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Exploring Properties of D-Lindelöf Bitopological Spaces and Their Relations to D-Compact Spaces

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ABSTRACT: This paper systematically introduces the fundamental concepts of D-sets and examines their essential properties, with particular emphasis on D-Lindelöfness and D-countably compactness. We present the basic notions of bitopological spaces, along with relevant definitions and examples, including compactness, P*-compactness, S*-compactness, and P*-countably compactness and their properties. The main objective of this study is to investigate the structural properties of D-Lindelöf bitopological spaces, as well as P*-D-Lindelöf, S*-D-Lindelöf and D-Lindelöf spaces. We also analyze D-countably compact, P*-D-countably compact and S*-D-countably compact spaces. In addition, we examine the relationships between these properties and provide examples to highlight their differences.

Keywords: D-sets, τ_1^*, τ_2^*-D-cover, P*-D-cover, D-Lindelöf, P*-D-Lindelöf, S*-D-Lindelöf, D-countably compact, P*-D-countably compact, S*-D-countably compact.

Contents

1. Introduction

The study of bitopological spaces, initiated by [1] , involves sets equipped with two topologies, denoted as (X, τ_1, τ_2). Many researchers,including [2,3,4,5,6] and [7] have explored various aspects, including compactness, connectedness, covering properties, and separation axioms. [8] extended this to pairwise Lindelöf spaces, while [9] introduced P*-Lindelöf spaces. [10] introduced D-sets and new separation axioms. In [11] the almost Lindelöf degree $aL(X)$ was defined as a new cardinal function. Researchers have explored various aspects of Lindelöf spaces. For example, [12] studied nearly regular-Lindelöf, almost regular-Lindelöf and weakly regular-Lindelöf spaces. In [13,14] investigated pairwise almost Lindelöf and pairwise weakly regular-Lindelöf bitopological spaces. Moreover, [15] introduced and studied m-paraLindelöf, countable paraLindelöf, m-semiparaLindelöf and countable semiparaLindelöf spaces. [16] introduced weakly linearly Lindelöf and almost discretely Lindelöf spaces.

[17] introduced the concept of almost countably compact spaces in bitopological settings, generalizing countably compact spaces, and investigated their properties and relationships with countably compact spaces, thereby enhancing the understanding of compactness in bitopological spaces. In general, researchers have made significant contributions to the theory of bitopological spaces.

[18] showed that contra second countable spaces are P_1^*-Lindelöf spaces in bitopology, but the reverse does not necessarily hold.In addition, the authors clarified the relationship between these types of space and studied the connections between P_1-normal spaces and pairwise regular spaces. [19] focused on the finite product of pairwise nearly Lindelöf, pairwise almost Lindelöf and pairwise weakly Lindelöf spaces, finding that these properties are not preserved under product formation, but identified necessary conditions for their preservation. [20] demonstrated that the product space of a sequence of nonempty

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compact metric spaces is compact in the product topology using fundamental properties of compact metric spaces without relying on the axiom of choice or Tychonoff's theorem. [21] introduced several new concepts related to Lindelöf spaces in bitopological settings, including pairwise strongly Lindelöf, pairwise nearly Lindelöf, pairwise almost Lindelöf, and pairwise weakly strongly Lindelöf spaces, exploring their properties and relationships. [22] explored almost cellular-Lindelöf spaces, establishing bounds and examples, particularly in relation to feebly Lindelöf and weakly Lindelöf spaces. [23] investigated set star-Lindelöf spaces, a class of spaces between Lindelöf and star-Lindelöf spaces, providing examples and analyzing their topological properties. [24] expanded on pairwise countably compact spaces in bitopological spaces, generalizing and exploring their properties. [25] introduced δ^*-Lindelöf tritopological spaces, defined with respect to the δ^*-open sets, and explored their key characteristics, including relationships between δ^*-Lindelöfness and fundamental concepts such as δ^*-compactness, δ^*-separability and δ^*-second countability.

[26] constructed examples of regular countably compact $\mathbb{R}$-rigid spaces with properties like separability and first countability, answering questions posed by Tzannes, Banakh, and Ravsky, and provided a consistent example of an $\mathbb{R}$-rigid Nyikos space. [27] explored the foundations of computably compact metric spaces, introducing new characterizations and applications in computable analysis and effective topology. [28] introduced I-compactness as a covering property through ideals of $\mathbb{N}$, comparing it to s-compactness, compactness and sequential compactness, and explored its properties in regular and separable spaces, establishing connections between I-compactness and sequential I-compactness.

[29] explored nearly pairwise Lindelöf spaces within regular covers, focusing on properties of these novel generalizations of pairwise Lindelöf spaces. [30] presented uniform analogues of strongly paracompact and Lindelöf spaces, examining connections to other compactness properties and providing characterizations using finitely additive open covers, compactifications and ω-mappings. [31] introduced D-Lindelöf spaces, derived from D-sets using a D-cover, establishing foundational principles, exploring properties and relationships with other topological spaces and examining connections between D-Lindelöf spaces and Lindelöf spaces, as well as related spaces like D-compact and D-countably compact, providing theoretical developments, examples and counterexamples to highlight key distinctions and novel insights.

[32] introduced D-paralindelöf spaces, a new class of topological spaces formed by integrating D-sets and paralindelöf spaces, providing rigorous definitions for both paralindelöf and D-paralindelöf spaces. Meanwhile, [33] explored α-continuous functions modulo (J, I) between ideal spaces, focusing on properties like α-compactness and α-pseudocompactness.

Here, firstly, we introduce some basic concepts of bitopological spaces, including Lindelöfness, P*-Lindelöfness, and S*-Lindelöfness. Secondly, we provide an overview of D-sets and D-Lindelöfness in topological spaces and D-countably compactness.

Then, we present a concept of D-sets in bitopological space and develop various new Lindelöfness properties by using D-sets, such as D-Lindelöfness, P*-D-Lindelöfness, S*-D-Lindelöfness, and D-countably compactness. The relationships between these concepts are investigated, supported by illustrative examples and proven results. Additionally, counterexamples are provided, and the impact of specific functions on these properties is also investigated, contributing to a deeper understanding of bitopological spaces.

Throughout this paper, we employ the following notations:

- $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, I_{rr}, \mathbb{N}, \mathbb{N}^e, \mathbb{N}^o$ denote the sets of real, rational, integer, irrational, natural, even, and odd numbers, respectively.
- $(\tau^*_{l.r}, \tau^*_{rr}, \tau^*_u, \tau^*_{cof}, \tau^*_{coc}, \tau^*_{dis}, \tau^*_s, \tau^*_{ind}, \tau^*_{sup})$ denote the left-ray, right-ray, usual, co-finite, co-countable, discrete, Sorgenfrey line, indiscrete, and supremum topologies, respectively.
- Notational conventions:
  - P*: pair
  - S*: semi
  - P*-D-Lindelöf: pairwise D-Lindelöf
  - S*-D-Lindelöf: semi D-Lindelöf

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2. Basic Concepts:D-Sets, D-Lindelöfness and D-Countably Compactness

We provide an overview of D-sets, D-Lindelöfness and D-countably compactness in topological spaces.

Definition 2.1 ([10]) *Let (X^*, τ^*) be a topological space and let $V \subseteq X^*$. If V can be expressed as*

$$V = v_1 - v_2,$$

where v_1, v_2 are two open sets in τ^* and $v_1 \neq X^*$, then V is said to be a D-set generated by v_1 and v_2.

Result: From the previous definition, it follows that every open set $v_1 \neq X^*$ is a D-set, as it can be represented as $V = v_1$ with $v_2 = \varnothing$.

Example 2.1 Let $X^* = \{z_1, z_2, z_3\}$ and $\tau^* = \{\varnothing, X^*, \{z_1\}, \{z_1, z_2\}\}$, then the D-sets in X^* are:

$$\varnothing, \{z_1\}, \{z_1, z_2\}, \{z_2\}.$$

Example 2.2 Consider $(X^*, \tau^*_{\text{dis}})$. Then any proper subset of X^* is a D-set, meaning all members of $\mathcal{P}(X^*)$ are D-sets in X^*.

Example 2.3 Consider $(\mathbb{R}, \tau^*_{rr})$, the following forms of D-sets can be observed:

$$\{\varnothing, (x, \infty)\} \quad \text{and} \quad (x, y].$$

Since

$$(x, \infty) - (y, \infty) = \begin{cases} (x, y] & \text{if } x < y, \\ \varnothing & \text{if } x \geq y. \end{cases}$$

Example 2.4 Consider $(\mathbb{R}, \tau^*_{l.r})$, the following forms of D-sets can be observed:

$$\{\varnothing, (-\infty, x)\} \quad \text{and} \quad [x, y).$$

Since

$$(-\infty, x) - (-\infty, \beta) = \begin{cases} [x, y) & \text{if } x > y, \\ \varnothing & \text{if } x \leq y. \end{cases}$$

Example 2.5 Consider $(\mathbb{R}, \tau^*_{cof})$, the following forms of D-sets can be observed:

$$\varnothing, \{\mathbb{R} - \text{finite set}\} \text{ and any finite set.}$$

For instance,

$$(\mathbb{R} - \{1, 2, 3\}) - (\mathbb{R} - \{1, 5, 6\}) = \{5, 6\}$$

is a finite set, which is a D-set.

Example 2.6 Consider $(\mathbb{R}, \tau^*_{coc})$, the following forms of D-sets can be observed:

$$\varnothing, \{\mathbb{R} - \text{countable set}\} \text{ and any countable set.}$$

Example 2.7 Consider $(\mathbb{R}, \tau^*_u)$. Various forms of D-sets can be identified, such as:

$$\tau^*_u - \{\mathbb{R}\}, \ (-\infty, x], \ [x, \infty), \ (x, y], \ [x, y], \ [x, y),$$

and combinations like $\{[x, y) \cup (z, d] : y < z\}$, $\{(-\infty, x] \cup [y, z) : x < y\}$, etc.

For example:

$$\begin{aligned}
(-\infty, 9) - (2, 5) &= (-\infty, 2] \cup [5, 9), \\
(-\infty, 9) - (7, 10) &= (-\infty, 7], \\
(0, \infty) - (2, \infty) &= (0, 2], \\
(0, \infty) - (0, 2) &= [2, \infty), \\
(0, 6) - (2, 4) &= (0, 2] \cup [4, 6), \\
(0, \infty) - (2, 5) &= (0, 2] \cup [5, \infty), \\
(5, 10) - ((5, 7) \cup (8, 10)) &= [7, 8].
\end{aligned}$$

Remark 2.1 Note that while every open set is a D-set, not every D-set is necessarily an open set.

Theorem 2.1 ([34]) *The intersection of two D-sets, D_1 and D_2, is also a D-set.*

Corollary 2.1 *In any topological space, the intersection of a finite number of D-sets is also a D-set, as can be shown by induction.*

Remark 2.2 Neither the union of a finite number of D-sets nor the difference of two D-sets is necessarily a D-set.

Example 2.8 Consider the topological space $(\mathbb{R}, \tau^*_{rr})$. The sets $(3, 4]$ and $(13, \infty)$ are D-sets, but their union $(3, 4] \cup (13, \infty)$ is not a D-set.

Also, the sets $[2, 9)$ and $[4, 6)$ are D-sets in $(\mathbb{R}, \tau^*_{l.r})$. However,

$$[2, 9) - [4, 6) = [2, 4) \cup [6, 9)$$

is not a D-set.

Definition 2.2 ([34]) *Let (X^*, τ^*) be a topological space. Then:*

1. *X^* is called D–Lindelöf if every D–cover of X^* has a countable subcover.*
2. *X^* is called D–countably compact if every countable D–cover of X^* has a finite subcover.*

Theorem 2.2 *Every D–Lindelöf space is a Lindelöf space.*

Theorem 2.3 *The countable union of D–Lindelöf sets is D–Lindelöf.*

Theorem 2.4 ([34]) *Every D–compact space is D–Lindelöf and D–countably compact.*

Theorem 2.5 ([34]) *If X^* is a countable set, then (X^*, τ^*) is a D–Lindelöf space for any topology τ^* on X^*.*

Note. A D–Lindelöf space need not be countable. For instance, the topological space $(\mathbb{R}, \tau^*_{\text{indis}})$ is D–Lindelöf, but not countable.

Theorem 2.6 ([34]) *Every D–Lindelöf and D–countably compact space (X^*, τ^*) is D–compact.*

Theorem 2.7 ([31]) *Every closed subspace of a D–Lindelöf space is also D–Lindelöf.*

Theorem 2.8 *Every D–countably compact space is hereditarily D–countably compact with respect to closed subspaces.*

Theorem 2.9 ([34]) *The continuous image of a D–Lindelöf space is D–Lindelöf.*

Theorem 2.10 ([34]) *The continuous image of a D–countably compact space is D–countably compact.*

3. Review of Lindelöfness in Bitopological Space

In this section, we review the properties of bitopological spaces, specifically focusing on Lindelöf, P*-Lindelöf, S*-Lindelöf and P*-countably compact spaces.

Definition 3.1 ([9]) *A space $(X^*, \tau_1^*, \tau_2^*)$ is said to be **Lindelöf** if each (X^*, τ_1^*) and (X^*, τ_2^*) are Lindelöf spaces.*

*A space $(X^*, \tau_1^*, \tau_2^*)$ is **P*-Lindelöf** if every P*–open cover of $(X^*, \tau_1^*, \tau_2^*)$ has a countable subcover.*

*A space $(X^*, \tau_1^*, \tau_2^*)$ is **S*-Lindelöf** if every $\tau_1^*\tau_2^*$–open cover of $(X^*, \tau_1^*, \tau_2^*)$ has a countable subcover.*

Theorem 3.1 *Every S*-Lindelöf space is a P*-Lindelöf space.*

Remark 3.1 The converse of the above theorem does not necessarily hold.

Example 3.1 Consider $(\mathbb{R}, \tau^*_{ind}, \tau^*_{cof})$. Any P*–open cover of $(\mathbb{R}, \tau^*_{ind}, \tau^*_{cof})$ must contain $\{\mathbb{R}\}$, so it has a countable subcover, namely $\{\mathbb{R}\}$. Therefore, it is P*-Lindelöf. Furthermore, it is also S*-Lindelöf and Lindelöf since both $(\mathbb{R}, \tau^*_{ind})$ and $(\mathbb{R}, \tau^*_{cof})$ are Lindelöf spaces.

Definition 3.2 ([5]) *A space $(X^*, \tau_1^*, \tau_2^*)$ is called **P*-countably compact** if every countable P*–open cover of $(X^*, \tau_1^*, \tau_2^*)$ has a finite subcover.*

Theorem 3.2 *If G and H are two S*-Lindelöf subsets of a bitopological space $(X^*, \tau_1^*, \tau_2^*)$, then $H \cup G$ is an S*-Lindelöf subset of X^*.*

Proof: To prove that $H \cup G$ is an S*-Lindelöf subset of X^*, we must show that for any $\tau_1^*\tau_2^*$–open cover of $H \cup G$, there exists a countable subcover.

Let $\{V_\alpha : \alpha \in \Omega\}$ be any $\tau_1^*\tau_2^*$–open cover of $H \cup G$. Then $H \cup G \subseteq \bigcup_{\alpha\in\Omega} V_\alpha$ and therefore $H \subseteq \bigcup_{\alpha\in\Omega} V_\alpha$ and $G \subseteq \bigcup_{\alpha\in\Omega} V_\alpha$.

Hence, $\{V_\alpha : \alpha \in \Omega\}$ is a $\tau_1^*\tau_2^*$–open cover for both H and G Since H and G are S*-Lindelöf subsets, there exist countable subsets $\Omega_1, \Omega_2 \subseteq \Omega$ such that $\{V_{\alpha_i} : i \in \Omega_1\}$ and $\{V_{\alpha_i} : i \in \Omega_2\}$ are countable subcovers of H and G, respectively.

Therefore, $\{V_{\alpha_i} : i \in \Omega_1\} \cup \{V_{\alpha_i} : i \in \Omega_2\}$ is a countable subcover of $H \cup G$. Hence, $H \cup G$ is an S*-Lindelöf subset of X^*. □

Remark 3.2 If G and H are two S*-Lindelöf subsets of a bitopological space $(X^*, \tau_1^*, \tau_2^*)$, then $G \cap H$ need not be S*-Lindelöf.

Example 3.2 Let $X^* = \mathbb{R} \cup \{i, -i\}$, where $i, -i$ are complex numbers and define

$$\tau_1^* = P(\mathbb{R}) \cup \{A \subseteq X^* : -i, i \in A \text{ and } (X^* - A) \text{ is countable}\}$$

and

$$\tau_2^* = \tau_1^* \cup \{A \subseteq X^* : (-i \in A) \text{ or } (i \in A) \text{ and } (X^* - A) \text{ is countable}\}.$$

Now, let $G = \mathbb{R} \cup \{-i\}$ and $H = \mathbb{R} \cup \{i\}$. Then both G and H are S*-Lindelöf subsets of $(X^*, \tau_1^*, \tau_2^*)$, but $G \cap H = \mathbb{R}$ is not an S*-Lindelöf set.

4. D-Lindelöf and D-Countably Compactness in Bitopological Spaces

In this section, we introduce the concept of bitopological spaces, along with definitions and examples related to D-Lindelöfness and D-countably compactness. Furthermore, we discuss various compactness properties, such as compactness, P*-compactness and S*-compactness, providing illustrative examples to demonstrate these concepts.

Definition 4.1 *A bitopological space $(X^*, \tau_1^*, \tau_2^*)$ is called:*

1. ***P*-D-Lindelöf** if every P*-D-cover has a countable subcover.*
2. ***S*-D-Lindelöf** if every $\tau_1^*\tau_2^*$-D-cover has a countable subcover.*
3. ***D-Lindelöf** if each (X^*, τ_1^*) and (X^*, τ_2^*) are D-Lindelöf spaces.*

Corollary 4.1 *Every S*-D-Lindelöf space is a P*-D-Lindelöf space. However, the converse need not hold.*

Example 4.1 Consider $(\mathbb{R}, \tau^*_{dis}, \tau^*_{ind})$. Then it is P*-D-Lindelöf but not S*-D-Lindelöf.

Example 4.2 Given $(\mathbb{R}, \tau^*_{dis}, \tau^*_{l.r.})$, this space is not P*-D-Lindelöf since the P*-D-cover $\{(-\infty, 0) \cup \{\{y\} : y > 0\}\}$ has no countable subcover. Also, it is not S*-D-Lindelöf.

Example 4.3 The space $(\mathbb{R}, \tau_u^*, \tau_s^*)$ is Lindelöf, but not S*-D-Lindelöf and not P*-D-Lindelöf.

Example 4.4 $(\mathbb{Z}, \tau_{dis}^*, \tau_{coc}^*)$ is Lindelöf, S*-D-Lindelöf and P*-D-Lindelöf.

Theorem 4.1 *If (X^*, τ_{sup}^*) is a D-Lindelöf space, then $(X^*, \tau_1^*, \tau_2^*)$ is S*-D-Lindelöf.*

Proof: Let $\tilde{C}$ be any $\tau_1^*\tau_2^*$-D-cover of X^*. Then $\tilde{C}$ is a τ_{sup}^*-D-cover of X^*. Since (X^*, τ_{sup}^*) is D-Lindelöf, it has a countable subcover. Hence $(X^*, \tau_1^*, \tau_2^*)$ is S*-D-Lindelöf. □

Theorem 4.2 *A space $(X^*, \tau_1^*, \tau_2^*)$ is S*-D-Lindelöf if and only if it is both P*-D-Lindelöf and D-Lindelöf.*

Proof: Let $\tilde{C}$ be a $\tau_1^*\tau_2^*$-D-cover of X^*.

1. If $\tilde{C}$ is a τ_1^*-D-cover, then it has a countable subcover since (X^*, τ_1^*) is D-Lindelöf.
2. If $\tilde{C}$ is a τ_2^*-D-cover, then it has a countable subcover since (X^*, τ_2^*) is D-Lindelöf.
3. If $\tilde{C}$ is a P*-D-cover, then it has a countable subcover since $(X^*, \tau_1^*, \tau_2^*)$ is P*-D-Lindelöf.

Therefore, $(X^*, \tau_1^*, \tau_2^*)$ is S*-D-Lindelöf. □

Theorem 4.3 *If X^* is countable, then $(X^*, \tau_1^*, \tau_2^*)$ is P*-D-Lindelöf.*

Proof: Let $\tilde{C} = \{D_\alpha : \alpha \in \Omega\}$ be a P*-D-cover of X^*. For each $b_i \in X^*$, choose $D_{\alpha_i} \in \tilde{C}$ such that $b_i \in D_{\alpha_i}$. Then $\tilde{C}^* = \{D_{\alpha_i} : i = 1, 2, 3, \ldots\}$ is a countable subcover of X^*. Hence the result. □

Theorem 4.4 *If X^* is countable, then $(X^*, \tau_1^*, \tau_2^*)$ is also S*-D-Lindelöf and D-Lindelöf.*

Theorem 4.5 *Every D-Lindelöf space $(X^*, \tau_1^*, \tau_2^*)$ is Lindelöf.*

Theorem 4.6 *Every P*-D-Lindelöf space is P*-Lindelöf.*

Proof: This follows from the fact that every P*-cover is a P*-D-cover. □

Theorem 4.7 *Every S*-D-Lindelöf space is S*-Lindelöf.*

Proof: Using the same reasoning as in the previous theorem, the result follows. □

Example 4.5 Consider $(\mathbb{R}, \tau_{dis}^*, \tau_{coc}^*)$. Then this space is not S*-Lindelöf and hence it is not S*-D-Lindelöf. It is P*-Lindelöf but it is not P*-D-Lindelöf. In addition, $(\mathbb{R}, \tau_{dis}^*, \tau_{coc}^*)$ is not Lindelöf and therefore not D-Lindelöf.

Example 4.6 Let τ_1^* and τ_2^* be two topologies on $\mathbb{R}$ defined by the bases

$$\beta_1 = \{(-\infty, b) : b > 0\} \cup \{\{y\} : y > 0\}, \quad \beta_2 = \{(b, \infty) : b < 0\} \cup \{\{y\} : y < 0\}.$$

Then $(\mathbb{R}, \tau_1^*, \tau_2^*)$ is not Lindelöf and hence it is neither D-Lindelöf nor S*-D-Lindelöf. It is also not P*-Lindelöf and thus not P*-D-Lindelöf.

Definition 4.2 *A space $(X^*, \tau_1^*, \tau_2^*)$ is called **P*-D-countably compact** if every countable P*-D-cover of $(X^*, \tau_1^*, \tau_2^*)$ has a finite subcover.*

Theorem 4.8 *Every D (respectively P*-D, S*-D) compact space is D (respectively P*-D, S*-D) Lindelöf and D (respectively P*-D, S*-D) countably compact.*

Proof: Let $\tilde{C} = \{V_\alpha : \alpha \in \Omega\} \cup \{U_\gamma : \gamma \in \Delta\}$ be an arbitrary D (respectively P*-D, S*-D) cover of $(X^*, \tau_1^*, \tau_2^*)$. Since $(X^*, \tau_1^*, \tau_2^*)$ is D (respectively P*-D, S*-D) compact, $\tilde{C}$ has a finite subcover. As every finite set is countable, $(X^*, \tau_1^*, \tau_2^*)$ is both D (respectively P*-D, S*-D) Lindelöf and D (respectively P*-D, S*-D) countably compact. □

Theorem 4.9 *Every D (respectively P*-D, S*-D) Lindelöf countably compact space $(X^*, \tau_1^*, \tau_2^*)$ is D (respectively P*-D, S*-D) compact.*

Example 4.7 The space $(\mathbb{Q}, \tau_{dis}^*, \tau_{coc}^*)$ is P*-D-Lindelöf and D-Lindelöf, hence S*-D-Lindelöf. However, it is not D-compact, P*-D-compact or S*-D-compact, nor countably compact in any of these senses.

Theorem 4.10 *Let $f : (X^*, \tau_1^*, \tau_2^*) \to (Y^*, T_1^*, T_2^*)$ be a P*–continuous, P*–closed, onto function and suppose Y^* is locally indiscrete. Then Y^* is P*–D–Lindelöf whenever X^* is P*–D–Lindelöf.*

Proof: Let $\tilde{C} = \{V_\alpha : \alpha \in \Omega\} \cup \{U_\gamma : \gamma \in \Delta\}$ be a P*–D–cover of Y^*. Since f is P*–continuous and onto, the set $\tilde{\beta} = \{f^{-1}(V_\alpha) : \alpha \in \Omega\} \cup \{f^{-1}(U_\gamma) : \gamma \in \Delta\}$ is a P*–open cover of X^*. As X^* is P*–D–Lindelöf, there exist countable subsets $\Omega^* \subseteq \Omega$, $\Delta^* \subseteq \Delta$ such that $\tilde{\beta}^* = \{f^{-1}(V_\alpha) : \alpha \in \Omega^*\} \cup \{f^{-1}(U_\gamma) : \gamma \in \Delta^*\}$ is a countable subcover. Hence, $\tilde{C}^* = \{V_\alpha : \alpha \in \Omega^*\} \cup \{U_\gamma : \gamma \in \Delta^*\}$ is a countable subcover of Y^*. Therefore, Y^* is P*–D–Lindelöf. □

Definition 4.3 *A bitopological space $(X^*, \tau_1^*, \tau_2^*)$ is called **hereditary Lindelöf (D-Lindelöf)** if both (X^*, τ_1^*) and (X^*, τ_2^*) are hereditary Lindelöf (D-Lindelöf).*

Example 4.8 Consider the two bases on $\mathbb{R}$ defined by $\beta_1^* = \{\{y\} : y \in \mathbb{R} - \{5\}\} \cup \{\emptyset, \mathbb{R}\}$ and $\beta_2^* = \{\{y\} : y \in \mathbb{R} - \{4\}\} \cup \{\emptyset, \mathbb{R}\}$. Let τ_1^* and τ_2^* be the topologies generated by these bases. Then $(\mathbb{R}, \tau_1^*, \tau_2^*)$ is neither hereditary Lindelöf nor hereditary D-Lindelöf. For $S = (1, 5)$, the subspace $(S, \tau_1^*(S))$ is neither Lindelöf nor D-Lindelöf.

Theorem 4.11 *If $(X^*, \tau_1^*, \tau_2^*)$ is hereditary D-Lindelöf, then it is S*-D-Lindelöf.*

Proof: Let $\tilde{C} = \{V_\alpha : \alpha \in \Omega\} \cup \{U_\gamma : \gamma \in \Delta\}$ be any $\tau_1^*\tau_2^*$–D–cover of X^*. Since $U = \bigcup_{\gamma \in \Delta} U_\gamma$ is τ_1^*–D-Lindelöf, there exists a countable subset $\Delta^* \subseteq \Delta$ such that $U = \bigcup_{\gamma \in \Delta^*} U_\gamma$. Similarly, since $V = \bigcup_{\alpha \in \Omega} V_\alpha$ is τ_2^*–D-Lindelöf, there exists a countable subset $\Omega^* \subseteq \Omega$ such that $V = \bigcup_{\alpha \in \Omega^*} V_\alpha$. Hence, $\{V_\alpha : \alpha \in \Omega^*\} \cup \{U_\gamma : \gamma \in \Delta^*\}$ is a countable subcover of X^*. Thus, $(X^*, \tau_1^*, \tau_2^*)$ is S*–D–Lindelöf. □

Corollary 4.2 *If $(X^*, \tau_1^*, \tau_2^*)$ is hereditary D-compact, then it is S*–D–Lindelöf.*

5. Conclusion

In this study, we have explored the concepts of D-Lindelöfness and D-countably compactness in bitopological spaces. We have introduced various definitions and properties, including P*-D-Lindelöf, S*-D-Lindelöf and D-Lindelöf spaces. By presenting multiple theorems and examples, we have examined the connections between these properties and established conditions under which they hold.

Our results showed that every S*-D-Lindelöf space is P*-D-Lindelöf, but the converse did not necessarily true. Furthermore, we proved that hereditary D-Lindelöf spaces are S*-D-Lindelöf.

The study of D-Lindelöfness and D-countably compactness in bitopological spaces has theoretical implications for understanding the properties of topological spaces. Overall, this research advances our understanding of bitopological spaces and provides a foundation for further exploration of these concepts.

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