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The Generalized q-Operator ${}_r\Phi_s$ and its Applications in q-Integrals

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ABSTRACT: In this paper, we use the generalzed q-operator ${}_r\Phi_s$ to generalize some well-known q-integrals such as Askey-beta Integral, Anderws-Askey Integral, Gasper integral and Askey-Roy integral.

Keywords: The q-operator ${}_r\Phi_s$, Askey-beta integral, Gasper integral, Andrews-Askey integral.

Contents

1. Defintions and Basic Results

We'll follow the notations that used in [17] with assuming that $|q| < 1$.
Let a be a complex variable. The q-shifted factorial is defined by [1,2,14]

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),$$

and adopting the following compact notation for the multiple q-shifted factorial [9,30,31]:

$$(a_1,\ldots,a_r;q)_n = (a_1;q)_n\ldots(a_r;q)_n,$$

where n is an integer or ∞.

The basic hypergeometric series ${}_r\phi_s$ is defined by [17,18,34]:

$$\begin{aligned}{}_r\phi_s(a_1,\ldots,a_r;b_1,\ldots,b_s;q,x) &= {}_r\phi_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,x\right)\\ &= \sum_{k=0}^{\infty}\frac{(a_1;q)_k\cdots(a_r;q)_k}{(q;q)_k\,(b_1;q)_k\cdots(b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}x^k,\end{aligned}$$

where $r,s\in\mathbb{N}$; $a_1,\ldots,a_r,b_1,\ldots,b_s\in\mathbb{C}$; and none of the denominator factors evaluate to zero. The above series is absolutely convergent for all $x\in\mathbb{C}$ if $r<s+1$, for $|x|<1$ if $r=s+1$ and for $x=0$ if $r>s+1$.

The most important case of the above series is when $r=s+1$ [6,7,8]

$$\begin{aligned}{}_{s+1}\phi_s(a_1,\ldots,a_{s+1};b_1,\ldots,b_s;q,x) &= {}_{s+1}\phi_s\left(\begin{matrix}a_1,\ldots,a_{s+1}\\ b_1,\ldots,b_s\end{matrix};q,x\right)\\ &= \sum_{k=0}^{\infty}\frac{(a_1;q)_k\ldots(a_{s+1};q)_k}{(q;q)_k\,(b_1;q)_k\cdots(b_s;q)_k}x^k, \quad |x|<1.\end{aligned}$$

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The q-binomial coefficient is given as [19,20,25]

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, & \text{if } 0 \le k \le n; \\ 0, & \text{otherwise,} \end{cases}$$

where $n, k \in \mathbb{N}$.

The following equation will be used in this paper [26,27,32]:

$$(a;q)_{n-k} = \frac{(a;q)_n}{(q^{1-n}/a;q)_k} \left(-\frac{q}{a}\right)^k q^{\binom{k}{2}-nk}, \tag{1.1}$$

where n and k are integers.

The Cauchy identity is given by [3,4,21]

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1. \tag{1.2}$$

Euler found the following special case of Cauchy identity [5,33,39]:

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} x^k = (x;q)_\infty. \tag{1.3}$$

q-Chu-Vandermonde's identity [17,28] is

$${}_2\phi_1\left(\begin{matrix} q^{-n}, b \\ c, \end{matrix} ; q, cq^n/b\right) = \frac{(c/b;q)_n}{(c;q)_n}. \tag{1.4}$$

The operator θ is defined as [29,37]

$$\theta\left\{f(a)\right\} = \frac{f(aq^{-1}) - f(a)}{aq^{-1}}.$$

In 1997, Chen and Liu [13] defined the q-exponential operator $E(b\theta)$ as follows:

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^{\binom{n}{2}}}{(q;q)_n}. \tag{1.5}$$

Chen and Liu [13] proved the following result:

$$E(b\theta)\{(at;q)_\infty\} = (at, btq)_\infty. \tag{1.6}$$

In 2007, Fang [15] defined the Cauchy operator ${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix} ; q, -c\theta\right)$ as follows:

$${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix} ; q, -c\theta\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} (-c\theta)^n. \tag{1.7}$$

Fang proved the following results:

Theorem 1.1 [15] *Let* ${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix} ; q, -c\theta\right)$ *be defined as in (1.7) then:*

$${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix} ; q, -c\theta\right)\left\{\frac{(at, as;q)_\infty}{(aw;q)_\infty}\right\} = \frac{(at, as, bct;q)_\infty}{(aw, ct;q)_\infty}\, {}_3\phi_2\left(\begin{matrix} b, s/w, q/at \\ q/ct, q/aw \end{matrix} ; q, q\right), \tag{1.8}$$

provided that $s/w = q^{-n}$ *and* $|cst/w| < 1$.

$${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix} ; q, -c\theta\right)\{(as;q)_\infty\} = \frac{(bcs, as;q)_\infty}{(cs;q)_\infty}, \tag{1.9}$$

provided that $|cs| < 1$.

The series ${}_r\varphi_s$ is defined as [11,38,43]:

$$ {}_r\varphi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix} ;q,x\right) = \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_r;q)_n}{(q,b_1,\dots,b_s;q)_n}\, x^n. $$

Note that when $r = s+1$, we have ${}_{s+1}\varphi_s = {}_{s+1}\phi_s$.

In 2010, Zhang and Yang [43] construct the finite q-Exponential Operator ${}_2\mathbb{E}_1\left[\begin{matrix} q^{-N},w \\ v \end{matrix} ;q,d\theta\right]$ with two parameters as:

$$ {}_2\mathbb{E}_1\left[\begin{matrix} q^{-N},w \\ v \end{matrix} ;q,d\theta\right] = \sum_{n=0}^{N} \frac{(q^{-N},w;q)_n}{(q,v;q)_n}(d\theta)^n. \tag{1.10} $$

Zhang and Yang [43] proved the following results:

Theorem 1.2 [43]. *Let* ${}_2\mathbb{E}_1\left[\begin{matrix} q^{-N},w \\ v \end{matrix} ;q,d\theta\right]$ *be defined as in* (1.10), *then*

$$ \begin{aligned} &{}_2\mathbb{E}_1\left[\begin{matrix} q^{-N},w \\ v \end{matrix} ;q,-\frac{v}{tw}q^N\theta\right]\left\{\frac{(as,at;q)_\infty}{(aw;q)_\infty}\right\} \\ &\quad = \frac{(as,at;q)_\infty}{(aw;q)_\infty}\frac{(v/w;q)_N}{(v;q)_N}\,{}_4\varphi_2\left(\begin{matrix} q^{-N},\ w,\ s/u,\ q/at \\ q/au,\ q^{1-N}w/v \end{matrix} ;q,q\right), \end{aligned} \tag{1.11} $$

provided that $|aw| < 1$.

$$ {}_2\mathbb{E}_1\left[\begin{matrix} q^{-N},w \\ v \end{matrix} ;q,d\theta\right]\{(at;q)_\infty\} = (at;q)_\infty\ {}_2\varphi_1\left(\begin{matrix} q^{-N},w \\ v \end{matrix} ;q,-dt\right), \tag{1.12} $$

provided that $|dt| < 1$.

In 2016, Li and Tan [24] introduced the generalized q-exponential operator $\mathbb{E}\left[\begin{matrix} v,u \\ w \end{matrix} \middle|q;t\theta\right]$ with three parameters as:

$$ \mathbb{E}\left[\begin{matrix} u,v \\ w \end{matrix} \middle|q;t\theta\right] = \sum_{n=0}^{\infty} \frac{(v,u;q)_n}{(q,w;q)_n}(t\theta)^n. \tag{1.13} $$

Li and Tan proved the following results:

Theorem 1.3 [24]. *Let* $\mathbb{E}\left[\begin{matrix} v,u \\ w \end{matrix} \middle|q;t\theta\right]$ *be defined as in (1.13), then*

$$ \begin{aligned} &\mathbb{E}\left[\begin{matrix} u,v \\ w \end{matrix} \middle|q;t\theta\right]\left\{\frac{(xa,xc;q)_\infty}{(xb;q)_\infty}\right\} \\ &\quad = \frac{(xc,xa;q)_\infty}{(xb;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(v,u;q)_{n+k}}{(w;q)_{n+k}(q;q)_n}\frac{(a/b,q/cx;q)_k}{(q,q/xb;q)_k}(-1)^n(tc)^{n+k}q^{-\binom{k}{2}-nk}, \end{aligned} \tag{1.14} $$

provided that $|xb| < 1$.

In 2019, Saad and Khalaf [35] defined the generalized q-operator ${}_r\Phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix} ;q,-c\theta\right)$ as follows:

$$ {}_r\Phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix} ;q,-c\theta\right) = \sum_{k=0}^{\infty} W_k \frac{(-c\theta)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}, \tag{1.15} $$

where $W_k = \dfrac{(a_1,\dots,a_r;q)_k}{(b_1,\dots,b_s;q)_k}$.

The following known operators can be obtained by specializing the values of the arguments in the q-operator ${}_r\Phi_s$ (1.15):

1. When $r = s = 0$ and $c = -b$, we get the q-exponential operator $E(b\theta)$ defined by Chen and Liu [13] in 1998.
2. Setting $r = 1$, $s = 0$ and $a_1 = b$, we obtain the q-exponential operator ${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix} ; q, -c\theta\right)$ defined by Fang [15] in 2007.
3. Letting $r = 2, s = 1$, $(a_1, a_2, b_1, c) = (q^{-N}, w, v, -d)$, we obtain the finite q-exponential operator with two parameters ${}_2\mathbb{E}_1\left[\begin{matrix} q^{-N}, w \\ v \end{matrix} ; q, d\theta\right]$ defined by Zhang and Yang [43] in 2010.
4. Finally, when $r = 2, s = 1$, $(a_1, a_2, b_1, c) = (u, v, w, -t)$, we get the generalized q-exponential operator with three parameters $\mathbb{E}\left[\begin{matrix} u, v \\ w \end{matrix} |q; t\theta\right]$ defined by Li and Tan [24] in 2016.

Saad and Khalaf [35] gave the following result:

Theorem 1.4 [35]. *Let* ${}_r\Phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta\right)$ *be defined as in* (1.15)*, then*

$$
{}_r\Phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta\right)\{(at;q)_\infty\} = (at;q)_\infty \sum_{k=0}^{\infty} W_k \frac{(ct)^k}{(q;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}. \tag{1.16}
$$

In 2023, Saad and Khalaf [36] obtained following results for the operator ${}_r\Phi_s$

Theorem 1.5 [36]. *Let* ${}_r\Phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta\right)$ *be defined as in* (1.15)*, then*

$$
\begin{aligned}
{}_r\Phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta\right)\left\{\frac{(au, at, ax;q)_\infty}{(av, aw;q)_\infty}\right\} &= \frac{(au, at, ax;q)_\infty}{(av, aw;q)_\infty} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} W_{k+j+i} \\
&\times (cx)^{k+j+i} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} \left[(-1)^{k+j+i} q^{\binom{k+j+i}{2}}\right]^{s-r} \frac{(u/w, q/at;q)_i}{(q, q/aw;q)_i} \left(\frac{t}{v}\right)^i \frac{(t/v;q)_j}{(q,q)_j} \\
&\times \frac{(q/ax;q)_{j+i}}{(q/av;q)_{j+i}},
\end{aligned} \tag{1.17}
$$

provided that $max\{|av|, |aw|\} < 1$.

Corollary 1.1 [36].

$$
\begin{aligned}
{}_r\Phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta\right)\left\{\frac{(au, at;q)_\infty}{(aw;q)_\infty}\right\} &= \frac{(au, at;q)_\infty}{(aw;q)_\infty} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} W_{k+j} \frac{(ct)^k}{(q;q)_k} \\
&\times \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} \frac{(u/w, q/at;q)_j}{(q, q/aw;q)_j} (ct)^j \left[(-1)^j q^{\binom{j}{2}}\right]^{s-r} q^{kj(s-r)},
\end{aligned} \tag{1.18}
$$

Provided that $|aw| < 1$.

The following operator identities can be obtained by specializing the parameter values in the operator identities (1.16) and (1.18):

- By setting $r = s = 0$ and $c = -b$ in (1.16), we get Theorem 2.9. obtained in Chen and Liu [13] (equation (1.6)).
- Setting $r = 1$, $s = 0$ and $a_1 = b$ in (1.16), we get Theorem 1.3. obtained in Fang [15] (equation (1.9)).

- Setting $r=2$, $s=1$, $a_1=q^{-N}$, $a_2=w$, $b_1=v$ and $c=-d$ in (1.18), we get Theorem 2.1 obtained by Zhang and Yang [43] (equation (1.12)).
- Setting $r=1$, $s=0$, $a_1=b$, $u=s$ in (1.18), we get Theorem 1.1. obtained by Fang [15] (equation (1.8)).
- Setting $r=2$, $s=1$, $(a_1,a_2,b_1,c)=(q^{-N},w,v,\frac{v}{tw}q^N)$ and $u=s$ in (1.18), then by using (1.4) and (1.1), we get Theorem 2.3 obtained by Zhang and Yang [43] (equation (1.11)).
- Setting $r=2$, $s=1$, $(a_1,a_2,b_1,c,a,u,t,w)=(u,v,w,-t,x,a,c,b)$ in (1.18), we get Theorem 3 obtained by Li and Tan [24] (equation(1.14).

2. The q-Integral

Jackson [22,23] and Thomae [40,41] introduced the q-integral

$$\int_0^1 f(t)d_qt=(1-q)\sum_{n=0}^{\infty}f(q^n)q^n,$$

and Jackson gave the more general definition

$$\int_a^b f(t)d_qt=\int_0^b f(t)d_qt-\int_0^a f(t)d_qt,$$

where

$$\int_0^a f(t)d_qt=a(1-q)\sum_{n=0}^{\infty}f(aq^n)q^n.$$

Jackson defined an integral on $(0,\infty)$ as

$$\int_0^{\infty} f(t)d_qt=(1-q)\sum_{n=-\infty}^{\infty}f(q^n)q^n.$$

The bilateral q-integral is defined by

$$\int_{-\infty}^{\infty} f(t)d_qt=(1-q)\sum_{n=-\infty}^{\infty}\left[f(q^n)+f(-q^n)\right]q^n.$$

The Askey-beta integral [10] is stated as follows:

$$\int_{-\infty}^{\infty}\frac{(at,bt;q)_\infty}{(-dt,et;q)_\infty}d_qt=\frac{2(1-q)(q^2;q^2)_\infty^2(de,q/de,a/e,-a/d,b/e,-b/d;q)_\infty}{(d^2,e^2,q^2/d^2,q^2/e^2;q^2)_\infty(q,-ab/deq;q)_\infty}. \tag{2.1}$$

In 1998, Chen and Liu [13] used the operator $E(c\theta)$ to find an extension for the Askey-beta integral (2.1) as follows:

Theorem 2.1 [13]. *Let $E(c\theta)$ be defined as in (1.5), then*

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{(at,bt,ct;q)_\infty}{(-dt,et,-abct/deq^2;q)_\infty}d_qt\\&\quad=\frac{2(1-q)(q^2;q^2)_\infty^2(q/de,de,a/e,-b/d,b/e,-a/d,c/e,-c/d;q)_\infty}{(e^2,d^2,q^2/e^2,q^2/d^2;q^2)_\infty(q,-ac/deq,-ab/deq,-bc/deq;q)_\infty}.\end{aligned} \tag{2.2}$$

Gasper integral [16] is given by

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\mu e^{i\theta}/d,qde^{-i\theta}/\mu,\mu ce^{-i\theta},qe^{i\theta}/c\mu,abcdfe^{i\theta};q)_\infty}{(ae^{i\theta},be^{i\theta},ce^{-i\theta},de^{-i\theta},fe^{i\theta};q)_\infty}d\theta\\&\quad=\frac{(abcd,\mu c/d,dq/\mu c,\mu,q/\mu,acdf,bcdf;q)_\infty}{(q,bc,bd,ac,ad,fc,fd;q)_\infty},\end{aligned} \tag{2.3}$$

provided that $\max\{|a|,|d|,|c|,|b|\}<1$.

In 2005, Zhang and Wang [42] used the operator $E(d\theta)$ to obtain an extension for Gasper integral (2.3) as follows:

Theorem 2.2 [42]. *Let $E(d\theta)$ be defined as in (1.5), then*

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\mu e^{i\theta}/d, qde^{-i\theta}/\mu, \mu ce^{-i\theta}, qe^{i\theta}/c\mu, abcdfe^{i\theta};q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}, fe^{i\theta};q)_\infty}\\&\times {}_3\phi_2\left(\begin{matrix}bcdf, q/ac, q/ad\\ qe^{-i\theta}/a, q^2/acdg\end{matrix};q;q\right)d\theta\\&=\frac{(abcd, \mu c/d, dq/\mu c, \mu, q/\mu, acdf, bcdf, bcdg, cdfg, acdg/q;q)_\infty}{(q, ac, ad, bc, bd, cf, df, cg, dg, abc^2d^2fg/q;q)_\infty}.\end{aligned}\tag{2.4}$$

Andrews-Askey integral [12] is stated as follows:

$$\int_c^d\frac{(qx/c, qx/d;q)_\infty}{(ax, bx;q)_\infty}d_qx=\frac{d(1-q)(q, c/d, dq/c, abcd;q)_\infty}{(bc, bd, ad, ac;q)_\infty},\tag{2.5}$$

provided that $\max\{|a|,|b|,|c|,|d|\}<1$.

In 2014, Cao [12] used the operator $\mathbb{E}\left[\begin{matrix}w, r\\ v\end{matrix};q,-\frac{v}{wrt}\right]$ to obtain an extension for Andrews-Askey integral (2.5) as follows:

Theorem 2.3 [12]. *Let $\mathbb{E}\left[\begin{matrix}w, r\\ v\end{matrix};q,-\frac{v}{wrt}\right]$ be defined as in (1.10), then*

$$\begin{aligned}&\int_c^d\frac{(qt/d, qt/c;q)_\infty}{(bt, at;q)_\infty}\,{}_4\phi_2\left[\begin{matrix}r, w, c/t, q/ad\\ q/ad, qrw/v\end{matrix};q,q\right]d_qt\\&=\frac{d(1-q)(q, dq/c, c/d, abcd, v/wr, v;q)_\infty}{(bc, bd, ac, ad, v/w, v/r;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}w, r\\ v\end{matrix};q,\frac{vbc}{wr}\right].\end{aligned}\tag{2.6}$$

In this paper, we use the generalized q-operator ${}_r\Phi_s$ to give a generalization for Askey-beta Integral, Anderws-Askey Integral and Gasper integral.

3. Applications of the q-Operator ${}_r\Phi_s$ in q-Integrals

In this section, we act the operator ${}_r\Phi_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,-c\theta\right)$ with respect to the parameter a to find a generalization for some well-known integrals.

3.1. Generalization of Askey-beta Integral

We construct a generalization of the Askey-beta integral (2.1) by applying the operator ${}_r\Phi_s$ to both sides of the Askey-beta integral and using (1.18).

Theorem 3.1 (Generalization of Askey-beta integral). *Given Askey-beta integral (2.1), then*

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{(bt;q)_\infty}{(-dt, et;q)_\infty}\frac{(at, -ab/deq;q)_\infty}{(aw;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b/wdeq, q/at;q)_j}{(q, q/aw;q)_j}(ft)^j\left[(-1)^jq^{\binom{j}{2}}\right]^{s-r}W_{k+j}\\&\times\frac{(ft)^k}{(q;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}q^{kj(s-r)}d_qt\\&=2(1-q)\frac{(q^2;q^2)_\infty^2(de, q/de, b/e, -b/d;q)_\infty}{(d^2, e^2, q^2/d^2, q^2/e^2;q^2)_\infty(q;q)_\infty}\frac{(a/e, -a/d;q)_\infty}{(aw;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{((-1)/dw, qe/a;q)_j}{(q, q/aw;q)_j}\\&\times(f/e)^j\left[(-1)^jq^{\binom{j}{2}}\right]^{s-r}W_{k+j}\frac{(f/e)^k}{(q;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}q^{kj(s-r)}.\end{aligned}\tag{3.1}$$

Proof: Rewrite Askey-beta integral (2.1) as follows:

$$\int_{-\infty}^{\infty} \frac{(bt;q)_\infty}{(-dt,et;q)_\infty}(at,-ab/deq;q)_\infty d_qt$$
$$= \frac{2(1-q)(q^2;q^2)_\infty^2(de,q/de,b/e,-b/d;q)_\infty}{(d^2,e^2,q^2/d^2,q^2/e^2;q^2)_\infty(q;q)_\infty}(a/e,-a/d;q)_\infty.$$

Multiplying both sides of the above equation by $\dfrac{1}{(aw;q)_\infty}$, we get

$$\int_{-\infty}^{\infty} \frac{(bt;q)_\infty}{(-dt,et;q)_\infty}\frac{(at,-ab/deq;q)_\infty}{(aw;q)_\infty} d_qt$$
$$= \frac{2(1-q)(q^2;q^2)_\infty^2(de,q/de,b/e,-b/d;q)_\infty}{(d^2,e^2,q^2/d^2,q^2/e^2;q^2)_\infty(q;q)_\infty}\frac{(a/e,-a/d;q)_\infty}{(aw;q)_\infty}.$$

Acting the operator ${}_r\Phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s\end{matrix};q,-f\theta\right)$ on both sides of the above equation yielding

$$\int_{-\infty}^{\infty} \frac{(bt;q)_\infty}{(-dt,et;q)_\infty}{}_r\Phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s\end{matrix};q,-f\theta\right)\left\{\frac{(at,-ab/deq;q)_\infty}{(aw;q)_\infty}\right\} d_qt$$
$$= 2(1-q)\frac{(q^2;q^2)_\infty^2(de,q/de,b/e,-b/d;q)_\infty}{(d^2,e^2,q^2/d^2,q^2/e^2;q^2)_\infty(q;q)_\infty}$$
$$\times\, {}_r\Phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s\end{matrix};q,-f\theta\right)\left\{\frac{(a/e,-a/d;q)_\infty}{(aw;q)_\infty}\right\}.$$

By using (1.18), we get

$$\int_{-\infty}^{\infty} \frac{(bt;q)_\infty}{(-dt,et;q)_\infty}\frac{(at,-ab/deq;q)_\infty}{(aw;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b/wdeq,q/at;q)_j}{(q,q/aw;q)_j}(ft)^j\left[(-1)^jq^{\binom{j}{2}}\right]^{s-r}W_{k+j}$$
$$\times\frac{(ft)^k}{(q;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}q^{kj(s-r)}d_qt$$
$$= 2(1-q)\frac{(q^2;q^2)_\infty^2(de,q/de,b/e,-b/d;q)_\infty}{(d^2,e^2,q^2/d^2,q^2/e^2;q^2)_\infty(q;q)_\infty}\frac{(a/e,-a/d;q)_\infty}{(aw;q)_\infty}$$
$$\times\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{((-1)/dw,qe/a;q)_j}{(q,q/aw;q)_j}(f/e)^j\left[(-1)^jq^{\binom{j}{2}}\right]^{s-r}W_{k+j}\frac{(f/e)^k}{(q;q)_k}$$
$$\times\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}q^{kj(s-r)}.$$

□

- Setting $r=s=0$ in (3.1) and by using (1.3), we get Theorem 6.3. obtained by Chen and Liu [13] (equation (2.2).

3.2. Generalization of Gasper Integral

The following generalization of the Gasper integral is obtained by applying the operator ${}_r\Phi_s$ to both sides of the Gasper integral (2.3) and utilizing the equations (1.17) and (1.18):

Theorem 3.2 (Generalization of Gasper Integral). *Given Andrews-Askey integral* (2.3)*, then*

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(abcdfe^{i\theta},ac,ad,\mu e^{i\theta}/d,qde^{-i\theta}/\mu,c\mu e^{-i\theta},qe^{i\theta}/c\mu;q)_\infty}{(av,ae^{i\theta},be^{i\theta},ce^{-i\theta},de^{-i\theta},fe^{i\theta};q)_\infty}d\theta\\
&\quad\times\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}W_{k+j+i}(gd)^{k+j+i}\frac{(-1)^kq^{\binom{k}{2}}}{(q;q)_k}\left[(-1)^{k+j+i}q^{\binom{k+j+i}{2}}\right]^{s-r}\frac{(bcdf,q/ac;q)_i}{(q,q/ae^{i\theta};q)_i}\\
&\quad\times\left(\frac{c}{v}\right)^i\frac{(c/v;q)_j}{(q,q)_j}\frac{(q/ad;q)_{j+i}}{(q/av;q)_{j+i}}\\
&=\frac{(\mu c/d,dq/\mu c,\mu,q/\mu,abcd,acdf,bcdf;q)_\infty}{(q,bc,bd,fc,fd,av;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(bcd/v,q/acdf;q)_j}{(q,q/av;q)_j}\\
&\quad\times(gcdf)^j\left[(-1)^jq^{\binom{j}{2}}\right]^{s-r}W_{k+j}\frac{(gcdf)^k}{(q;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}q^{kj(s-r)},
\end{aligned}\tag{3.2}$$

provided that $\max\{|a|,|b|,|c|,|d|,|v|\}<1, cd\mu\neq 0.$

Proof: Rewrite Gasper integral (2.3) as

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\mu e^{i\theta}/d,qde^{-i\theta}/\mu,c\mu e^{-i\theta},qe^{i\theta}/c\mu,;q)_\infty}{(be^{i\theta},ce^{-i\theta},de^{-i\theta},fe^{i\theta};q)_\infty}\frac{(abcdfe^{i\theta},ac,ad;q)_\infty}{(ae^{i\theta};q)_\infty}d\theta\\
&\quad=\frac{(\mu c/d,dq/\mu c,\mu,q/\mu,acdf;q)_\infty}{(q,bc,bd,fc,fd;q)_\infty}(abcd,bcdf)_\infty.
\end{aligned}$$

Multiplying both sides of the above equation by $\frac{1}{(av;q)_\infty}$ and by acting the operator ${}_r\Phi_s\left(\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{matrix};q,-g\theta\right)$ on both sides of the above equation, we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\mu e^{i\theta}/d,qde^{-i\theta}/\mu,c\mu e^{-i\theta},qe^{i\theta}/c\mu;q)_\infty}{(be^{i\theta},ce^{-i\theta},de^{-i\theta},fe^{i\theta};q)_\infty}\\
&\quad\times{}_r\Phi_s\left(\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{matrix};q,-g\theta\right)\left\{\frac{(abcdfe^{i\theta},ac,ad;q)_\infty}{(av,ae^{i\theta};q)_\infty}\right\}d\theta\\
&=\frac{(\mu c/d,dq/\mu c,\mu,q/\mu,bcdf;q)_\infty}{(q,bc,bd,fc,fd;q)_\infty}{}_r\Phi_s\left(\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{matrix};q,-g\theta\right)\left\{\frac{(abcd,acdf;q)_\infty}{(av;q)_\infty}\right\}.
\end{aligned}$$

By using (1.17) and (1.18), we get

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\mu e^{i\theta}/d, qde^{-i\theta}/\mu, c\mu e^{-i\theta}, qe^{i\theta}/c\mu;q)_\infty}{(be^{i\theta},ce^{-i\theta},de^{-i\theta},fe^{i\theta};q)_\infty}\frac{(abcdfe^{i\theta},ac,ad;q)_\infty}{(av,ae^{i\theta};q)_\infty}d\theta\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}W_{k+j+i}\\
&\quad\times (gd)^{k+j+i}\left[(-1)^{k+j+i}q^{\binom{k+j+i}{2}}\right]^{s-r}\frac{(-1)^kq^{\binom{k}{2}}}{(q;q)_k}\frac{(bcdf,q/ac;q)_i}{(q,q/ae^{i\theta};q)_i}\left(\frac{c}{v}\right)^i\\
&\quad\times\frac{(c/v;q)_j}{(q,q)_j}\frac{(q/ad;q)_{j+i}}{(q/av;q)_{j+i}}\\
&=\frac{(\mu c/d,dq/\mu c,\mu,q/\mu,bcdf;q)_\infty}{(q,bc,bd,fc,fd;q)_\infty}\frac{(abcd,acdf;q)_\infty}{(av;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(bcd/v,q/acdf;q)_j}{(q,q/av;q)_j}\\
&\quad\times (gcdf)^j\left[(-1)^jq^{\binom{j}{2}}\right]^{s-r}W_{k+j}\frac{(gcdf)^k}{(q;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}q^{kj(s-r)}.\\
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(abcdfe^{i\theta},ac,ad,\mu e^{i\theta}/d,qde^{-i\theta}/\mu,c\mu e^{-i\theta},qe^{i\theta}/c\mu;q)_\infty}{(av,ae^{i\theta},be^{i\theta},ce^{-i\theta},de^{-i\theta},fe^{i\theta};q)_\infty}d\theta\\
&\quad\times\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}W_{k+j+i}(gd)^{k+j+i}\frac{(-1)^kq^{\binom{k}{2}}}{(q;q)_k}\left[(-1)^{k+j+i}q^{\binom{k+j+i}{2}}\right]^{s-r}\frac{(bcdf,q/ac;q)_i}{(q,q/ae^{i\theta};q)_i}\\
&\quad\times\left(\frac{c}{v}\right)^i\frac{(c/v;q)_j}{(q,q)_j}\frac{(q/ad;q)_{j+i}}{(q/av;q)_{j+i}}\\
&=\frac{(\mu c/d,dq/\mu c,\mu,q/\mu,abcd,acdf,bcdf;q)_\infty}{(q,bc,bd,fc,fd,av;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(bcd/v,q/acdf;q)_j}{(q,q/av;q)_j}\\
&\quad\times (gcdf)^j\left[(-1)^jq^{\binom{j}{2}}\right]^{s-r}W_{k+j}\frac{(gcdf)^k}{(q;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}q^{kj(s-r)}.
\end{aligned}$$

□

- Letting $v=r=s=0$ in (3.2) and by using (1.2) and (1.3) we obtain Theorem 4.1. obtained in Zhang and Wang [42] (equation (2.4).

3.3. Generalization of Andrews-Askey Integral

By using the operator ${}_r\Phi_s$ to both sides of the Andrews-Askey integral (2.5), and by employing (1.18) and (1.16), we obtain the following generalization of the Andrews-Askey integral:

Theorem 3.3 (Generalization of Andrews-Askey integral). *Given Andrews-Askey integral (2.5), then*

$$\begin{aligned}
&\int_c^d\frac{(ac,qx/c,qx/d;q)_\infty}{(ax,bx;q)_\infty}\sum_{k=0}^{\infty}\left(\frac{gq}{a}\right)^k\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}W_k\frac{(c/x;q)_k}{(q/ax;q)_k}d_qx\\
&\quad=\frac{d(1-q)(q,dq/c,c/d,abcd;q)_\infty}{(ad,bc,bd,cd;q)_\infty}\sum_{k=0}^{\infty}\left(\frac{gq}{a}\right)^k\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}W_k\frac{(bc;q)_k}{(q/ad;q)_k},
\end{aligned}\tag{3.3}$$

provided that $\max\{|ac|,|ad|,|bc|,|bd|\}<1$.

Proof: Rewrite Andrews-Askey integral (2.5) as follows:

$$\int_c^d\frac{(qx/c,qx/d;q)_\infty}{(bx;q)_\infty}\frac{(ac,ad;q)_\infty}{(ax;q)_\infty}d_qx=\frac{d(1-q)(q,dq/c,c/d;q)_\infty}{(bc,bd;q)_\infty}(abcd;q)_\infty.$$

Acting the operator ${}_r\Phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,-g\theta\right)$ on both sides of the above equation yielding

$$\begin{aligned}&\int_c^d \frac{(qx/c,qx/d;q)_\infty}{(bx;q)_\infty}\,{}_r\Phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,-g\theta\right)\left\{\frac{(ac,ad;q)_\infty}{(ax;q)_\infty}\right\}d_qx \\ &\quad= \frac{d(1-q)(q,dq/c,c/d;q)_\infty}{(bc,bd;q)_\infty}\,{}_r\Phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,-g\theta\right)\{(abcd;q)_\infty\}.\end{aligned}$$

By using (1.18) and (1.16), we obtain

$$\begin{aligned}&\int_c^d \frac{(qx/c,qx/d;q)_\infty}{(bx;q)_\infty}\frac{(ac,ad;q)_\infty}{(ax;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c/x,q/ad;q)_j}{(q,q/ax;q)_j}(gd)^j \\ &\quad\times\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r} W_{k+j}\frac{(gd)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} q^{kj(s-r)}d_qx \\ &= \frac{d(1-q)(q,dq/c,c/d;q)_\infty}{(bc,bd;q)_\infty}(abcd;q)_\infty\sum_{k=0}^{\infty}\frac{(gbcd)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}W_k. \\ &\int_c^d \frac{(qx/c,qx/d;q)_\infty}{(ax,bx;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c/x,q/ad;q)_j}{(q,q/ax;q)_j}(gd)^j\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r} \\ &\quad\times W_{k+j}\frac{(gd)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} q^{kj(s-r)}d_qx \\ &= \frac{d(1-q)(q,dq/c,c/d,abcd;q)_\infty}{(ac,ad,bc,bd;q)_\infty}\sum_{k=0}^{\infty}\frac{(gbcd)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}W_k.\end{aligned}$$

□

- Setting $r=2$, $s=1$, $a_1=r$, $a_2=w$, $b_1=v$ and $c=-v/wrt$ in (3.3), then by using (1.1) and (1.4) we get Theorem 15 obtained by Cao [12] (equation (2.6).

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