g_n\right](f_n)^2 \tag{2.24}$$
$$y_n = 2\,\mathrm{As}\,(1 - R_n)\,z_n^2 = \frac{As}{4\sqrt{4A-1}}\left[2\sqrt{4A-1} - g_n\right](f_n)^2$$

Thus , (2.22) & (2.24) satisfy (2.1).

Remark 2.4 *It is worth to mention that , apart from (2.14), the expression* $\mathrm{k}^2 + (4\ \mathrm{A} - 1)\mathrm{s}^2$ *is a perfect square when*
(i) $\mathrm{s} = 2\ \mathrm{d} - 1, \mathrm{k} = 2\ \mathrm{A} - \left(2\ \mathrm{d}^2 - 2\ \mathrm{d} + 1\right)$ *and* $\mathrm{s} = 2\mathrm{pq}, \mathrm{k} = (4\ \mathrm{A} - 1)\mathrm{p}^2 - \mathrm{q}^2$. *Following the above procedure , one may obtain some more patterns of integer solutions to (2.1).*

3. Conclusion

In this paper, an attempt has been made to determine the non-zero distinct integer solutions to the non-homogeneous sextic diophantine equation with three unknowns given in the title through employing transformations. The researchers in this area may search for other choices of transformations to obtain integer solutions to the sextic diophantine equation with three or more unknowns.

Acknowledgments

We thank the referee for the suggestions.

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